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# Differentiation of Solutions of Dynamic Equations on Time Scales with Respect to Parameters

**Roman Hilscher**
Masaryk University
Department of Mathematics and Statistics, Faculty of Science
Kotlářská 2, CZ-61137 Brno, Czech Republic
hilscher@math.muni.cz

**Vera Zeidan**
Michigan State University
Department of Mathematics
East Lansing, MI 48824, USA
zeidan@math.msu.edu

**Werner Kratz**
University of Ulm
Institute of Applied Analysis
D-89069 Ulm, Germany
werner.kratz@uni-ulm.de

**Abstract**

In this paper we prove the differentiability properties of solutions of nonlinear dynamic equations on time scales with respect to parameters. This complements the previous work of the first and third authors regarding the existence and continuity of solutions with respect to parameters. In addition, we treat separately time scale dynamic equations which are linear with respect to the unknown function and the parameter. For this case we derive an improved result which says that the solution is an entire function of the parameter.

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**Keywords:** Time scale, embedding theorem, differentiation with respect to parameters, entire function, complex domain.

## 1 Introduction

In this paper we study the differentiability properties of solutions of dynamic equations on the so-called time scales $\mathbb{T}$, which are defined to be any nonempty and closed subsets of $\mathbb{R}$. This allows one to unify the traditional results for the differential equations (which

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we call the *continuous case*) and the difference equations (which we call the *discrete case*), explain the differences between the continuous and discrete theories, and extend such results to arbitrary time scales.

In this paper we consider the nonlinear time scale dynamic system

$$x^\Delta = f(t, x, \lambda), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.1}$$
$$x(a) = x^0(\lambda), \tag{1.2}$$

where the nonlinearity $f$ and the initial value $x^0$ may depend on a parameter $\lambda \in \mathbb{R}^r$. The data and the solutions are considered to be *real* valued, however, the results of this paper easily extend to the complex setting.

Assuming that there exists a solution $x(t, 0)$ on $[a, b]_{\mathbb{T}}$ of the problem (1.1), (1.2) with $\lambda = 0$, the first and third authors proved a time scale embedding theorem in [16, Theorem 3.2] saying that the solution $x(t, \lambda)$ of (1.1), (1.2) exists on $[a, b]_{\mathbb{T}}$ and is continuous in $\big(t, x^0(\lambda), \lambda\big)$. In particular, the function $x(t, \cdot)$ is continuous in $\lambda$. Note that the embedding theorem in [16, Theorem 3.2] was proved for the case when $\lambda \in \mathbb{R}$, but the same proof remains valid when $\lambda \in \mathbb{R}^r$.

In the present work we continue in this direction and show that, under suitable assumptions, the solutions $x(t, \lambda)$ of (1.1), (1.2) are differentiable in $\lambda$, and the partial derivative $x_\lambda(t, \lambda) := \dfrac{\partial}{\partial \lambda} x(t, \lambda)$, satisfies at $\lambda = 0$ the linearized dynamic equation

$$Z^\Delta = A(t)\, Z + P(t), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \tag{1.3}$$
$$Z(a) = Dx^0(0), \tag{1.4}$$

where $Dx^0(\cdot)$ denotes the Jacobian of the function $x^0(\cdot)$, $A(t)$ is the $n \times n$ matrix and $P(t)$ is the $n \times r$ matrix defined on $[a, \rho(b)]_{\mathbb{T}}$ by

$$A(t) := f_x\big(t, x(t, 0), 0\big), \quad P(t) := f_\lambda\big(t, x(t, 0), 0\big). \tag{1.5}$$

The proof is based on the Gronwall inequality on time scales (see Lemma 2.3 below).

An expected consequence of this is that the time scale differentiation of $x(t, \lambda)$ with respect to $t$ and the usual (partial) differentiation of $x(t, \lambda)$ with respect to $\lambda$ can be interchanged, i.e., the function $x^\Delta(t, \cdot)$ is differentiable in $\lambda$ and

$$\frac{\partial}{\partial \lambda} \big[x^\Delta(t, \lambda)\big]_{\lambda=0} = \big[x_\lambda(t, 0)\big]^\Delta \quad \text{for all } t \in [a, \rho(b)]_{\mathbb{T}}. \tag{1.6}$$

Of course, such a result is very desirable, for instance, this issue arises for $r = 1$ when studying eigenvalue problems on time scales, see e.g. [2, Lemma 4] or [7, formula (3)], although in [2] it is not explicitly stated that such a property should be satisfied. This problem can be viewed from the perspective of partial dynamic equations, where the solution $x$ is defined on $\mathbb{T} \times \mathbb{R}^r$, the product of $r + 1$ time scales, or on its subset $[a, b]_{\mathbb{T}} \times \bar{B}_\gamma$, where $\bar{B}_\gamma$ is the closed ball in $\mathbb{R}^r$ of radius $\gamma$. Then formula (1.6) represents the equality of the mixed second order partial derivatives of $x(t, \lambda)$. As in the continuous time case, this result is indeed guaranteed once these mixed second order partial derivatives are *continuous* in the topology of the given time scale, see [6, Theorem 6.1]. However, as we shall see, in our case the functions $\frac{\partial}{\partial\lambda}[x^{\Delta}(t,\lambda)]_{\lambda=0}$ and $[x_{\lambda}(t,0)]^{\Delta}$ are *not continuous* in $t$ but merely (piecewise) rd-continuous, which makes the conclusion of [6, Theorem 6.1] in our case inaccessible. On the other hand, we will prove that formula (1.6) nicely follows from the theory of time scale dynamic equations.

The idea of interchanging the $\Delta$-derivative and the usual derivative $\frac{\partial}{\partial\lambda}$ for a function $x(t,\lambda)$ on $\mathbb{T}\times\mathbb{R}^r$ is not completely new. In [14, Lemma 2] we encounter a *one*-parameter family $x(t,\lambda)$ of functions defined on $[a,b]_{\mathbb{T}}\times(-\gamma,\gamma)$ for which formula (1.6) is satisfied. In this reference as well as in [17, Theorem 5.1] the question is to show that for any given solution $\eta(\cdot)$ of the linearized equation, there exists a family $x(t,\lambda)$ solving the original nonlinear equation and satisfying the given boundary conditions such that $x(t,\cdot),\ x^{\Delta}(t,\cdot)\in \mathrm{C}^1$ (or even $\mathrm{C}^2$) and $x_{\lambda}(t,0)=\eta(t)$ for all $t\in[a,b]_{\mathbb{T}}$, and that (1.6) holds. In [14, Lemma 2] this family is explicitly constructed to have readily the differentiability requirements including identity (1.6). While in [17, Theorem 5.1] this construction is done through the application of the natural extension of the embedding theorem [16, Theorem 3.2] to vector parameters. In this case, the needed differentiability of this family with respect to the vector-parameter $\lambda$ is a consequence of the results of this paper. Therefore, in the present paper we start with the nonlinear equation (1.1) and the initial condition (1.2), both may depend on the vector-parameter $\lambda$, and then show that the solution $x(t,\cdot),\ x^{\Delta}(t,\cdot)\in \mathrm{C}^1$ and that (1.6) holds.

The question of the differentiability of solutions of nonlinear equations on *time scales* has been studied in the literature in [20, Theorem 2.7.1]. In this reference we can find conditions which guarantee that the solution $x(t,\lambda)$ is differentiable in $\lambda$, but only under a rather restrictive assumption that the Lipschitz constant $K$ of the nonlinearity $f$ satisfies $K(b-a)<1$. Our result below (Theorem 3.1) does not require this.

Our result extends to time scales the corresponding continuous time result, i.e., for the classical *differential* equation $x'=f(t,x,\lambda)$, see e.g. [12, Theorem 7.1, Appendix] when $\lambda\in\mathbb{R}^r$, or [21, Theorem 1.10.1] for the one-parameter case. Finally, let us mention that the methods we employed in [16, Section 3] to prove the time scale embedding theorem as well as the methods in this paper extend directly to the time scale dynamic equations

$$x^{\Delta}=f(t,x^{\sigma},\lambda),\quad t\in[a,\rho(b)]_{\mathbb{T}}, \tag{1.7}$$

$$x^{\Delta}=f(t,x,x^{\sigma},\lambda),\quad t\in[a,\rho(b)]_{\mathbb{T}}, \tag{1.8}$$

or even to time scale dynamic equations of all stated forms with *complex-valued* data and solutions (see Remark 3.8).

For the special case when $r=1$ and the right-hand side $f$ is *linear* in $x$ and $\lambda$ we obtain in Theorem 4.3 a better result, namely that the solution $x(t,\cdot)$ is an *entire function* in $\lambda$. This is proven along the hints provided in [21, pg. 79] for the continuous time case. Such a result is again important especially in the theory of eigenvalue problems for time scale symplectic systems.

In this paper we use a common time scale notation and terminology as in [9], with which the reader can also get acquainted in [16, Section 2].

The paper is divided as follows. In the next section we recall the assumptions and the statement of the time scale embedding theorem from [16]. In Section 3 we establish our main result (Theorem 3.1) regarding the differentiability of solutions of problem (1.1), (1.2) with respect to the parameter $\lambda$. Finally, in Section 4 we treat the linear case and prove that the solutions are entire functions in $\lambda$.

## 2 Elementary Time Scale Notation

Let $\mathbb{T}$ be a bounded time scale. Then $\mathbb{T}$ can be identified with the time scale interval $[a,b]_{\mathbb{T}}$, where $a := \min\mathbb{T}$ and $b := \max\mathbb{T}$ both exist and belong to $\mathbb{T}$. The forward and backward jump operators are denoted by $\sigma(t)$ and $\rho(t)$, respectively, and the graininess function by $\mu(t) := \sigma(t) - t$. A point $t \in [a,\rho(b)]_{\mathbb{T}}$ is *right-scattered,* if $\sigma(t) > t$, while $t \in [\sigma(a), b]_{\mathbb{T}}$ is *left-scattered,* if $\rho(t) < t$. Similarly, a point $t \in [a,b)_{\mathbb{T}}$ is *right-dense,* if $\sigma(t) = t$, while $t \in (a,b]_{\mathbb{T}}$ is *left-dense,* if $\rho(t) = t$.

A function $f$ on $\mathbb{T}$ (with values in a Banach space) is *regulated* if the right-hand limit $f(t^+)$ exists (finite) at all right-dense points $t \in [a,b)_{\mathbb{T}}$ and the left-hand limit $f(t^-)$ exists (finite) at all left-dense points $t \in (a,b]_{\mathbb{T}}$. A function $f$ is *rd-continuous* (we write $f \in \mathrm{C}_{\mathrm{rd}}$) if it is regulated and if it is continuous at all right-dense points $t \in [a,b)_{\mathbb{T}}$. A function $f$ is *piecewise rd-continuous* (we write $f \in \mathrm{C}_{\mathrm{prd}}$) if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in [a,b)_{\mathbb{T}}$. At the right-dense points $\{t_1,\cdots,t_k\}$ where a given $\mathrm{C}_{\mathrm{prd}}$-function $f$ is not continuous, the statements and conditions involving the values $f(t_i)$, $i \in \{1,\dots,k\}$, simply mean that these statements and conditions hold when the value $f(t_i)$ is replaced by $f(t_i^+)$. This convention will be assumed throughout the paper without further recall. A matrix-function $f$ is *regressive* if $I + \mu(t)\, f(t)$ is invertible for all $t \in [a,\rho(b)]_{\mathbb{T}}$.

A function $f$ is *rd-continuously $\Delta$-differentiable* (we write $f \in \mathrm{C}^1_{\mathrm{rd}}$) if $f^\Delta(t)$ exists for all $t \in [a,\rho(b)]_{\mathbb{T}}$ and $f^\Delta \in \mathrm{C}_{\mathrm{rd}}$. A continuous function $f$ is *piecewise rd-continuously $\Delta$-differentiable* (we write $f \in \mathrm{C}^1_{\mathrm{prd}}$) if $f$ is continuous and $f^\Delta$ exists at all, except possibly at finitely many, $t \in [a,\rho(b)]_{\mathbb{T}}$ and $f^\Delta \in \mathrm{C}_{\mathrm{prd}}$.

The compositions of a function $f$ with the jump operators are denoted by $f^\sigma(t) := f\big(\sigma(t)\big)$ and $f^\rho(t) := f\big(\rho(t)\big)$.

Let us fix $n \in \mathbb{N}$, $\varepsilon \in (0, b-a)$, and $\gamma_0 > 0$, and let $\bar{x} : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be a given $\mathrm{C}^1_{\mathrm{prd}}$ vector function. Similarly to [23, Section 2], assume that we are given open sets $X \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^r$ such that $\overline{T_{2\varepsilon}(\bar{x})} \times \bar{B}_{\gamma_0} \subseteq [a,\rho(b)]_{\mathbb{T}} \times X \times \Lambda$, where

$$T_{2\varepsilon}(\bar{x}) := \big\{(t,x) \in [a,\rho(b)]_{\mathbb{T}} \times \mathbb{R}^n \ \text{ such that } \big|x - \bar{x}(t)\big| < 2\varepsilon\big\}$$

is the $2\varepsilon$-tube about the function $\bar{x}(\cdot)$, $\overline{T_{2\varepsilon}(\bar{x})}$ is its closure, $\bar{B}_{\gamma_0}$ is the closed ball in $\mathbb{R}^r$ of radius $\gamma_0$, and where $|\cdot|$ is the Euclidean norm. For a continuous function $x(\cdot)$ on $[a,b]_{\mathbb{T}}$ the notation $x \in T_{2\varepsilon}(\bar{x})$ means that $\big(t, x(t)\big) \in T_{2\varepsilon}(\bar{x})$ for all $t \in [a, \rho(b)]_{\mathbb{T}}$.

Let be given functions

$$f : [a, \rho(b)]_{\mathbb{T}} \times X \times \Lambda \to \mathbb{R}^n, \quad x^0 : \Lambda \to \mathbb{R}^n.$$

In this paper we will assume that

(H0) system (1.1), (1.2) with $\lambda = 0$ has a solution $\bar{x}(\cdot)$ on $[a,b]_{\mathbb{T}}$,

and that the data in problem (1.1), (1.2) satisfy the following hypotheses:

(H1) $f$ is $\mathrm{C}_{\mathrm{prd}} \times \mathrm{C} \times \mathrm{C}$-continuous on its domain, see [14, Definition 3],

(H2) $f$ is Lipschitz continuous in $x$ uniformly in $(t,\lambda)$, that is, there exists a constant $K > 0$ such that for all $(t,x), (t,y) \in T_{2\varepsilon}(\bar{x})$ and $\lambda \in B_{\gamma_0}$ we have

$$\big|f(t,x,\lambda) - f(t,y,\lambda)\big| \le K\,|x-y|,$$

(H3) $x^0$ is continuously differentiable in $\lambda$,

(H4) $f_x$ and $f_\lambda$ exist and are $\mathrm{C}_{\mathrm{prd}} \times \mathrm{C} \times \mathrm{C}$-continuous on their domain.

The following time scale embedding theorem is a straightforward extension of the one proven in [16, Theorem 3.2], in which proof the uniqueness is implicit and $\lambda$ being in $\mathbb{R}^r$ is admitted.

**Theorem 2.1 (Embedding theorem).** *Assume that* (H0)–(H2) *hold. Then there exist constants $\alpha > 0$ and $\gamma > 0$ such that for any parameter $\lambda$ and initial value $x^0(\lambda)$ such that $|\lambda| < \gamma$ and $|x^0(\lambda) - \bar{x}(0)| < \gamma$ there is a unique solution $x(\cdot,\lambda)$ on $[a,b]_{\mathbb{T}}$ of* (1.1), (1.2) *satisfying $x(t,0) = \bar{x}(t)$ for all $t \in [a,b]_{\mathbb{T}}$, $x(\cdot,\cdot)$ is continuous in $(t,\lambda)$, and*

$$\big|x(t,\lambda) - \bar{x}(t)\big| \le \alpha \left\{ \big|x^0(\lambda) - \bar{x}(0)\big| + \int_a^b \big|\, f\big(t, \bar{x}(t), \lambda\big) - f\big(t, \bar{x}(t), 0\big)\big|\, \Delta t \right\}$$

*for all $t \in [a,b]_{\mathbb{T}}$.*

*Remark* 2.2. Consider a linear system on $[a, \rho(b)]_{\mathbb{T}}$

$$x^{\Delta} = S(t)\, x, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad x(t_0) = x^0, \tag{2.1}$$

where $t_0 \in [a,b]_{\mathbb{T}}$, and $S(t)$ is an $s \times s$ matrix with $\mathrm{C}_{\mathrm{prd}}$-entries.

(i) When $t_0 = a$, Theorem 2.1 yields that for any $x^0 \in \mathbb{R}^s$ the system (2.1) has a unique solution on $[a,b]_{\mathbb{T}}$. To see this, just notice that $x \equiv 0$ solves the system (2.1) on $[a, \rho(b)]_{\mathbb{T}}$ for $x^0 = 0$. Thus, by Theorem 2.1, there exists $\gamma > 0$ such that for $|x^0| < \gamma$, system (2.1) has a unique solution. Using the linearity of the system, it follows that for any $x^0$ the system has a unique solution.

(ii) If $t_0 \in (a,b]_{\mathbb{T}}$, then for any $x^0 \in \mathbb{R}^s$ the system (2.1) has a unique solution on $[a,b]_{\mathbb{T}}$ whenever on $[a,t_0)_{\mathbb{T}}$ the matrix $S(t)$ is regressive, that is, $I + \mu(t)\, S(t)$ is invertible.

(iii) Consider the linear system

$$x^{\Delta} = S(t)\, x^{\sigma}, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad x(t_0) = x^0. \tag{2.2}$$

Then for any $x^0 \in \mathbb{R}^s$ the system (2.2) has a unique solution on $[a,b]_{\mathbb{T}}$ whenever $I - \mu(t)\, S(t)$ is invertible on $[t_0, \rho(b)]_{\mathbb{T}}$. This follows from [16, Remark 3.9], the version of Theorem 2.1 corresponding to (2.2) and that $x \equiv 0$ solves the system (2.2) on $[a, \rho(b)]_{\mathbb{T}}$ for $x^0 = 0$.

(iv) Consider the linear system

$$x^{\Delta} = S(t)\, x + T(t)\, x^{\sigma}, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad x(t_0) = x^0, \tag{2.3}$$

where $S(t)$ and $T(t)$ are $s \times s$ matrices with $\mathrm{C_{prd}}$-entries. Combining parts (ii) and (iii) above yields that the system (2.3) has a unique solution on $[a, \rho(b)]_{\mathbb{T}}$ whenever $I + \mu(t)\, S(t)$ is invertible on $[a, t_0)_{\mathbb{T}}$ and $I - \mu(t)\, T(t)$ is invertible on $[t_0, \rho(b)]_{\mathbb{T}}$.

The following Gronwall inequality on time scales will be used in the proof of the main result of this paper. Here $e_p(t,s)$ is the time scale exponential function, i.e., the function $e_p(\cdot, s)$ is the unique solution of the initial value problem $x^{\Delta} = p(t)\, x$, $x(s) = 1$, see [9, Section 2.2].

**Lemma 2.3.** *Let* $y, g, p \in \mathrm{C_{prd}}$ *be real-valued scalar functions on* $[a, \rho(b)]_{\mathbb{T}}$ *such that* $p(\cdot) \geq 0$ *and*

$$y(t) \leq g(t) + \int_a^t y(\tau)\, p(\tau)\, \Delta\tau \quad \textit{for all } t \in [a,b]_{\mathbb{T}}.$$

*Then*

$$y(t) \leq g(t) + \int_a^t e_p\big(t, \sigma(\tau)\big)\, g(\tau)\, p(\tau)\, \Delta\tau \quad \textit{for all } t \in [a,b]_{\mathbb{T}}.$$

*Proof.* See [1, Theorem 5.6] or [9, Theorem 6.4]. □

# 3 Main Results and Proofs

In this section we state and prove the following main result of this paper.

**Theorem 3.1.** *Suppose that* (H0)–(H4) *hold. Then there exists* $\delta > 0$ *such that for* $|\lambda| < \delta$ *there exists a unique solution* $x(t,\lambda)$ *of problem* (1.1), (1.2) *such that the function* $x(\cdot,\cdot)$ *is continuous in* $(t,\lambda)$ *on* $[a,b]_{\mathbb{T}} \times B_\delta$, *and* $x(t,\cdot)$ *is continuously differentiable at* $\lambda = 0$ *uniformly in* $t$, *and the derivative* $x_\lambda(t,\lambda) := \dfrac{\partial}{\partial\lambda} x(t,\lambda)$ *satisfies at* $\lambda = 0$ *the linearized system* (1.3)–(1.5). *Furthermore,* $x^{\Delta}(t,\cdot)$ *is continuously differentiable at* $\lambda = 0$ *uniformly in* $t$ *and formula* (1.6) *holds.*

*Remark* 3.2. (i) Although we stated the above differentiability property of $x(t,\cdot)$ at $\lambda = 0$, it is clear that one may replace the point $\lambda = 0$ by any $\lambda \in B_{\gamma_0}$.

(ii) From the proof of Theorem 3.1 it will follow that if we replace the assumptions (H1), (H3), and (H4) by the hypothesis

(H5) $f$ and its partial derivatives with respect to $x$ and $\lambda$ up to order $m$ are $\mathrm{C_{prd}} \times \mathrm{C} \times \mathrm{C}$-continuous on their domain, and $x^0 \in \mathrm{C}^m$ on its domain,

then the solution $x(t,\cdot)$ of (1.1), (1.2) and $x^\Delta(t,\cdot)$ will be of the class $\mathrm{C}^m$, i.e., the partial derivatives of $x(t,\cdot)$ and $x^\Delta(t,\cdot)$ with respect to the $\lambda_i$'s up to order $m$ are continuous.

Before proving Theorem 3.1 we shall discuss some implications of the hypotheses (H1)–(H4) made on the data $f$ and $x^0$.

(C1) Assumption (H4) yields that the function $f\big(t,\bar{x}(t),\cdot\big)$ is differentiable at $\lambda = 0$ uniformly in $t$, see [14, Definition 2 and Proposition 2]. That is, for any given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $0 < |\lambda| < \delta_1$ implies

$$\frac{\big|\, f\big(t,\bar{x}(t),\lambda\big) - f\big(t,\bar{x}(t),0\big) - f_\lambda\big(t,\bar{x}(t),0\big)\,\lambda\big|}{|\lambda|} < \varepsilon \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}}. \tag{3.1}$$

(C2) Assumption (H4) implies that the function $f(t,\cdot,\lambda)$ is differentiable at $\bar{x}(t)$ uniformly in $(t,\lambda)$. That is, for given $\varepsilon > 0$ there exists $\delta_2 \in (0,\delta_1)$ such that for all $t \in [a,\rho(b)]_{\mathbb{T}}$ and $0 < |x - \bar{x}(t)| < \delta_2$ we have

$$\frac{\big|\, f\big(t,x,\lambda\big) - f\big(t,\bar{x}(t),\lambda\big) - f_x\big(t,\bar{x}(t),\lambda\big)\,[x - \bar{x}(t)]\,\big|}{\big|x - \bar{x}(t)\big|} < \varepsilon.$$

And since, by Theorem 2.1 , $x(t,\cdot)$ is continuous at $\lambda = 0$ *uniformly* in $t$, then for the specified $\delta_2 > 0$ there exists $\delta_3 \in (0,\delta_2)$ such that $|\lambda| < \delta_3$ implies $\big|x(t,\lambda) - \bar{x}(t)\big| < \delta_2$ for all $t \in [a,b]_{\mathbb{T}}$. Hence, for such $|\lambda| < \delta_3$ we have that

$$\frac{\big|\, f\big(t,x(t,\lambda),\lambda\big) - f\big(t,\bar{x}(t),\lambda\big) - f_x\big(t,\bar{x}(t),\lambda\big)\,[x(t,\lambda) - \bar{x}(t)]\,\big|}{\big|x(t,\lambda) - \bar{x}(t)\big|} < \varepsilon \tag{3.2}$$

for all $t \in [a,\rho(b)]_{\mathbb{T}}$.

(C3) Assumption (H4) implies that the functions $f_x(t,\cdot,\cdot)$ and $f_\lambda(t,\cdot,\cdot)$ are continuous at $\lambda = 0$ uniformly in $t$. Thus, for any $\varepsilon > 0$ there is $\delta_4 \in (0,\delta_3)$ such that for all $t \in [a,\rho(b)]_{\mathbb{T}}$ and $0 < |x - \bar{x}(t)| < \delta_4$ we have

$$\big\|f_x\big(t,x,\lambda\big) - f_x\big(t,\bar{x}(t),0\big)\big\| < \varepsilon \quad \text{and} \quad \big\|f_\lambda\big(t,x,\lambda\big) - f_\lambda\big(t,\bar{x}(t),0\big)\big\| < \varepsilon,$$

where $\|\cdot\|$ is any matrix norm compatible with the vector norm $|\cdot|$, see [4, Sections 9.3–9.4]. The *$t$-uniform* continuity of $x(t,\cdot)$ at $\lambda = 0$ implies the existence of there exists

$\delta_5 \in (0, \delta_4)$ such that $|\lambda| < \delta_5$ implies $\big|x(t,\lambda) - \bar{x}(t)\big| < \delta_4$ for all $t \in [a,b]_{\mathbb{T}}$. Hence, for all $t \in [a, \rho(b)]_{\mathbb{T}}$ and for all $|\lambda| < \delta_5$ we have that

$$\big\|f_x(t, x(t,\lambda), \lambda\big) - f_x(t, \bar{x}(t), 0)\big\| < \varepsilon, \qquad \big\|f_\lambda(t, x(t,\lambda), \lambda\big) - f_\lambda(t, \bar{x}(t), 0)\big\| < \varepsilon, \tag{3.3}$$

and

$$\big\|f_x(t, \bar{x}(t), \lambda\big) - f_x(t, \bar{x}(t), 0)\big\| < \varepsilon. \tag{3.4}$$

(C4) The function $f_x\big(\cdot, \bar{x}(\cdot), 0\big)$ is bounded by $M_1 > 0$. Thus, using (3.3)(i), it follows that for $|\lambda| < \delta_5$ and for $t \in [a, \rho(b)]_{\mathbb{T}}$ we have

$$\big\|f_x(t, \bar{x}(t), 0)\big\| \le M_1 \quad \text{and} \quad \big\|f_x(t, x(t,\lambda), \lambda)\big\| < \varepsilon + M_1. \tag{3.5}$$

(C5) By hypothesis (H3), the function $x^0(\cdot)$ is continuously differentiable at $\lambda = 0$. Furthermore, by (1.2) we have $x(a, \cdot) = x^0(\cdot)$. Thus, for a given $\varepsilon > 0$ there exists $\delta_0 \in (0, \delta_5)$ such that $0 < |\lambda| < \delta_0$ implies

$$\frac{\big|x(a,\lambda) - x(a,0) - Dx^0(0)\,\lambda\,\big|}{|\lambda|} < \varepsilon \quad \text{and} \quad \Big|\, Dx^0(\lambda) - Dx^0(0) \Big| < \varepsilon. \tag{3.6}$$

Therefore, we just proved the following assertion.

**Lemma 3.3.** *Suppose that* (H0)–(H4) *hold. Then for every* $\varepsilon > 0$ *there exists* $\delta_0 > 0$ *such that* $0 < |\lambda| < \delta_0$ *implies that conditions* (3.1)–(3.6) *hold true.*

*Proof.* We take $\delta_0$ from the above conclusion (C5). Then, by construction of $\delta_0$ through (C1)–(C5) above it follows that all the inequalities in (3.1)–(3.6) are satisfied for $|\lambda| < \delta_0$. □

**Lemma 3.4.** *Assume that for some* $\varepsilon > 0$ *there exists* $\delta_0 > 0$ *such that for* $0 < |\lambda| < \delta_0$ *and* $t \in [a, \rho(b)]_{\mathbb{T}}$*, conditions* (3.1), (3.2), (3.4), *and* (3.5)*(i) hold. Then for any continuous function* $Z : [a,b]_{\mathbb{T}} \to \mathbb{R}^{n \times r}$ *and for any* $t \in [a, \rho(b)]_{\mathbb{T}}$ *we have*

$$\begin{aligned} &\frac{\big|\, f\big(t, x(t,\lambda), \lambda\big) - f\big(t, x(t,0), 0\big) - f_x\big(t, x(t,0), 0\big)\, Z(t)\,\lambda - f_\lambda\big(t, x(t,0), 0\big)\,\lambda\,\big|}{|\lambda|} \\ &\qquad \le (2\varepsilon + M_1)\,\frac{\big|\, x(t,\lambda) - x(t,0) - Z(t)\,\lambda\,\big|}{|\lambda|} + 2\varepsilon\,\|Z(t)\| + \varepsilon. \end{aligned} \tag{3.7}$$

*If for* $0 < |\lambda| < \delta_0$ *and* $t \in [a, \rho(b)]_{\mathbb{T}}$*, we have* (3.3) *and* (3.5)*(ii) hold, then for any pair of continuous matrix functions* $Z, Z_1 : [a,b]_{\mathbb{T}} \to \mathbb{R}^{n \times r}$ *we have*

$$\begin{aligned} &\big\|\, f_x\big(t, x(t,\lambda), \lambda\big)\, Z_1(t) - f_x\big(t, x(t,0), 0\big)\, Z(t) + f_\lambda\big(t, x(t,\lambda), \lambda\big) - f_\lambda\big(t, x(t,0), 0\big)\big\| \\ &\qquad \le (\varepsilon + M_1)\,\big\|\, Z_1(t) - Z(t)\,\big\| + \varepsilon\,\|Z(t)\| + \varepsilon. \end{aligned} \tag{3.8}$$

*Proof.* Denote the left-hand side of (3.7) as $\psi(t,\lambda)$, and put

$$\xi(t,\lambda) := \frac{x(t,\lambda)-x(t,0)-Z(t)\lambda}{|\lambda|}. \tag{3.9}$$

Then we get

$$\begin{aligned}
\psi(t,\lambda) \le{}& \frac{\big|\, f\big(t,x(t,\lambda),\lambda\big)-f\big(t,x(t,0),\lambda\big)-f_x\big(t,x(t,0),\lambda\big)\,[x(t,\lambda)-x(t,0)]}{|\lambda|}\\
&+\frac{f_x\big(t,x(t,0),\lambda\big)\,[x(t,\lambda)-x(t,0)]-f_x\big(t,x(t,0),0\big)\,Z(t)\,\lambda\big|}{|\lambda|}\\
&+\frac{\big|\, f\big(t,x(t,0),\lambda\big)-f\big(t,x(t,0),0\big)-f_\lambda\big(t,x(t,0),0\big)\,\lambda\,\big|}{|\lambda|}\\
\overset{(3.1)}{\le}{}& \frac{\big|\, f\big(t,x(t,\lambda),\lambda\big)-f\big(t,x(t,0),\lambda\big)-f_x\big(t,x(t,0),\lambda\big)\,[x(t,\lambda)-x(t,0)]\,\big|}{\big|x(t,\lambda)-x(t,0)\big|}\times\\
&\times\ \frac{\big|x(t,\lambda)-x(t,0)\big|}{|\lambda|}\\
&+\frac{\big|\big[f_x\big(t,x(t,0),\lambda\big)-f_x\big(t,x(t,0),0\big)\big]\,[x(t,\lambda)-x(t,0)]\,\big|}{|\lambda|}\\
&+\big|\, f_x\big(t,x(t,0),0\big)\,\xi(t,\lambda)\,\big|+\varepsilon\\
\overset{(3.2)}{\le}{}& \varepsilon\,\big|\xi(t,\lambda)\big|+\varepsilon\,\frac{|Z(t)\,\lambda|}{|\lambda|}\\
&+\big\|\, f_x\big(t,x(t,0),\lambda\big)-f_x\big(t,x(t,0),0\big)\,\big\|\cdot\frac{\big|\,x(t,\lambda)-x(t,0)\,\big|}{|\lambda|}\\
&+\big\|\, f_x\big(t,x(t,0),0\big)\,\big\|\cdot\big|\xi(t,\lambda)\big|+\varepsilon\\
\overset{(3.4),\,(3.5)(\mathrm{i})}{\le}{}& (2\varepsilon+M_1)\,\big|\xi(t,\lambda)\big|+2\varepsilon\,\|Z(t)\|+\varepsilon.
\end{aligned}$$

Therefore, estimate (3.7) is established.

Denote the left-hand side of (3.8) as $\phi(t,\lambda)$. Then we get

$$\begin{aligned}
\phi(t,\lambda) \le{}& \big\|\, f_x\big(t,x(t,\lambda),\lambda\big)\,[Z_1(t)-Z(t)]+\big[f_x\big(t,x(t,\lambda),\lambda\big)-f_x\big(t,x(t,0),0\big)\big]\,Z(t)\big\|\\
&+\big\|f_\lambda\big(t,x(t,\lambda),\lambda\big)\ -f_\lambda\big(t,x(t,0),0\big)\big\|\\
\overset{(3.3),\,(3.5)(\mathrm{ii})}{\le}{}& (\varepsilon+M_1)\,\big\|Z_1(t)-Z(t)\big\|+\varepsilon\,\|Z(t)\|+\varepsilon,
\end{aligned}$$

whence, estimate (3.8) holds. $\square$

Now we have all the preparatory material in order to prove Theorem 3.1.

*Proof of Theorem 3.1.* First note that, by [14, Proposition 1], the assumption (H1) implies that for any function $x \in T_{2\varepsilon}(\bar{x})$ the composition $f\big(\cdot, x(\cdot), \lambda\big) \in \mathrm{C_{prd}}$, and hence it is $\Delta$-integrable.

Let $x(t,\lambda)$ be the solution of (1.1), (1.2), which exists for $t \in [a,b]_{\mathbb{T}}$ and $|\lambda| < \gamma$ by the embedding theorem (Theorem 2.1). Let $Z(\cdot)$ be the solution of the linear equation (1.3) satisfying (1.4). By Remark 2.2(i), [15, Remark 2.1(ii)] or [13, Theorem 5.7], the solution $Z(\cdot)$ indeed exists and is continuous on $[a,b]_{\mathbb{T}}$. Hence, there exists $M_2 > 0$ such that

$$\|Z(t)\| \le M_2 \quad \text{for all } t \in [a,b]_{\mathbb{T}}. \tag{3.10}$$

Let $\varepsilon > 0$ be arbitrary and take $\delta := \min\{\gamma, \delta_0\}$, where $\delta_0 > 0$ is from Lemma 3.3. Then for $0 < |\lambda| < \delta$ all the inequalities in (3.1)–(3.8) are satisfied. Consequently, with the notation from (1.5) and (3.9) we have

$$\begin{aligned}
\big|\xi(t,\lambda)\big| &\le \big|\xi(a,\lambda)\big| \\
&\quad + \int_a^t \frac{\big|\, f(\tau, x(\tau,\lambda),\lambda\big) - f\big(\tau, x(\tau,0),0\big) - A(\tau)\, Z(\tau)\,\lambda - P(\tau)\,\lambda\big|}{|\lambda|}\,\Delta\tau \\
&\overset{(3.6)(\mathrm{i}),\,(3.7)}{\le} \varepsilon + \int_a^t \big\{\, (2\varepsilon + M_1)\,\big|\xi(\tau,\lambda)\big| + 2\varepsilon\,\|Z(\tau)\| + \varepsilon \,\big\}\,\Delta\tau \\
&\overset{(3.10)}{\le} g_0 + \int_a^t p_0\,\big|\xi(\tau,\lambda)\big|\,\Delta\tau,
\end{aligned}$$

where the positive constants $p_0$ and $g_0$ are given by

$$p_0 := 2\varepsilon + M_1, \quad g_0 := \varepsilon\,[1 + (2\,M_2 + 1)\,(b-a)].$$

Since $p_0 > 0$, it follows that $1 + \mu(t)\,p_0 > 0$, i.e., $p_0$ is positively regressive. Then the corresponding time scale exponential function $e_{p_0}(t,s) > 0$ for all $t, s \in [a,b]_{\mathbb{T}}$, by [9, Theorem 2.44(i)]. Therefore, by the Gronwall inequality on time scales (Lemma 2.3),

$$\big|\xi(t,\lambda)\big| \le g_0 + \int_a^t e_{p_0}\big(t,\sigma(\tau)\big)\, g_0\, p_0\,\Delta\tau \quad \text{for all } t \in [a,b]_{\mathbb{T}}. \tag{3.11}$$

Since $e_{p_0}(\cdot,\cdot)$ is continuous in its arguments, it is bounded, i.e., there exists $M_3 > 0$ such that

$$e_{p_0}\big(t,\sigma(\tau)\big) \le M_3 \quad \text{for } t \in [a,b]_{\mathbb{T}},\ \tau \in [a,\rho(t)]_{\mathbb{T}}.$$

Then, from (3.11) we get for any $t \in [a,b]_{\mathbb{T}}$

$$\begin{aligned}
\big|\xi(t,\lambda)\big| &\le g_0 + \int_a^b M_3\, g_0\, p_0\,\Delta\tau \\
&= \varepsilon\,[1 + (2\,M_2 + 1)\,(b-a)]\,[1 + M_3\,(2\varepsilon + M_1)\,(b-a)] \\
&\to 0 \quad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

Therefore, the function $x(t,\cdot)$ is differentiable at $\lambda = 0$ *uniformly in* $t$ and $x_\lambda(t,0) = Z(t)$ for all $t \in [a,b]_{\mathbb{T}}$.

Since the solution $x(t,\lambda)$ exists for all $\lambda \in B_\delta$ and the assumptions (H0)–(H4) are independent of the position of the point $\lambda = 0$ in the ball $B_\delta$, we can conclude that at any $\lambda \in B_\delta$, the function $x(t,\cdot)$ is differentiable *uniformly in* $t$, and that $Z(\cdot,\lambda) := x_\lambda(\cdot,\lambda)$ solves

$$Z^\Delta = A(t,\lambda)\,Z + P(t,\lambda), \quad t \in [a,\rho(b)]_{\mathbb{T}}, \tag{3.12}$$

$$Z(a) = Dx^0(\lambda), \tag{3.13}$$

where $A(t,\lambda)$ is the $n \times n$ matrix and $P(t,\lambda)$ is the $n \times r$ matrix defined on $[a,\rho(b)]_{\mathbb{T}}$ by

$$A(t,\lambda) := f_x\big(t, x(t,\lambda), \lambda\big), \quad P(t,\lambda) := f_\lambda\big(t, x(t,\lambda), \lambda\big). \tag{3.14}$$

Clearly, $Z(t,0) = Z(t)$ holds.

We now show that $x_\lambda(t,\cdot)$ is continuous at $\lambda = 0$ *uniformly in* $t$. Let $\varepsilon > 0$ be given and $\delta := \min\{\gamma, \delta_0\}$, where $\delta_0 > 0$ is from Lemma 3.3. Then for $0 < |\lambda| < \delta$ all the inequalities in (3.3), (3.5)(ii), and (3.6)(ii) are satisfied. Hence, by Lemma 3.4, estimate (3.8) holds with $Z_1(t) := Z(t,\lambda)$. Thus, denote

$$\begin{aligned}\Gamma(t,\lambda) := f_x(t, x(t,\lambda), \lambda\big)\,Z(t,\lambda) - f_x(t, x(t,0), 0\big)\,Z(t) \\ + f_\lambda(t, x(t,\lambda), \lambda\big) - f_\lambda(t, x(t,0), 0\big).\end{aligned} \tag{3.15}$$

Then (3.12) and (3.13) yield

$$\begin{aligned}\|Z(t,\lambda) - Z(t)\| &\le \big\|Dx^0(\lambda) - x^0(0)\big\| + \int_a^t \|\,\Gamma(\tau,\lambda)\,\|\,\Delta\tau \\ &\overset{(3.6)(ii),\,(3.8)}{\le} \varepsilon + \int_a^t \big\{\,(\varepsilon + M_1)\,\|Z_1(t) - Z(t)\| + \varepsilon\,\|Z(t)\| + \varepsilon\,\big\}\,\Delta\tau \\ &\overset{(3.10)}{\le} g_1 + \int_a^t p_1\,\big\|Z(\tau,\lambda) - Z(\tau)\big\|\,\Delta\tau,\end{aligned}$$

where the positive constants $p_1$ and $g_1$ are given by

$$p_1 := \varepsilon + M_1, \quad g_1 := \varepsilon\,[1 + (M_2 + 1)\,(b-a)].$$

Continue similarly to the argument for $\xi(t,\lambda)$ above, and after applying the time scale Gronwall inequality (Lemma 2.3) we conclude that for any $t \in [a,b]_{\mathbb{T}}$

$$\begin{aligned}\|Z(t,\lambda) - Z(t)\| &\le \varepsilon\,[1 + (M_2 + 1)\,(b-a)]\,[1 + M_4\,(\varepsilon + M_1)\,(b-a)] \\ &\to 0 \quad \text{as } \varepsilon \to 0^+,\end{aligned}$$

where $M_4$ stands for the bound of $e_{p_1}(\cdot,\cdot)$ on $[a,b]_{\mathbb{T}}$. Therefore, $Z(t,\cdot) = x_\lambda(t,\cdot)$ is continuous at $\lambda = 0$ uniformly in $t$.

Now there are two ways to prove that $x^\Delta(t,\cdot)$ is also continuously differentiable in $\lambda$ at $\lambda = 0$ uniformly in $t$.

1st method: By using the estimate (3.7) and notation (3.9) we have for any $t \in [a, \rho(b)]_{\mathbb{T}}$ and $0 < |\lambda| < \delta$ the estimate

$$\big|\xi^{\Delta}(t,\lambda)\big| = \frac{\big|\, x^{\Delta}(t,\lambda) - x^{\Delta}(t,0) - Z^{\Delta}(t)\,\lambda\,\big|}{|\lambda|} \le (2\varepsilon + M_1)\,\big|\xi(t,\lambda)\big| + \varepsilon\,(2\,M_2 + 1). \tag{3.16}$$

Since we have already proven that $x(t,0)$ is differentiable at $\lambda = 0$ uniformly in $t$ and that $x_\lambda(t,0) = Z(t)$, there exists $\delta_5 \in (0,\delta)$ such that for $0 < |\lambda| < \delta_5$ we have $\big|\xi(t,\lambda)\big| < \varepsilon$, for all $t \in [a,\rho(b)]_{\mathbb{T}}$. Thus, for such $\lambda$ we conclude from (3.16) that

$$\big|\xi^{\Delta}(t,\lambda)\big| \le (2\varepsilon + M_1)\,\varepsilon + \varepsilon\,(2\,M_2 + 1) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Hence, $x^{\Delta}(t,\cdot)$ is differentiable at $\lambda = 0$ *uniformly in* $t$, and $\frac{\partial}{\partial\lambda}\big[x^{\Delta}(t,\lambda)\big]_{\lambda=0} = Z^{\Delta}(t)$ for all $t \in [a,\rho(b)]_{\mathbb{T}}$. The fact that $Z(t) = x_\lambda(t,0)$ yields that formula (1.6) holds.

Of course, similar arguments show that $x^{\Delta}(t,\cdot)$ is differentiable at any $\lambda \in B_\delta$ *uniformly in* $t$, and that $\frac{\partial}{\partial\lambda}\big[x^{\Delta}(t,\lambda)\big] = Z^{\Delta}(t,\lambda)$, where $Z(t,\lambda)$ is the solution of (3.12) and (3.13). To show the continuity of $\frac{\partial}{\partial\lambda}\big[x^{\Delta}(t,\lambda)\big]$ at $\lambda = 0$ uniformly in $t$, we use the estimates (3.8) and (3.15). It follows that for $t \in [a,\rho(b)]_{\mathbb{T}}$ and $0 < |\lambda| < \delta$ we have

$$\big\|Z^{\Delta}(t,\lambda) - Z^{\Delta}(t)\big\| = \|\,\Gamma(t,\lambda)\,\| \overset{(3.8),\,(3.10)}{\le} (\varepsilon + M_1)\,\|\,Z(t,\lambda) - Z(t)\| + \varepsilon\,M_2 + \varepsilon, \tag{3.17}$$

Since we have already proven that $Z(t,\cdot) = x_\lambda(t,\cdot)$ is continuous at $\lambda = 0$ uniformly in $t$, then there exists $\delta_6 \in (0,\delta)$ such that for $0 < |\lambda| < \delta_6$ we have $\|Z(t,\lambda) - Z(t)\| < \varepsilon$ for all $t \in [a,\rho(b)]_{\mathbb{T}}$. Thus, for such $\lambda$ we conclude from (3.17) that

$$\big\|Z^{\Delta}(t,\lambda) - Z^{\Delta}(t)\big\| \le (\varepsilon + M_1)\,\varepsilon + \varepsilon\,(M_2 + 1) \to 0 \quad \text{as } \varepsilon \to 0^+,$$

proving the continuity of $\frac{\partial}{\partial\lambda}\big[x^{\Delta}(t,\lambda)\big]$ at $\lambda = 0$ uniformly in $t$.

2nd method: We know that the function $x(t,\lambda)$ satisfies the equation

$$x^{\Delta}(t,\lambda) = f\big(t, x(t,\lambda), \lambda\big), \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}},\ |\lambda| < \delta. \tag{3.18}$$

Since we have shown that $x(t,\cdot)$ is differentiable uniformly in $t$ when $\lambda$ is near $0$ and that $x_\lambda(t,\lambda)$ is continuous at $\lambda = 0$ uniformly in $t$, then the right-hand side of equation (3.18) is continuously differentiable in $\lambda$ at $\lambda = 0$ uniformly in $t$, and thus $x^{\Delta}(t,\cdot)$ is also continuously differentiable at $\lambda = 0$ uniformly in $t$ and

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\big[x^{\Delta}(t,\lambda)\big] &= \frac{\partial}{\partial\lambda}\big[f\big(t, x(t,\lambda), \lambda\big)\big] \\
&= A(t,\lambda)\,x_\lambda(t,\lambda) + P(t,\lambda) = A(t,\lambda)\,Z(t,\lambda) + P(t,\lambda) \\
&= Z^{\Delta}(t,\lambda) = \big[x_\lambda(t,\lambda)\big]^{\Delta}
\end{aligned}$$

for all $t \in [a,\rho(b)]_{\mathbb{T}}$, where $Z(t,\lambda)$, $A(t,\lambda)$, and $P(t,\lambda)$ are defined in (3.12)–(3.14). Whence, identity (1.6) holds. The proof of Theorem 3.1 is now complete. □

*Remark* 3.5. (i) In the proof above we obtained that (1.6) holds not only at $\lambda = 0$ but for all $\lambda$ near $0$. However, since the solution $x(t,\lambda)$ exists for all $\lambda \in B_\delta$ and the assumptions (H0)–(H4) are independent of the position of the point $\lambda = 0$ in the ball $B_\delta$, we can conclude that the rule for interchanging the derivatives holds for any $\lambda \in B_\delta$, i.e.,

$$\frac{\partial}{\partial\lambda}\big[x^\Delta(t,\lambda)\big] = \big[x_\lambda(t,\lambda)\big]^\Delta \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}},\ \lambda \in B_\delta.$$

(ii) When $r = 1$, one may think to replace the parameter interval $[-\lambda_0, \lambda_0]$ by some *time scale* interval $[\alpha,\beta]_{\tilde{\mathbb{T}}}$, where $\tilde{\mathbb{T}}$ is another time scale (possibly different from $\mathbb{T}$). However, the proof of formula (1.6) uses the chain rule on $\mathbb{R}^n$ and it is well known that such a chain rule does not work on general time scales, see e.g. [9, Section 1.5].

*Remark* 3.6. In the theory of dynamic equations on time scales one often encounters the situation when the right-hand side of the equation depends on $x^\sigma$ instead of $x$. Thus, we can consider the problem (1.7), (1.2). Then the corresponding embedding theorem holds under the additional assumption that the matrix $I - \mu(t)\, f_x\big(t, \bar{x}^\sigma(t), 0\big)$ is invertible for all $t \in [a,\rho(b)]_{\mathbb{T}}$, see [16, Remark 3.9]. In this case the conclusion of Theorem 3.2 remains true with the linearized system

$$Z^\Delta = f_x\big(t, \bar{x}^\sigma(t), 0\big)\, Z^\sigma + f_\lambda\big(t, \bar{x}^\sigma(t), 0\big), \quad t \in [a,\rho(b)]_{\mathbb{T}},$$

instead of equation (1.3). We refer to [17, Section 3] for a general transformation method between the two types of problems (1.1) and (1.7). Moreover, see [16, Remarks 3.8, 3.9] for the discussion about the above regressivity-type condition on the matrix $f_x\big(\cdot, \bar{x}^\sigma(\cdot), 0\big)$ and the position of the given initial condition.

*Remark* 3.7. A most general form of the problem is then the dynamic equation (1.8), (1.2), in which both $x$ and $x^\sigma$ is present in $f$. Then the linearized equation takes the form

$$Z^\Delta = f_x(t)\, Z + f_y(t)\, Z^\sigma + f_\lambda(t), \quad t \in [a,\rho(b)]_{\mathbb{T}},$$

where $f_x$ and $f_y$ denote the partial derivatives of $f$ with respect to second and third variables, respectively, and the partial derivatives are evaluated at $\big(t, \bar{x}(t), \bar{x}^\sigma(t), 0\big)$. In this case the result requires the invertibility of the matrix $I - \mu(t)\, f_y\big(t, \bar{x}(t), \bar{x}^\sigma(t), 0\big)$ on $[a,\rho(b)]_{\mathbb{T}}$.

*Remark* 3.8. Upon replacing the involved norms of real-valued vectors and matrices by the corresponding norms of *complex-valued* vectors and matrices, one can easily check that the methods of proof of the embedding theorem (Theorem 2.1) in [16] and the proof of the differentiability theorem (Theorem 3.1) extend directly to *complex-valued* data

$$f : [a,\rho(b)]_{\mathbb{T}} \times X \times \Lambda \to \mathbb{C}^n, \quad x^0 : \Lambda \to \mathbb{C}^n,$$

where now $X \subseteq \mathbb{C}^n$, $\Lambda \subseteq \mathbb{C}^r$,

$$T_{2\varepsilon}(\bar{x}) := \big\{(t,x) \in [a,\rho(b)]_{\mathbb{T}} \times \mathbb{C}^n \ \text{ such that } \big|x - \bar{x}(t)\big| < 2\varepsilon\big\},$$

and where $B_{\gamma_0} := \{\lambda \in \mathbb{C}^r,\ |\lambda| < \gamma_0\}$ is the open ball with radius $\gamma_0$. In this case we may replace the assumptions (H3) and (H4) by the requirement that $f$ is holomorphic in its (complex) arguments $x$ and $\lambda$ and that $x^0$ is holomorphic in $\lambda$. Then we can conclude that the complex-valued solution $x(t,\lambda)$, which is now defined on $[a,b]_{\mathbb{T}} \times B_\delta$, is also holomorphic in $\lambda$. Note that in this case the independent variable $t \in [a,b]_{\mathbb{T}}$ is still real.

# 4 Linear Systems

In this section we deal with linear dynamic systems. In the first part we apply Theorem 3.1, and in particular formula (1.6), to an eigenvalue problem associated with a special linear system, called a time scale symplectic system. In the second part of this section we prove for the case of general linear systems that the solutions are entire functions of the parameter $\lambda$.

Consider the eigenvalue problem

$$X^\Delta = \mathcal{A}(t)\,X + \mathcal{B}(t)\,U, \quad U^\Delta = \mathcal{C}(t)\,X + \mathcal{D}(t)\,U - \lambda\,W(t)\,X^\sigma, \quad t \in [a,\rho(b)]_{\mathbb{T}}, \tag{4.1}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},W : [a,\rho(b)]_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ are given $\mathrm{C_{prd}}$ matrices, $W(t)$ is symmetric, and $\lambda$ is a *scalar* parameter. We assume that the $2n \times 2n$ coefficient matrix $\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}$ satisfies the identity

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0 \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}}, \tag{4.2}$$

where $\mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ has $n \times n$ block entries. Linear systems whose coefficient matrix satisfies (4.2) are in the literature called *time scale symplectic* (or *Hamiltonian*) *systems*, see e.g. [3, 10, 15].

*Remark* 4.1. (i) Observe that one can write the eigenvalue problem (4.1) as a linear matrix system in which the right hand side has no shift in $(X,U)$. Indeed, by using the identity $X^\sigma - X = \mu X^\Delta$ in the first equation of (4.1), we obtain that the eigenvalue problem (4.1) is equivalent to

$$\begin{pmatrix} X \\ U \end{pmatrix}^\Delta = \mathcal{S}(t,\lambda) \begin{pmatrix} X \\ U \end{pmatrix} \quad t \in [a,\rho(b)]_{\mathbb{T}}, \tag{4.3}$$

where

$$\mathcal{S}(t,\lambda) := \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} - \lambda\,W(I+\mu\mathcal{A}) & \mathcal{D} - \lambda\,\mu\,W\mathcal{B} \end{pmatrix}(t) = \mathcal{S}(t) + \lambda\mathcal{Q}(t), \tag{4.4}$$

with

$$\mathcal{Q}(t) := \begin{pmatrix} 0 & 0 \\ -W(I+\mu\mathcal{A}) & -\mu\,W\mathcal{B} \end{pmatrix}(t). \tag{4.5}$$

(ii) Suppose that an initial point $t_0 \in [a,b]_{\mathbb{T}}$ and initial data $X^0, U^0 \in \mathbb{R}^{n\times n}$ are given. Then, by part (i) above and by Remark 2.2, we can assert that for every $\lambda \in \mathbb{R}$, there exists on $[a,b]_{\mathbb{T}}$ a unique solution $\big(X(\cdot,\lambda), U(\cdot,\lambda)\big)$ of (4.1) satisfying

$$\big(X(t_0,\lambda), U(t_0,\lambda)\big) = (X_0, U_0)$$

as long as the following conditions hold:

(a) the $2n \times 2n$ matrix $I + \mu(t)\,\mathcal{S}(t)$ is invertible for all $t \in [a, t_0)_{\mathbb{T}}$ (void if $t_0 = a$),

(b) the $2n \times 2n$ matrix $I - \mu(t)\,\mathcal{T}(t)$ is invertible for all $t \in [t_0, \rho(b)]_{\mathbb{T}}$ (void if $t_0 = b$),

where $\mathcal{T}(t) := \big(\begin{smallmatrix} 0 & 0 \\ -\lambda W(t) & 0 \end{smallmatrix}\big)$. Notice however that condition (b) above is always satisfied. While condition (a) is known to hold under condition (4.2), because in this case the matrix $I + \mu(t)\,\mathcal{S}(t)$ is symplectic. Note also that if (4.2) holds, i.e., if (4.1) with $\lambda = 0$ is a time scale symplectic system, then (4.1) is a time scale symplectic system for all $\lambda \in \mathbb{R}$, i.e., the matrix $\mathcal{S}(t,\lambda)$ given in (4.4) satisfies the identity (4.2) as well.

*Remark* 4.2. The question whether the solutions $\big(X(\cdot,\lambda), U(\cdot,\lambda)\big)$ of (4.1) are differentiable with respect to $\lambda$ and whether the equalities

$$\frac{\partial}{\partial\lambda}\big[X^{\Delta}(t,\lambda)\big]_{\lambda=0} = \big[X_{\lambda}(t,0)\big]^{\Delta}, \quad \frac{\partial}{\partial\lambda}\big[U^{\Delta}(t,\lambda)\big]_{\lambda=0} = \big[U_{\lambda}(t,0)\big]^{\Delta} \tag{4.6}$$

are satisfied on any time scale was posed e.g., in [7, formula (3)]. Of course, this property holds for the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, and it is often used in the oscillation and eigenvalue theories for continuous time linear Hamiltonian systems and discrete symplectic systems, see [5, 11, 19, 22]. As a consequence of Theorem 3.1 and Remark 3.7 we can now conclude that the identities in (4.6) indeed hold on any time scale and that the functions $X_{\lambda}(\cdot,0)$, $U_{\lambda}(\cdot,0)$, and $X(\cdot,0)$ satisfy the linearized system

$$(X_{\lambda})^{\Delta} = \mathcal{A}(t)\,X_{\lambda} + \mathcal{B}(t)\,U_{\lambda}, \quad (U_{\lambda})^{\Delta} = \mathcal{C}(t)\,X_{\lambda} + \mathcal{D}(t)\,U_{\lambda} - \lambda\,W(t)\,X_{\lambda}{}^{\sigma} - W(t)\,X^{\sigma},$$

in which we suppress the argument $(t,0)$ in the solution.

Since the eigenvalue problem (4.1) is *linear* in $\lambda$, it is expected that its solutions as functions of the parameter $\lambda$ enjoy "nicer" properties than continuous differentiability. Let us now turn our attention to a general linear system

$$y^{\Delta} = [A(t) + \lambda\,B(t)]\,y, \quad t \in [a,\rho(b)]_{\mathbb{T}},$$

where $A(\cdot)$ and $B(\cdot)$ are given $n \times n$ matrix functions. The following result is motivated by [21, Problem 1.10.4, pg. 79].

**Theorem 4.3.** *Let $A, B : [a,\rho(b)]_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ are* $\mathrm{C_{prd}}$ *functions and $\lambda \in \mathbb{R}$ a parameter, and assume that $A(\cdot)$ is regressive on $[a,\rho(b)]_{\mathbb{T}}$. Then the fundamental matrix $\Phi(t,\lambda)$ of the system*

$$Y^{\Delta} = [A(t) + \lambda\,B(t)]\,Y, \quad t \in [a,\rho(b)]_{\mathbb{T}}, \quad Y(a) = I, \tag{4.7}$$

*is an entire function of the parameter $\lambda$.*

*Proof.* Let $Y_0(\cdot)$ be the unique solution, i.e., the fundamental matrix, of the system

$$Y^\Delta = A(t)\,Y, \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad Y(a) = I.$$

By induction, if the matrix function $Y_{k-1}(t)$ is defined for some $k \in \mathbb{N}$, then we let $Y_k(\cdot)$ to be the unique solution of the system

$$Y^\Delta = A(t)\,Y + B(t)\,Y_{k-1}(t), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad Y(a) = 0.$$

The existence of these solutions is guaranteed e.g. by [15, Remark 2.1(ii)] or by [13, Theorem 5.7]. In addition, by the variation of constants formula, see [9, Theorem 5.24], we have

$$Y_k(t) = Y_0(t) \int_a^t Y_0^{-1}\big(\sigma(\tau)\big)\, B(\tau)\, Y_{k-1}(\tau)\, \Delta\tau, \quad t \in [a, b]_{\mathbb{T}},\ k \in \mathbb{N}. \tag{4.8}$$

Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\big\|\, Y_0(t) \,\big\| \le \alpha \quad \text{for all } t \in [a, b]_{\mathbb{T}}, \tag{4.9}$$

$$\big\|\, Y_0(t)\, [Y_0^\sigma(\tau)]^{-1} B(\tau) \,\big\| \le \beta \quad \text{for all } t, \tau \in [a, \rho(b)]_{\mathbb{T}}, \tag{4.10}$$

where $\|\cdot\|$ is the spectral matrix norm. It follows by induction that

$$\big\|\, Y_k(t) \,\big\| \le \alpha\, \beta^k\, h_k(t, a) \quad \text{for } t \in [a, b]_{\mathbb{T}},\ k \in \mathbb{N} \cup \{0\}, \tag{4.11}$$

where $h_k(t, a)$ are the time scale polynomials, i.e., $h_0(t, a) \equiv 1$, $h_1(t, a) = t - a$, and in general $h_{k+1}(t, a) := \int_a^t h_k(\tau, a)\, \Delta\tau$, see [9, Section 1.6]. Note that $h_k(t, a) \ge 0$ for all $t \in [a, b]_{\mathbb{T}}$. Indeed, for $k = 0$ inequality (4.11) reduces to (4.9), and if we assume that $\big\|\, Y_{k-1}(t) \,\big\| \le \alpha\, \beta^{k-1}\, h_{k-1}(t, a)$, then identity (4.8) yields

$$\begin{aligned}
\big\|\, Y_k(t) \,\big\| &\le \int_a^t \big\|\, Y_0(t)\, [Y_0^\sigma(\tau)]^{-1} B(\tau) \,\big\| \cdot \big\|\, Y_{k-1}(\tau) \,\big\|\, \Delta\tau \\
&\le \alpha\, \beta^k \int_a^t h_{k-1}(\tau, a)\, \Delta\tau = \alpha\, \beta^k\, h_k(t, a).
\end{aligned}$$

Now the result of [8, Theorem 4.1] shows that $h_k(t, a) \le \dfrac{(t-a)^k}{k!}$ for $t \ge a$. Consequently, for any $t \in [a, b]_{\mathbb{T}}$ we have

$$\begin{aligned}
\Big\| \sum_{k=0}^{\infty} \lambda^k\, Y_k(t) \Big\| &\le \sum_{k=0}^{\infty} |\lambda|^k \big\|\, Y_k(t) \,\big\| \le \sum_{k=0}^{\infty} |\lambda|^k\, \alpha\, \beta^k\, h_k(t, a) \\
&\le \alpha \sum_{k=0}^{\infty} |\lambda|^k\, \beta^k\, \frac{(t-a)^k}{k!} = \alpha\, e^{|\lambda|\, \beta\, (t-a)}.
\end{aligned}$$

Hence, the series $\sum_{k=0}^{\infty} \lambda^k Y_k(t)$ converges uniformly on $[a,b]_{\mathbb{T}}$ to a continuous function $\Phi(t,\lambda)$. We now multiply equation (4.8) by $\lambda^k$ and add up all these equations for $k \in \mathbb{N} \cup \{0\}$ and get

$$Y_0(t) + \sum_{k=1}^{\infty} \lambda^k Y_k(t) = Y_0(t) + Y_0(t) \int_a^t Y_0^{-1}\big(\sigma(\tau)\big)\, B(\tau) \sum_{k=1}^{\infty} \lambda^k Y_{k-1}(\tau)\, \Delta\tau,$$

$t \in [a,b]_{\mathbb{T}}$. Hence, by shifting the summation index in the series on the right-hand side, it follows that the function $\Phi(t,\lambda) := \sum_{k=0}^{\infty} \lambda^k Y_k(t)$ satisfies the equation

$$\Phi(t,\lambda) = Y_0(t) + Y_0(t) \int_a^t Y_0^{-1}\big(\sigma(\tau)\big)\, B(\tau)\, \lambda\, \Phi(\tau,\lambda)\, \Delta\tau, \quad t \in [a,b]_{\mathbb{T}}.$$

Therefore, again by the time scale variations of constants formula,

$$\Phi^{\Delta}(t,\lambda) = [A(t) + \lambda\, B(t)]\, \Phi(t,\lambda) \quad \text{for all } t \in [a,\rho(b)]_{\mathbb{T}}.$$

Moreover, $\Phi(a,\lambda) = Y_0(a) + \sum_{k=1}^{\infty} Y_k(a) = I$, i.e., $\Phi(\cdot,\lambda)$ is the fundamental matrix of the dynamic equation in (4.7). From the series representation of $\Phi(t,\lambda)$ it follows that $\Phi(t,\cdot)$ is an entire function in $\lambda$. □

*Remark* 4.4. From the proof of Theorem 4.3 and from the estimate

$$h_k(t,s) \le \frac{(t-s)^k}{k!} \quad \text{for all } t,s \in [a,b]_{\mathbb{T}},\ t \ge s$$

in [8, Theorem 4.1] one can see that if we replace the initial condition $Y(a) = I$ in Theorem 4.3 by $Y(s) = I$ for some given point $s \in [a,b]_{\mathbb{T}}$, then the fundamental matrix (and hence all the solutions) of the equation $Y^{\Delta} = [A(t) + \lambda\, B(t)]\, Y$ is an entire function in $\lambda$ for all points $t \in [s,b]_{\mathbb{T}}$.

The following consequence of Theorem 4.3 presents the nice property enjoyed by the solutions of the eigenvalue problem (4.1).

**Corollary 4.5.** *Assume that for $\lambda = 0$ the coefficient matrix $\mathcal{S}(t) = \mathcal{S}(t,0)$ of system* (4.1) *satisfies identity* (4.2). *Then, the solutions of* (4.1) *are entire functions of $\lambda$.*

*Proof.* First note that the system being symplectic at $\lambda = 0$, yields that $I + \mu(t)\, \mathcal{S}(t)$ is invertible on $[a,b]_{\mathbb{T}}$, and thus for any initial condition at $t_0 \in [a,b]_{\mathbb{T}}$, and for any $\lambda \in \mathbb{R}$, the system (4.1) has a unique solution $\big(X(t,\lambda), U(t,\lambda)\big)$ on $[a,b]_{\mathbb{T}}$ that is differentiable in $\lambda$, see Remarks 4.1(ii) and 4.2. Now write the system (4.1) in the form (4.3)–(4.5).

Then, it has the form of (4.7), where $Y := \left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right)$, $A := \mathcal{S}$, and $B := \mathcal{Q}$. Since $\mathcal{S}$ is regressive, then by Theorem 4.3, the fundamental matrix $\Phi(t,\lambda)$ of (4.3)–(4.5) starting at $t_0 = a$ is an entire function in $\lambda$. Therefore, every solution of system (4.1) is also an entire function of $\lambda$, because it is of the form

$$\begin{pmatrix} X(t,\lambda) \\ U(t,\lambda) \end{pmatrix} = \Phi(t,\lambda) \begin{pmatrix} M \\ N \end{pmatrix} \quad \text{on } [a,b]_{\mathbb{T}},$$

where $M, N \in \mathbb{R}^{n\times n}$ are constant matrices. □

*Remark* 4.6. The result of Corollary 4.5 is used e.g. in [24, pg. 290] in the context of the second order Sturm–Liouville dynamic equation

$$-\big(p(t)\, y^{\Delta}\big)^{\Delta} + q(t)\, y^{\sigma} = \lambda\, w(t)\, y^{\sigma}. \tag{4.12}$$

In this reference, the differentiability of the solution $y(t,\lambda)$ with respect to $\lambda$ is also used in the proof of [24, Lemma 2.5]. Note that for the same equation (4.12) the differentiability of $y(t,\lambda)$ with respect to $\lambda$ is proven in [18, Lemma 3.2]. Equation (4.12) is a special case of our symplectic system (4.1).

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# References

[1] R. P. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: a survey, *Math. Inequal. Appl.* **4** (2001), no. 4, 535–557.

[2] R. P. Agarwal, M. Bohner, P. J. Y. Wong, Sturm–Liouville eigenvalue problems on time scales, *Appl. Math. Comput.* **99** (1999), no. 2–3, 153–166.

[3] C. D. Ahlbrandt, M. Bohner, J. Ridenhour, Hamiltonian systems on time scales, *J. Math. Anal. Appl.* **250** (2000), no. 2, 561–578.

[4] D. S. Bernstein, *Matrix Mathematics. Theory, Facts, and Formulas with Application to Linear Systems Theory,* Princeton University Press, Princeton, 2005.

[5] M. Bohner, O. Došlý, W. Kratz, Sturmian and spectral theory for discrete symplectic systems, *Trans. Amer. Math. Soc.* **361** (2009), no. 6, 3109–3123.

[6] M. Bohner, G. S. Guseinov, Partial differentiation on time scales, *Dynam. Systems Appl.* **13** (2004), 351–379.

[7] M. Bohner, R. Hilscher, An eigenvalue problem for linear Hamiltonian dynamic systems, *Fasc. Math.* **35** (2005), 35–49.

[8] M. Bohner, D. A. Lutz, Asymptotic expansion and analytic time scales, *ZAMM Z. Angew. Math. Mech.* **86** (2006), no. 1, 37–45.

[9] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications,* Birkhäuser, Boston, 2001.

[10] O. Došlý, S. Hilger, R. Hilscher, Symplectic dynamic systems, in: "Advances in Dynamic Equations on Time Scales", M. Bohner and A. Peterson, editors, pp. 293–334, Birkhäuser, Boston, 2003.

[11] O. Došlý, W. Kratz, Oscillation theorems for symplectic difference systems, *J. Difference Equ. Appl.* **13** (2007), no. 7, 585–605.

[12] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory,* John Wiley & Sons, New York, 1966.

[13] S. Hilger, Analysis on measure chains – a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1–2, 18–56.

[14] R. Hilscher, V. Zeidan, Calculus of variations on time scales: weak local piecewise $\mathrm{C}^1_{\mathrm{rd}}$ solutions with variable endpoints, *J. Math. Anal. Appl.* **289** (2004), no. 1, 143–166.

[15] R. Hilscher, V. Zeidan, Time scale symplectic systems without normality, *J. Differential Equations* **230** (2006), no. 1, 140–173.

[16] R. Hilscher, V. Zeidan, Time scale embedding theorem and coercivity of quadratic functionals, *Analysis (Munich)* **28** (2008), no. 1, 1–28.

[17] R. Hilscher, V. Zeidan, Weak maximum principle and accessory problem for control problems on time scales, *Nonlinear Anal.* **70** (2009), no. 9, 3209–3226.

[18] Q. Kong, Sturm–Liouville problems on time scales with separated boundary conditions, *Results Math.* **52** (2008), no. 1–2, 111–121.

[19] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, 1995.

[20] V. Lakshmikantham, S. Sivasundaram, B. Kaymakçalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.

[21] W. T. Reid, *Ordinary Differential Equations*, Wiley, New York, 1971.

[22] M. Wahrheit, Eigenvalue problems and oscillation of linear Hamiltonian systems, *Internat. J. Difference Equ.* **2** (2007), no. 2, 221–244.

[23] V. Zeidan, P. Zezza, The conjugate point condition for smooth control sets, *J. Math. Anal. Appl.* **132** (1988), no. 2, 572–589.

[24] C. Zhang, Y. Shi, Eigenvalues of second-order symmetric equations on time scales with periodic and antiperiodic boundary conditions, *Appl. Math. Comput.* **203** (2008), no. 1, 284–296.