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Global Hopf Bifurcation for an Internet Congestion Model

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Abstract

In this paper, our attention is focused on the global existence of bifurcating periodic solutions. We show that periodic solutions exist after the second critical value of time delay. Furthermore, a numerical example is given.

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1 Introduction

Recently, there has been an increasing interest in the dynamics of internet congestion models, see [1,3,4,6] and the references therein. If we understand more about the bifurcation behaviors of internet congestion control systems, we can explain the parameter sensitivity observed in practice and achieve some desirable system behaviors.

In this paper, we consider an internet model with a single link and single source, which was described in [2] as

$$\dot{x}(t) = k[w - x(t-D)p(x(t-D))]. \tag{1.1}$$

Here $x(t)$ denotes the sending rate of the source at time t, k is a positive gain parameter, $D > 0$ is the sum of forward and return delays, w is a target number of markets per unit time. The congestion indication function $p(\cdot)$ is increasing, nonnegative, and not identically zero, which can be viewed as the probability that a packet at the resource receives a "mark" – a feedback congestion indication signal.

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There have been many significant results about equation (1.1). For instance, the existence and properties of local Hopf bifurcation were studied by considering k and D as bifurcation parameters in [3] and [4], respectively. Furthermore, Hopf bifurcation control was investigated in [1, 6]. However, the bifurcating periodic solutions obtained in [3, 4] are generally local. That is to say, those periodic solutions only exist in the one-sided small neighborhood of the first critical value. Therefore, it is necessary and significant to investigate the global existence of these periodic solutions. In this paper, by applying the global Hopf bifurcation result in Wu [5] for functional differential equations which was established using a purely topological argument, we can get that the local Hopf bifurcation of (1.1) implies the global extension under certain conditions.

The paper is organized as follows. In the next section, we give some preliminary results about the local Hopf bifurcation of (1.1). In Section 3, the global existence of bifurcating periodic solutions will be established. In Section 4, an example is analyzed and some numerical simulations are presented.

2 Preliminary Results

We first present the existence of local Hopf bifurcation for system (1.1), more details can be found in [4]. We assume that the congestion indication function $p(\cdot)$ is a nonlinear function and has at least third-order continuous derivative.

Let x^* be the equilibrium point of (1.1). Then x^* satisfies $w = x^* p(x^*)$. The linear part of (1.1) at x^* is

$$\dot{u}(t) = -k(p(x^*) + x^* p'(x^*))u(t - D), \tag{2.1}$$

and the corresponding characteristic equation is given by the transcendental equation

$$\lambda + k(p(x^*) + x^* p'(x^*))e^{-\lambda D} = 0. \tag{2.2}$$

Assume equation (2.2) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$ ($\omega_0 > 0$), we insert $\lambda = i\omega_0$ into (2.2) and obtain

$$\begin{cases} -k(p(x^*) + x^* p'(x^*)) \cos(\omega_0 D) = 0, \\ \omega_0 - k(p(x^*) + x^* p'(x^*)) \sin(\omega_0 D) = 0. \end{cases} \tag{2.3}$$

From this we can get

$$\omega_0 D = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

and

$$\omega_0 - k(p(x^*) + x^* p'(x^*))(-1)^n = 0.$$

Thus,

$$\omega_0 = k(p(x^*) + x^* p'(x^*)) \tag{2.4}$$

and

$$D_c(n) = \frac{(2n+1)\pi}{2k(p(x^*) + x^*p'(x^*))}, \quad n = 0, 2, 4, \ldots. \tag{2.5}$$

Let $\lambda(D) = \alpha(D) + i\beta(D)$ be the root of (2.2) satisfying

$$\alpha(D_c(n)) = 0, \qquad \beta(D_c(n)) = \omega_0, \quad n = 0, 2, 4, \ldots.$$

Proposition 2.1. $\dfrac{\mathrm{d}}{\mathrm{d}D}\mathrm{Re}\lambda(D_c(0)) = \dfrac{(k(p(x^*) + x^*p'(x^*)))^2}{1 + \pi^2/4} > 0.$

Proposition 2.2. *If the delay parameter D is smaller than the critical value $D_c(0)$, then the equilibrium x^* of system* (1.1) *is asymptotically stable. When D passes through $D_c(0)$, a Hopf bifurcation will occur in the system at x^*.*

3 Existence of Global Hopf Bifurcation

In this section, we mainly prove that the local Hopf bifurcation of (1.1) in Proposition 2.2 can be extended for large values by applying the global Hopf bifurcation theorem in [5]. Following the work of Wu [5], we make the following definitions:

$$X = C([-D, 0], \mathbb{R}),$$
$$\Sigma = Cl\{(x, D, T) \in X \times \mathbb{R}^+ \times \mathbb{R}^+ : x \text{ is a } T\text{-periodic solution of (1.1)}\},$$
$$N = \{(\bar{x}, D, T) : w - \bar{x}p(\bar{x}) = 0\}.$$

Let $\ell(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))$ denote the connected component through

$$(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))$$

in Σ, where $D_c(n)$ and ω_0 are defined in the previous section.

Lemma 3.1. *System* (1.1) *has no nontrivial D-periodic solution.*

Proof. For a contradiction, suppose that system (1.1) has a nontrivial D-periodic solution. Then the ordinary differential equation

$$\dot{x}(t) = k[w - x(t)p(x(t))] \tag{3.1}$$

has a nontrivial periodic solution. According to the properties of the function $p(\cdot)$, we can easily obtain that if $x(t) < x^*$, then $\dot{x}(t) > 0$. If $x(t) > x^*$, then $\dot{x}(t) < 0$. So system (3.1) has no nontrivial periodic solution. Thus the proof is complete. □

Lemma 3.2. *If the congestion indication function $p(\cdot)$ is bounded, then all nontrivial periodic solutions of* (1.1) *are uniformly bounded.*

Proof. If the congestion indication function $p(\cdot)$ is bounded, then for any t, there exist $p_1 < p(x(t)) < p_2$, where p_1 and p_2 are constants. Let x be a nontrivial periodic solution of system (1.1). We define

$$x(\eta) = \max\{x(t)\}, \quad x(\xi) = \min\{x(t)\}.$$

Thus $x(\eta - D)p(x(\eta - D)) = w = x(\xi - D)p(x(\xi - D))$, and from (1.1) we can get

$$x(t) = x(0)\exp\left\{\int_0^t k\left[\frac{w}{x(s)} - \frac{x(s-D)}{x(s)}p\left(x(s-D)\right)\right]ds\right\},$$

which implies that either $x(t) > 0$ or $x(t) < 0$ if $x(0) \neq 0$. If $x(t) > 0$, then $\dot{x}(t) < kw$, which induces $x(t) < x(t-D) + kwD$. Thus

$$\begin{aligned} x(\eta) &< x(\eta - D) + kwD \\ &= \frac{w}{p(x(\eta - D))} + kwD \\ &< \frac{w}{p_1} + kwD. \end{aligned}$$

If $x(t) < 0$, then we set $y(t) = -x(t)$. Then $y(\eta) = -x(\xi)$ and (1.1) is equivalent to the equation

$$\dot{y}(t) = -k[w + y(t-D)p(-y(t-D))]. \tag{3.2}$$

By the standard comparison theorem, we have

$$x(\xi) > \frac{w}{p_2} + kwD.$$

From above, we can know that if x is a nontrivial periodic solution of (1.1), then

$$\frac{w}{p_2} + kwD < x(t) < \frac{w}{p_1} + kwD.$$

As a result, the nontrivial periodic solutions of (1.1) are uniformly bounded. $\square$

Theorem 3.3. *If the function $p(\cdot)$ is bounded, then for any $D > D_c(n)$, $n = 2, 4, 6, \ldots$, system (1.1) has at least $\dfrac{n+2}{2}$ periodic solutions.*

Proof. At first, we shall prove that $\ell(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))$ is unbounded. Following the definition of an isolated center in Wu [5], we can easily show that $(x^*, D_c(n), T)$ is the only isolated center. The characteristic equation

$$\Delta(x^*, D_c(n), T) = \lambda + k(p(x^*) + x^*p'(x^*))e^{-\lambda D}$$

is continuous in $(\bar{x}, D, T) \in C \times \mathbb{R}^+ \times \mathbb{R}^+$.

Obviously, from $\Delta(x^*, D_c(n), T)(i\omega_0) = 0$, $\dfrac{\partial\Delta(x^*, D_c(n), T)}{\partial D}|_{D=D_c(n),\lambda=i\omega_0} \neq 0$ and Lemma 3.1 and 3.2, we have that, for fixed n, there exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda(D) : (D_c(n) - \delta, D_c(n) + \delta) \to C$ such that $\Delta(\lambda(D)) = 0$, $|\lambda(D) - i\omega_0| < \varepsilon$ for all $D \in [D_c(n) - \delta, D_c(n) + \delta]$ and $\lambda(D_c(n)) = i\omega_0$, $\dfrac{\mathrm{dRe}\lambda(D)}{\mathrm{d}D}|_{D=D_c(n)} > 0$. Define

$$\Omega_{\varepsilon, \frac{2\pi}{D_c(n)\omega_0}} = \left\{(\eta, T) :\ 0 < \eta < \varepsilon, \left|T - \frac{2\pi}{D_c(n)\omega_0}\right| < \varepsilon\right\}.$$

It is not difficult to show that on $[D_c(n) - D, D_c(n) + D] \times \partial\Omega_{\varepsilon, 2\pi/(D_c(n))\omega_0}$,

$$\Delta(x^*, D_c(n), T)\left(\eta + \frac{2\pi}{T}i\right) = 0 \quad \text{if and only if} \quad \eta = 0, D = D_c(n), T = \frac{2\pi}{D_c(n)\omega_0}.$$

Moreover, if we define

$$H^{\pm}\left(x^*, D_c(n), \frac{2\pi}{D_c(n)\omega_0}\right)(\eta, T) = \Delta(x^*, D_c(n) \pm \delta, T)\left(\eta + i\frac{2\pi}{T}\right),$$

then we can compute the crossing number of the isolated center $\left(x^*, D_c(n), \dfrac{2\pi}{D_c(n)\omega_0}\right)$ as follows:

$$\begin{aligned}
\gamma\left(x^*, D_c(n), \frac{2\pi}{D_c(n)\omega_0}\right) &= \deg_B\left(H^-\left(x^*, D_c(n), \frac{2\pi}{D_c(n)\omega_0}\right), \Omega_{\varepsilon, \frac{2\pi}{D_c(n)\omega_0}}\right) \\
&\quad - \deg_B\left(H^+\left(x^*, D_c(n), \frac{2\pi}{D_c(n)\omega_0}\right), \Omega_{\varepsilon, \frac{2\pi}{D_c(n)\omega_0}}\right) \\
&= -1.
\end{aligned}$$

We have

$$\sum_{(\bar{x}, \bar{D}, \bar{T}) \in \ell(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))} \gamma(\bar{x}, \bar{D}, \bar{T}) < 0,$$

and thus, from [5, Theorem 3.3], the connected component $\ell(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))$ in Σ is unbounded.

Next, we shall prove that the projection of ℓ onto D-space has no upper bound. From (2.5), we have

$$D_c(n)\omega_0 = \frac{(2n+1)\pi}{2}, \qquad n = 0, 2, 4, \ldots.$$

Thus there exists an $n_i \in N$, such that

$$\frac{1}{n_i} < \frac{2\pi}{D_c(n)\omega_0} < 4.$$

Therefore the projection of $\ell(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))$ onto T-space is bounded. From Lemma 3.2, we can also get that the projection of $\ell(x^*, D_c(n), 2\pi/(D_c(n)\omega_0))$ onto x-space is bounded. According to Lemma 3.1, we can easily obtain that the projection on D-space must be unbounded and includes $[D_c(n), +\infty)$. As a result, for any $D > D_c(n)$, system (1.1) has at least $\frac{n+2}{2}$ periodic solutions with a period in $\left(\frac{1}{n_i}, 4\right)$. The proof is now complete. $\square$

Remark 3.4. If the congestion indication function $p(\cdot)$ is bounded, then the periodic solution still exists even if the parameter D is far from the second critical value.

4 Numerical Simulation

Consider the system

$$\dot{x}(t) = \frac{3}{2}\left[1 - x(t-D)\left(\tanh(x-D)+1\right)\right], \tag{4.1}$$

where $p(x) = \tanh x + 1$ is bounded. We can compute that system (4.1) has the unique equilibrium $x^* = 0.639232$, and $\omega_0 = 3$, $D_c(n) = \frac{(2n+1)\pi}{6}$, $D_c(0) = 0.523599$, $D_c(2) = 2.61799, \ldots$, $D_c(22) = 23.5619, \ldots$. From Figures 4.1–4.4, we can easily see that when D is far from the second critical value $D_c(2)$, large periodic solutions exist.

Figure 4.1: Waveform plot of system (4.1) when $D = 0.5 < D_c(0)$

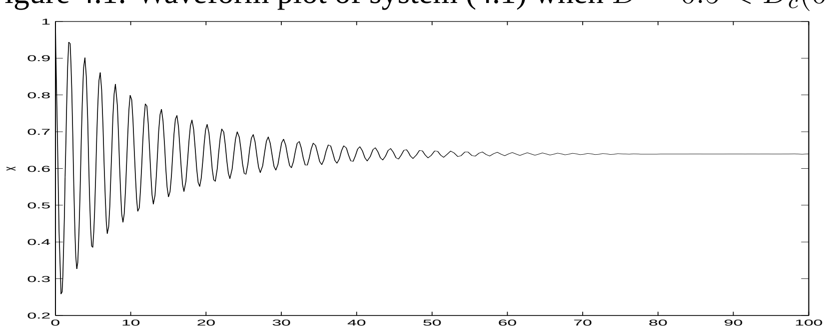

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Figure 4.2: Waveform plot of system (4.1) when $D = 0.55 > D_c(0)$

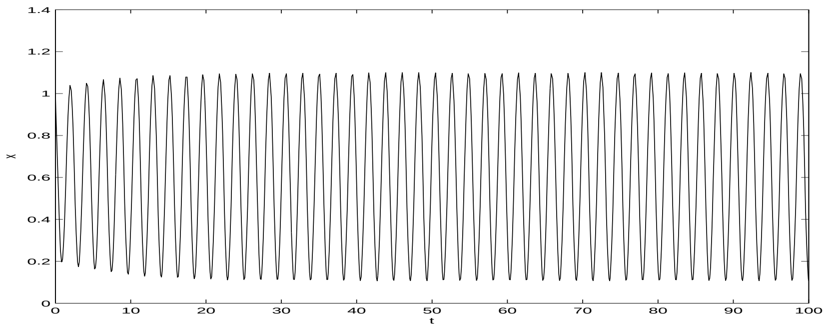


Figure 4.3: Waveform plot of system (4.1) when $D = 2.7 > D_c(2)$

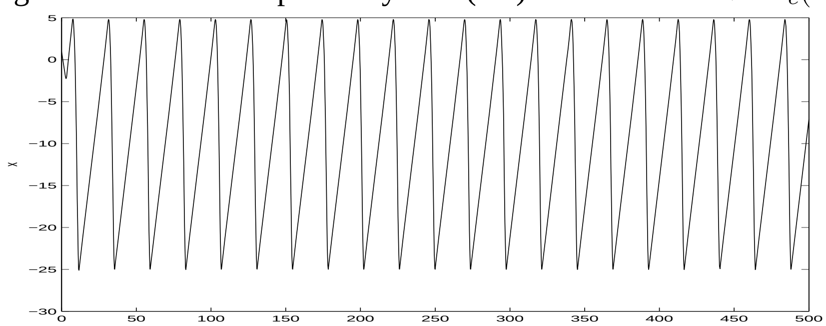
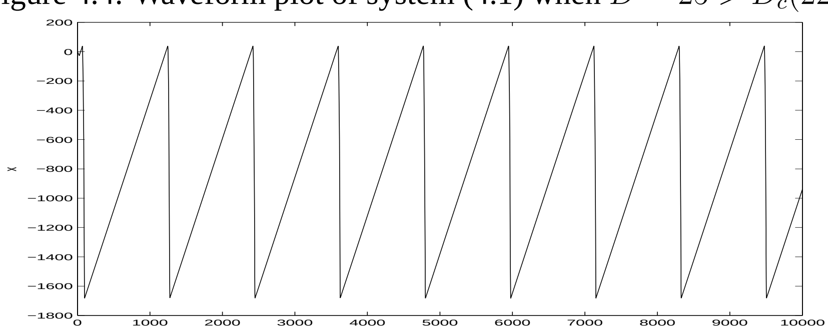


Figure 4.4: Waveform plot of system (4.1) when $D = 25 > D_c(22)$



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