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Singular Radial Positive Solutions for Nonlinear Elliptic Systems

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Abstract

In this paper we study the existence and nonexistence of positive singular radial solutions of the Dirichlet p-Laplacian system

$$\begin{cases} \Delta_p u + f(x,u,v) = 0 & \text{in } \Omega \\ \Delta_q v - g(x,u,v) = 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $N > \max(p,q)$, $p, q > 1$, as well as the solution's behavior near zero. Here, Ω is the unit ball of $\mathbb{R}^N$ except for the center zero, and f, g are nonnegative continuous functions. We use Leray–Schauder's theorem and a method of monotone iterations to prove existence, and we will be concerned with the study of the asymptotic behavior of solutions. Also, we will present some sufficient conditions for nonexistence of positive solutions.

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1 Introduction and Main Results

We prove the existence of radial positive solutions of the nonlinear system with elliptic equations

$$\begin{cases} -\Delta_p u = a(x)u^\alpha v^\beta + h_1(x) & \text{in } B' \\ \Delta_q v = b(x)u^\gamma v^\delta + h_2(x) & \text{in } B' \\ u = v = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

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where B is the unit ball of $\mathbb{R}^N$, $B' = B\backslash\{0\}$, $N > \max(p,q)$, $p,q > 1$, and $a,b,h_1,h_2 \in L^\infty(B)$ are given nonnegative functions. A positive solution (u,v) of (1.1) is said to be singular at zero (or just singular) if

$$\limsup_{r\to 0^+}(u(r)+v(r)) = \infty.$$

Set

$$l = \limsup_{r\to 0^+}\left(r^{\frac{N-p}{p-1}}u(r) + r^{\frac{N-q}{q-1}}v(r)\right).$$

If $l = \infty$, then the solution (u,v) is called strongly singular; while if $0 < l < \infty$, this singularity is said to be fundamental.

The problem (1.1) does not have a variational structure and much attention has been given to the existence of solutions for such systems by using different approaches. The asymptotic behavior of the solutions was an interesting question to solve, especially when the solution presents some singularities around the origin. The previous articles which are treating the question of the existence are numerous, see C. Cid and C. Yarur [4]; concerning the behavior of the solution near zero and nonexistence of nonnegative solutions without boundary conditions see M.F. Bidaut–Veron and P. Grillot [3]. Note that for the system

$$\begin{cases} -\Delta u = u^\alpha, \quad u > 0 \quad \text{in } B' \\ \qquad u = 0 \qquad \text{on } \partial B, \end{cases} \tag{1.2}$$

nonnegative singular solutions exist under the condition $\alpha < \dfrac{N+2}{N-2}$. In such a case, the solution u with a singularity at zero satisfies

$$0 \le \limsup_{r\to 0^+}|r|^{N-2}u(r) < \infty,$$

and thus the singularity is of fundamental type. See S.S. Lin [6], P.L. Lions [7], W.M. Ni and P. Sacks [8].

Our method includes two fundamental cases (the sublinear and superlinear class) as done by A. Ahammou [1, 2] for the system

$$\begin{cases} -\Delta_p u = a(x)|u|^{\alpha-1}u + b(x)|v|^{\beta-1}v \qquad \text{in } B \\ -\Delta_q v = c(x)|u|^{\gamma-1}u + d(x)|v|^{\delta-1}v \qquad \text{in } B \\ \qquad u = v = 0 \qquad \text{on } \partial B, \end{cases} \tag{1.3}$$

where he treated the question of existence in two principal classes:

$$\beta\gamma < (p-1)(q-1) \quad \text{and} \quad \beta\gamma > (p-1)(q-1),$$

by using techniques of the topological degree theory combined with the blow-up method introduced by B. Gidas and J. Spruck [5]. In this work we suppose that $\beta\gamma \neq (p-1-\alpha)(q-1-\delta)$ and we formulate our main results as follows.

Theorem 1.1 (Sublinear Case). *Assume that*

$$\beta\gamma < (p-1-\alpha)(q-1-\delta), \quad \alpha < p-1, \quad \delta < q-1$$

and suppose furthermore that

i- $$\gamma > \frac{N(p-1)}{N-p} \text{ or}$$

ii- $$\frac{N-p}{N(p-1)}\alpha + \frac{N-q}{N(q-1)}\beta < 1 \quad \text{and} \quad \frac{N-p}{N(p-1)}\gamma + \frac{N-q}{N(q-1)}\delta < 1.$$

Then, for any $\lambda \in [0,\infty[$*, there exists a nonnegative nontrivial solution* (u,v) *to the problem* (1.1) *such that*

$$\lim_{r\to 0^+} r^{\frac{N-p}{p-1}} u(r) = \left(\frac{N-p}{p-1}\right)^{-1} \lambda^{\frac{1}{p-1}}.$$

Theorem 1.2 (Superlinear Case). *Assume that*

$$\beta\gamma > (p-1-\alpha)(q-1-\delta), \quad \alpha < p-1, \quad \delta < q-1,$$

$h_1 \equiv h_2 \equiv 0$ *and suppose furthermore that*

i- $$\gamma > \frac{N(p-1)}{N-p} \text{ and } \frac{N-p}{N(p-1)}\alpha + \frac{\frac{N-p}{p-1}\gamma - q}{N(q-1-\delta)}\beta < 1 \text{ or}$$

ii- $$\frac{N-p}{N(p-1)}\alpha + \frac{N-q}{N(q-1)}\beta < 1 \quad \text{and} \quad \frac{N-p}{N(p-1)}\gamma + \frac{N-q}{N(q-1)}\delta < 1.$$

Then there exists a positive constant $\lambda_{N,p}$ *such that for any* $\lambda \in [0,\lambda_{N,p}]$*, the problem* (1.1) *has a nonnegative nontrivial solution* (u,v) *satisfying*

$$\lim_{r\to 0^+} r^{\frac{N-p}{p-1}} u(r) = \left(\frac{N-p}{p-1}\right)^{-1} \lambda^{\frac{1}{p-1}}.$$

2 Preliminaries

For all $r \in [0,1]$ let f, g be the functions defined by

$$f(r,u(r),v(r)) = a(r)u^{\alpha}(r)v^{\beta}(r) + h_1(r)$$

$$g(r,u(r),v(r)) = b(r)u^{\gamma}(r)v^{\delta}(r) + h_2(r).$$

Set

$$a = \sup_{r\in[0,1]} a(r), \quad b = \sup_{r\in[0,1]} b(r)$$

$$h_1=\sup_{r\in[0,1]} h_1(r) \quad \text{and } h_2=\sup_{r\in[0,1]} h_2(r).$$

Since this work is concerned with radial solutions, system (1.1) may be written as

$$\left\{\begin{array}{ll} -\left(r^{N-1}|u'|^{p-2}u'\right)'=r^{N-1}f(r,u(r),v(r)), & \text{in } (0,1)\\ \left(r^{N-1}|v'|^{q-2}v'\right)'=r^{N-1}g(r,u(r),v(r)), & \text{in } (0,1)\\ u(1)=v(1)=0. & \end{array}\right. \tag{2.1}$$

Define the solution operator $S=(S_1,S_2)$ by

$$S_1(u,v)(r)=\int_r^1 s^{\frac{1-N}{p-1}}\left(\lambda_1+\int_0^s t^{N-1}f(t,u(t),v(t))dt\right)^{\frac{1}{p-1}}ds, \tag{2.2}$$

$$S_2(u,v)(r)=\int_r^1 s^{\frac{1-N}{q-1}}\left(\lambda_2+\int_s^1 t^{N-1}g(t,u(t),v(t))dt\right)^{\frac{1}{q-1}}ds, \tag{2.3}$$

where λ_1 and λ_2 are nonnegative numbers. Then, a nonnegative fixed point (u,v) of the operator S is a nonnegative solution to (2.1).

3 Existence Results and Asymptotic Behavior

3.1 Sublinear Case: $\beta\gamma<(p-1-\alpha)(q-1-\delta)$

First of all, we show the following result.

Proposition 3.1. *Consider the hypothesis of Theorem 1.1. Then, there exists an invariant set $\mathcal{A}$ under the operator S:*

$$S(\mathcal{A})\subset\mathcal{A}.$$

Proof. Here we detail the proof under the case -i- of Theorem 1.1; the case -ii- can be treated similarly. Recall that:

$i-\qquad \gamma>\dfrac{N(p-1)}{N-p}.$

Let $(\lambda_1,\lambda_2)\in([0,\infty[)^2$, and consider a set $\mathcal{A}$ of $(C(0,1])^2$ such that

$$\mathcal{A}=\left\{(u,v)\in(C(0,1])^2:0\le u\le\rho^l r^{\frac{p-N}{p-1}},\ 0\le v\le\rho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}\right\},$$

where ρ is a positive constant and l,k satisfy

$$\frac{\beta}{p-1-\alpha}<\frac{l}{k}<\frac{q-1-\delta}{\gamma},$$

and set

$$N'=\frac{N-p}{p-1}\gamma.$$

Next, we prove that $\mathcal{A}$ is invariant under S. We have from the expression (2.2) of S_1

$$S_1(u,v) = \int_r^1 s^{\frac{1-N}{p-1}} \left(\lambda_1 + \rho^{\alpha l+\beta k} \int_0^s a(t) t^{N-\frac{N-p}{p-1}\alpha - \frac{N'-q}{q-1-\delta}\beta - 1} dt + \frac{h_1}{N} \right)^{\frac{1}{p-1}} ds.$$

Now, $\beta\gamma < (p-1-\alpha)(q-1-\delta)$ implies $N - \dfrac{N-p}{p-1}\alpha - \dfrac{N'-q}{q-1-\delta}\beta > 0$. We obtain

$$\begin{aligned} S_1(u,v) &\leq \left(\lambda_1 + \frac{a\rho^{\alpha l+\beta k}}{N - \frac{N-p}{p-1}\alpha - \frac{N'-q}{q-1}\beta} + \frac{h_1}{N} \right) \int_r^1 s^{\frac{1-N}{p-1}} ds \\ &\leq \left(\lambda_1 + \frac{a\rho^{\alpha l+\beta k}}{N - \frac{N-p}{p-1}\alpha - \frac{N'-q}{q-1-\delta}\beta} + \frac{h_1}{N} \right)^{\frac{1}{p-1}} \left(\frac{N-p}{p-1} \right)^{-1} r^{\frac{p-N}{p-1}}. \end{aligned}$$

Let $\lambda_{1\rho} = \dfrac{1}{2} \left(\dfrac{N-p}{p-1} \right)^{p-1} \rho^{l(p-1)}$. Then we have for any $\lambda_1 \in \left[0, \lambda_{1\rho}\right]$

$$\begin{aligned} &S_1(u,v) \leq \\ &\rho^l \left(\frac{1}{2} + \left(\frac{a\rho^{\alpha l+\beta k - l(p-1)}}{N - \frac{N-p}{p-1}\alpha - \frac{N'-q}{q-1-\delta}\beta} + \frac{h_1}{N}\rho^{-l(p-1)} \right) \left(\frac{N-p}{p-1} \right)^{-(p-1)} \right)^{\frac{1}{p-1}} r^{\frac{p-N}{p-1}}. \end{aligned}$$

Since $\alpha l + \beta k - l(p-1) < 0$, we get for ρ sufficiently large

$$S_1(u,v) \leq \rho^l r^{\frac{p-N}{p-1}}. \tag{3.1}$$

From another side, the condition -i- implies that $N - \dfrac{N-p}{p-1}\gamma - \dfrac{N'-q}{q-1-\delta}\delta < 0$, and from the expression (2.3) of S_2 we get

$$\begin{aligned} S_2(u,v) &= \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \rho^{\gamma l+\delta k} \int_s^1 b(t) t^{N-\frac{N-p}{p-1}\gamma - \frac{N'-q}{q-1-\delta}\delta - 1} dt + \frac{h_2}{N} \right)^{\frac{1}{q-1}} ds \\ &\leq \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \frac{b\rho^{\gamma l+\delta k}}{\frac{N-p}{p-1}\gamma + \frac{N'-q}{q-1-\delta}\delta - N} s^{N-\frac{N-p}{p-1}\gamma - \frac{N'-q}{q-1-\delta}\delta} + \frac{h_2}{N} \right)^{\frac{1}{q-1}} ds \\ &\leq \left(\lambda_2 + \frac{b\rho^{\gamma l+\delta k}}{\frac{N-p}{p-1}\gamma + \frac{N'-q}{q-1-\delta}\delta - N} + \frac{h_2}{N} \right)^{\frac{1}{q-1}} \int_r^1 s^{\frac{q-N'}{q-1-\delta}-1} ds \\ &\leq \left(\lambda_2 + \frac{b\rho^{\gamma l+\delta k}}{\frac{N-p}{p-1}\gamma + \frac{N'-q}{q-1-\delta}\delta - N} + \frac{h_2}{N} \right)^{\frac{1}{q-1}} \left(\frac{N'-q}{q-1-\delta} \right)^{-1} r^{\frac{q-N'}{q-1-\delta}}. \end{aligned}$$

Let $\lambda_{2\rho} = \dfrac{1}{2}\left(\dfrac{N'-q}{q-1}\right)^{q-1}\rho^{k(q-1)}$. Then we obtain for all $\lambda_2 \in \left[0, \lambda_{2\rho}\right]$

$$S_2(u,v) \leq$$

$$\rho^k \left(\frac{1}{2} + \left(\frac{b\rho^{\gamma l+\delta k-k(q-1)}}{\frac{N-p}{p-1}\gamma + \frac{N'-q}{q-1-\delta}\delta - N} + \frac{h_2}{N}\rho^{-k(q-1)}\right)\left(\frac{N'-q}{q-1-\delta}\right)^{-(q-1)}\right)^{\frac{1}{q-1}} r^{\frac{q-N'}{q-1-\delta}}.$$

Since $\gamma l + \delta k - k(q-1) < 0$ and $-k(q-1) < 0$, we deduce that for ρ large enough

$$S_2(u,v) \leq \rho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}. \tag{3.2}$$

In this way, (3.1) and (3.2) justify the invariance of $\mathcal{A}$ under S. □

Note that the operator S may be not compact since it is not necessarily equicontinuous. Therefore we have to transfer S into another continuous and compact operator T. We will treat only the case -i- of Theorem 1.1; concerning -ii- the arguments are analogue.

We now construct the operator T. Consider the positive numbers ρ, l, k as constructed in Proposition 3.1, and let $(\lambda_1, \lambda_2) \in ([0,\infty[)^2$. Let T be an operator defined on a subset $\mathcal{B}$ of $(C[0,1])^2$ such that

$$\mathcal{B} = \left\{(y,z) \in (C[0,1])^2 : \ 0 \leq y \leq \rho^l r^\varepsilon, 0 \leq z \leq \rho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}} v_\rho(r)\right\},$$

where ε is a positive number and v_ρ is defined by

$$v_\rho(r) = \left(\rho^{-k} r^{(\frac{N-p}{p-1}\gamma-q)(q-1-\delta)^{-1}}\right)^{1+\varepsilon}, \quad r \in [0,1].$$

We define $T = (T_1, T_2)$ by:

$$T_1(y,z)(r) = r^{\frac{N-p}{p-1}+\varepsilon} S_1\left(r^{\frac{p-N}{p-1}-\varepsilon}y, v_\rho^{-1}z\right)(r)$$

and

$$T_2(y,z)(r) = v_\rho(r) S_2\left(r^{\frac{p-N}{p-1}-\varepsilon}y, v_\rho^{-1}z\right)(r),$$

i.e.,

$$T_1(y,z)(r) = r^{\frac{N-p}{p-1}+\varepsilon}\int_r^1 s^{\frac{1-N}{p-1}}\left(\lambda_1 + \int_0^s t^{N-1} f\left(t, r^{\frac{p-N}{p-1}-\varepsilon}y, v_\rho^{-1}z\right)dt\right)^{\frac{1}{p-1}} ds \tag{3.3}$$

and

$$T_2(y,z)(r) = v_\rho(r)\int_r^1 s^{\frac{1-N}{q-1}}\left(\lambda_2 + \int_s^1 t^{N-1} g\left(t, r^{\frac{p-N}{p-1}-\varepsilon}y, v_\rho^{-1}z\right)dt\right)^{\frac{1}{q-1}} ds. \tag{3.4}$$

In view of applying Leray–Schauder's fixed point theorem, the next result will be useful.

Proposition 3.2. *Under the case -i- of Theorem 1.1, we have:*

- *$\mathcal{B}$ is a closed convex bounded subset;*
- *$\mathcal{B}$ is invariant under T;*
- *T is a continuous and compact operator on $\mathcal{B}$.*

Proof. The first point is evident. From Proposition 3.1, $\mathcal{A}$ is invariant under S, so $T(\mathcal{B}) \subset \mathcal{B}$.

To prove continuity of T in $\mathcal{B}$, let (y_n, z_n) be a sequence converging on $\mathcal{B}$ to (y, z). Then the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} T\left(y_n, z_n\right)(r) = T(y,z)(r)$$

for all $r \in [0,1]$. Moreover, the family $\{T\left(y_n, z_n\right), n \geq 1\} \cup \{T(y,z)\}$ is equicontinuous since $\mathcal{B}$ is closed and $T(\mathcal{B})$ is equicontinuous. Hence, the convergence of $T\left(y_n, z_n\right)$ to $T(y,z)$ is uniform with respect to y, z. Therefore T is a continuous operator.

Now, we prove that $T(\mathcal{B})$ is a relatively compact subset of $\left(C[0,1]\right)^2$. According to the Ascoli–Arzela theorem, and since $T(\mathcal{B})$ is bounded, it remains to prove that $T(\mathcal{B})$ is an equicontinuous subset of $\left(C[0,1]\right)^2$. Using (3.3) and (3.4), it follows

$$\begin{aligned}\frac{d}{dr}T_1(y,z)(r) &= \left(\frac{N-p}{p-1}+\varepsilon\right) r^{-1} T_1(y,z)(r) \\ &\quad - r^{\varepsilon-1}\left(\lambda_1 + \int_0^r t^{N-1} f\left(t, r^{\frac{p-N}{p-1}-\varepsilon} y, v_\rho^{-1} z\right) dt\right)^{\frac{1}{p-1}},\end{aligned}$$

$$\begin{aligned}\frac{d}{dr}T_2(y,z)(r) =& \frac{N'-q}{q-1-\delta}(1+\varepsilon) r^{-1} T_2(y,z)(r) \\ &- \rho^{-k(1+\varepsilon)} r^{\frac{N'-q}{q-1-\delta}(1+\varepsilon)+\frac{1-N}{q-1}} \left(\lambda_2 + \int_r^1 g(t, t^{\frac{p-N}{p-1}-\varepsilon} y, v_\rho^{-1} z) dt\right)^{\frac{1}{q-1}}.\end{aligned}$$

Since $\mathcal{B}$ is invariant under T, we deduce that

$$\begin{aligned}\left|\frac{d}{dr}T_1(y,z)(r)\right| &\leq \left(\frac{N-p}{p-1}+\varepsilon\right)\rho^l r^{\varepsilon-1} + \frac{N-p}{p-1}\rho^l r^{\varepsilon-1} \\ \left|\frac{d}{dr}T_2(y,z)(r)\right| &\leq \frac{N'-q}{q-1-\delta}(1+\varepsilon)\rho^{-k\varepsilon} r^{\frac{N'-q}{q-1-\delta}\varepsilon-1} + \frac{N'-q}{q-1-\delta}\rho^{-k\varepsilon} r^{\frac{N'-q}{q-1-\delta}\varepsilon-1},\end{aligned}$$

and for a suitable $\varepsilon > 0$, we get

$$\left|\frac{d}{dr}T_1(y,z)(r)\right| \leq C_1 \quad \text{and} \quad \left|\frac{d}{dr}T_2(y,z)(r)\right| \leq C_2$$

uniformly with respect to y and z. Then, the set $T(\mathcal{B})$ is equicontinuous, and consequently it is a compact operator. □

Now, we are ready to prove the first main result.

Proof of Theorem 1.1. By Proposition 3.2, we deduce by Leray–Schauder's fixed point theorem that, for all $(\lambda_1, \lambda_2) \in ([0, \infty[)^2$, the operator T has a fixed point (y, z) in $\mathcal{B}$. Furthermore, (y, z) is a fixed point of T if and only if $(u, v) = \left(r^{\frac{p-N}{p-1}-\varepsilon} y, v_\alpha^{-1} z\right)$ is a fixed point of S. Hence we derive that $(u, v) \in \mathcal{A}$ such that

$$u(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left(\lambda_1 + \int_0^s t^{N-1} \left(a(t) u^\alpha v^\beta + h_1(t)\right) dt\right)^{\frac{1}{p-1}} ds$$

and

$$v(r) = \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \int_s^1 t^{N-1} \left(b(t) u^\gamma v^\delta + h_2(t)\right) dt\right)^{\frac{1}{q-1}} ds$$

with $(\lambda_1, \lambda_2) \in ([0, \infty[)^2$, and the Theorem 1.1 follows. We may use similar arguments to show the result for the case -ii-. Concerning the asymptotic behavior of u, we apply the L'Hôpital rule to the expression of u. Then for all $\lambda_1 \in [0, \infty[$, we have

$$\begin{aligned}
\lim_{r \to 0^+} r^{\frac{N-p}{p-1}} u(r) &= \left(\frac{N-p}{p-1}\right)^{-1} \lim_{r \to 0^+} \frac{-u'(r)}{r^{\frac{1-N}{p-1}}} \\
&= \left(\frac{N-p}{p-1}\right)^{-1} \lim_{r \to 0^+} \left(\lambda_1 + \int_0^r t^{N-1} \left(a(t) u^\alpha v^\beta + h_1(t)\right) dt\right)^{\frac{1}{p-1}} \\
&= \left(\frac{N-p}{p-1}\right)^{-1} \lambda_1^{\frac{1}{p-1}}.
\end{aligned}$$

This completes the proof. □

3.2 Superlinear Case: $\beta\gamma > (p-1-\alpha)(q-1-\delta)$

Proof of Theorem 1.2. In this proof, we study only the case -i- of Theorem 1.2 because the arguments are the same for the case -ii-. Recall that $h_1 \equiv h_2 \equiv 0$ and

$$i - \qquad \gamma > \frac{N(p-1)}{N-p} \quad and \quad \frac{N-p}{N(p-1)}\alpha + \frac{N'-q}{N(q-1-\delta)}\beta < 1.$$

Now, we construct nonnegative functions by monotone iterations under the operator S. Let $\lambda_1, \lambda_2, \lambda_{1,0}, \lambda_{2,0}, \varrho$ be nonnegative numbers and $(u_1, v_1) = (\lambda_{1,0} u_{0,p}, \lambda_{2,0} v_{0,q})$, where

$$\lambda_{1,0} = 2^{\frac{1}{p-1}} \left(\frac{N-p}{p-1}\right)^{-1} \lambda_1^{\frac{1}{p-1}}, \quad \lambda_{2,0} = 2^{\frac{1}{q-1}} \left(\frac{N'-q}{q-1-\delta}\right)^{-1} \lambda_2^{\frac{1}{q-1}},$$

$$u_{0,p}(r) = r^{\frac{p-N}{p-1}} - 1, \quad v_{0,q}(r) = r^{\frac{q-N'}{q-1-\delta}} - 1$$

and

$$N' = \frac{N-p}{p-1}\gamma,$$

and let $(u_n, v_n)_{n\in\mathbb{N}^*}$ be the sequence defined by

$$(u_{n+1}, v_{n+1}) = S\,(u_n, v_n)\,. \tag{3.5}$$

Then we obtain

$$\begin{aligned} &S_1\left(\varrho^l r^{\frac{p-N}{p-1}}, \varrho^k r^{\frac{q-N'}{q-1-\delta}}\right) \\ &\qquad = \int_r^1 s^{\frac{1-N}{p-1}}\left(\lambda_1 + \varrho^{\alpha l+\beta k}\int_0^s a(t)t^{N-\frac{N-p}{p-1}\alpha-\frac{N'-q}{q-1-\delta}\beta-1}dt\right)^{\frac{1}{p-1}} ds. \end{aligned}$$

From -i- we have $N - \dfrac{N-p}{p-1}\alpha - \dfrac{N'-q}{q-1-\delta}\beta > 0$. Then

$$\begin{aligned} S_1\left(\varrho^l r^{\frac{p-N}{p-1}}, \varrho^k r^{\frac{q-N'}{q-1-\delta}}\right) &\le \left(\lambda_1 + \frac{a\varrho^{\alpha l+\beta k}}{N-\frac{N-p}{p-1}\alpha-\frac{N'-q}{q-1}\beta}\right)\int_r^1 s^{\frac{1-N}{p-1}}ds \\ &\le \left(\lambda_1 + \frac{a\varrho^{\alpha l+\beta k}}{N-\frac{N-p}{p-1}\alpha-\frac{N'-q}{q-1-\delta}\beta}\right)^{\frac{1}{p-1}}\left(\frac{N-p}{p-1}\right)^{-1} r^{\frac{p-N}{p-1}}. \end{aligned}$$

Let $\lambda'_{1_\varrho} = \dfrac{1}{2}\left(\dfrac{N-p}{p-1}\right)^{p-1}\varrho^{l(p-1)}$. Then we have for any $\lambda_1 \in \left[0, \lambda'_{1_\varrho}\right]$

$$\begin{aligned} &S_1\left(\varrho^l r^{\frac{p-N}{p-1}}, \varrho^k r^{\frac{q-N'}{q-1-\delta}}\right) \\ &\qquad \le \varrho^l\left(\frac{1}{2} + \left(\frac{a\varrho^{\alpha l+\beta k-l(p-1)}}{N-\frac{N-p}{p-1}\alpha-\frac{N'}{q-1-\delta}\beta}\right)\left(\frac{N-p}{p-1}\right)^{-(p-1)}\right)^{\frac{1}{p-1}} r^{\frac{p-N}{p-1}}. \end{aligned}$$

Moreover, if we choose l and k such that

$$\frac{\beta}{p-1-\alpha} > \frac{l}{k} > \frac{q-1-\delta}{\gamma}$$

and for ϱ sufficiently small, then we obtain

$$S_1\left(\varrho^l r^{\frac{p-N}{p-1}}, \varrho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}\right) \le \varrho^l r^{\frac{p-N}{p-1}}. \tag{3.6}$$

From the hypothesis -i- we have $N-\dfrac{N-p}{p-1}\gamma-\dfrac{N'-q}{q-1-\delta}\delta<0$, and by (2.3) we obtain

$$\begin{aligned}
S_2\left(\varrho^l r^{\frac{p-N}{p-1}},\varrho^k r^{\frac{q-N'}{q-1-\delta}}\right) &= \int_r^1 s^{\frac{1-N}{q-1}}\left(\lambda_2+\varrho^{\gamma l+\delta k}\int_s^1 b(t)t^{N-N'-\frac{N'-q}{q-1-\delta}\delta-1}dt\right)^{\frac{1}{q-1}}ds\\
&\le \int_r^1 s^{\frac{1-N}{q-1}}\left(\lambda_2+\frac{b\varrho^{\gamma l+\delta k}}{N'+\frac{N'-q}{q-1-\delta}\delta-N}s^{N-N'-\frac{N'-q}{q-1-\delta}\delta}\right)^{\frac{1}{q-1}}ds\\
&\le \left(\lambda_2+\frac{b\varrho^{\gamma l+\delta k}}{N'+\frac{N'-q}{q-1-\delta}\delta-N}\right)^{\frac{1}{q-1}}\int_r^1 s^{(q-1-\delta)^{-1}(1+\delta-N')}ds\\
&\le \left(\lambda_2+\frac{b\varrho^{\gamma l+\delta k}}{N'+\frac{N'-q}{q-1-\delta}\delta-N}\right)^{\frac{1}{q-1}}\left(\frac{N'-q}{q-1-\delta}\right)^{-1}r^{\frac{q-N'}{q-1-\delta}}.
\end{aligned}$$

Let $\widetilde{\lambda}_{2\varrho}=\dfrac{1}{2}\left(\dfrac{N'-q}{q-1-\delta}\right)^{q-1}\varrho^{k(q-1)}$. Then we obtain for any $\lambda_2\in\left[0,\widetilde{\lambda}_{2\varrho}\right]$

$$\begin{aligned}
&S_2\left(\varrho^l r^{\frac{p-N}{p-1}},\varrho^k r^{\frac{q-N'}{q-1-\delta}}\right)\\
&\qquad\le \varrho^k\left(\frac{1}{2}+\left(\frac{b\varrho^{\gamma l+\delta k-k(q-1)}}{N'+\frac{N'-q}{q-1-\delta}\delta-N}\right)\left(\frac{N'-q}{q-1-\delta}\right)^{-(q-1)}\right)^{\frac{1}{q-1}}r^{\frac{q-N'}{q-1-\delta}}.
\end{aligned}$$

Since $\gamma l+\delta k-k(q-1)>0$, for ϱ small enough, we deduce that

$$S_2\left(\varrho^l r^{\frac{p-N}{p-1}},\varrho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}\right)\le \varrho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}. \tag{3.7}$$

Moreover, for all $(\lambda_1,\lambda_2)\in[0,\lambda'_{1\varrho}]\times[0,\widetilde{\lambda}_{2\varrho}]$, we have

$$\begin{aligned}
u_1(r) &= 2^{\frac{1}{p-1}}\left(\frac{N-p}{p-1}\right)^{-1}\lambda_1^{\frac{1}{p-1}}u_{0,p}(r)\le \varrho^l r^{\frac{p-N}{p-1}}\\
v_1(r) &= 2^{\frac{1}{q-1}}\left(\frac{\frac{N-p}{p-1}\gamma-q}{q-1-\delta}\right)^{-1}\lambda_2^{\frac{1}{q-1}}v_{0,q}(r)\le \varrho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}.
\end{aligned}\tag{3.8}$$

Thus, from (3.6), (3.7), (3.8) and using the monotony of the operator S, we deduce

$$\begin{aligned}
S_1(u_1,v_1) &\le \varrho^l r^{\frac{p-N}{p-1}}\\
S_2(u_1,v_1) &\le \varrho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}},
\end{aligned}\tag{3.9}$$

and by iteration we derive

$$\begin{aligned}
u_{n+1} &= S_1(u_n,v_n)\le \varrho^l r^{\frac{p-N}{p-1}}\\
v_{n+1} &= S_2(u_n,v_n)\le \varrho^k r^{(q-\frac{N-p}{p-1}\gamma)(q-1-\delta)^{-1}}
\end{aligned}\qquad \forall n\in\mathbb{N}^*. \tag{3.10}$$

Now, from (3.10) the sequence $(y_n, z_n)_n = \left(r^{\frac{N-p}{p-1}} u_n \ , \ r^{\frac{\frac{N-p}{p-1}\gamma - q}{q-1}} v_n \right)_n$ is bounded. So there exists a subsequence $(y_{n_k}, z_{n_k})_k$ converging to $(y, z) = \left(r^{\frac{N-p}{p-1}} u \ , \ r^{\frac{\frac{N-p}{p-1}\gamma - q}{q-1}} v \right)$ in $(C[0,1])^2$. That is to say, there exists a subsequence $(u_{n_k}, v_{n_k})_k$ which converges to (u, v) in $(C[0,1])^2$ as $k \to \infty$, and satisfies

$$u_{n_{k+1}}(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left(\lambda_1 + \int_0^s t^{N-1} f\left(t, u_{n_k}(t), v_{n_k}(t)\right) dt \right)^{\frac{1}{p-1}} ds$$

and

$$v_{n_{k+1}}(r) = \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \int_s^1 t^{N-1} g\left(t, u_{n_k}(t), v_{n_k}(t)\right) dt \right)^{\frac{1}{q-1}} ds,$$

and by the Lebesgue dominated convergence theorem, this becomes

$$u(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left(\lambda_1 + \int_0^s t^{N-1} a(t) u^{\alpha} v^{\beta} dt \right)^{\frac{1}{p-1}} ds$$

and

$$v(r) = \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \int_s^1 t^{N-1} b(t) u^{\gamma} v^{\delta} dt \right)^{\frac{1}{q-1}} ds$$

with $(\lambda_1, \lambda_2) \in [0, \lambda'_{1\varrho}] \times [0, \widetilde{\lambda}_{2\varrho}]$. Hence, (u, v) is a nonnegative nontrivial solution to the problem (2.1). We may proceed similarly about the case -ii- of Theorem 1.2. Similarly as in the proof of Theorem 1.1, we apply the L'Hôpital rule to deduce

$$\forall \lambda_1 \in [0, \lambda'_{1\varrho}] : \qquad \lim_{r \to 0^+} r^{\frac{N-p}{p-1}} u(r) = \left(\frac{N-p}{p-1} \right)^{-1} \lambda_1^{\frac{1}{p-1}}.$$

This concludes the proof. □

3.3 Asymptotic Behavior of Solutions

In this subsection, we present two results describing the solution's singularity which may be strong or fundamentally singular under some sufficient conditions.

Theorem 3.3. *Assume that the hypotheses of Theorem 1.1 or Theorem 1.2 hold, and let* (u, v) *be a nonnegative nontrivial solution to the system* (2.1) *such that*

$$\lim_{r \to 0^+} r^{\frac{N-p}{p-1}} u(r) \neq 0.$$

Suppose that the function $r \mapsto b(r)$ *does not vanish near zero and assume furthermore that*

$$\frac{N-p}{N(p-1)} \gamma + \frac{N-q}{N(q-1)} \delta > 1.$$

Then, the solution (u, v) *has a strong singularity.*

Proof. Consider the solution (u,v) of the problem (2.1). Then the function v can be written as

$$v(r)=\int_r^1 s^{\frac{1-N}{q-1}}\left(\lambda_2+\int_s^1 t^{N-1}\left(b(t)u^\gamma v^\delta+h_2(t)\right)dt\right)^{\frac{1}{q-1}}ds,\quad r\in(0,1],$$

with $\lambda_2\geq 0$. Using the L'Hôpital rule we get

$$\liminf_{r\to 0^+} r^{\frac{N-q}{q-1}}v(r)\geq\left(\frac{N-q}{q-1}\right)^{-1}\left(\int_0^1 t^{N-1}b(t)u^\gamma(t)v^\delta(t)dt\right)^{\frac{1}{q-1}}.$$

Since $\lim_{r\to 0^+} r^{\frac{N-p}{p-1}}u(r)>0$ and $\lim_{r\to 0^+} r^{\frac{N-q}{q-1}}v(r)>0$, there exist positive constants K_1 and K_2 such that for r near zero

$$u(r)\geq K_1 r^{\frac{p-N}{p-1}}\quad\text{and}\quad v(r)\geq K_2 r^{\frac{q-N}{q-1}}.$$

Then, for ϵ small enough, we get

$$\int_0^\epsilon t^{N-1}b(t)u^\gamma(t)v^\delta(t)dt\geq K_3\int_0^\epsilon t^{N-\frac{N-p}{p-1}\gamma-\frac{N-q}{q-1}\delta-1}dt=\infty$$

because $N-\dfrac{N-p}{p-1}\gamma-\dfrac{N-q}{q-1}\delta<0$. Hence, (u,v) is strongly singular. □

Theorem 3.4. *Assume that the hypotheses of Theorem 1.1 or Theorem 1.2 hold, and let (u,v) be a nonnegative nontrivial solution to* (2.1) *such that*

$$\lim_{r\to 0^+} r^{\frac{N-p}{p-1}}u(r)<\infty. \tag{3.11}$$

Suppose that the function $r\mapsto a(r)$ does not vanish near zero, $\alpha<p-1$, and

$$\beta\gamma-\alpha\delta<\frac{N(p-1)}{N-p}(\beta-\delta). \tag{3.12}$$

Then, the solution (u,v) has a fundamental singularity.

Before presenting the proof of this theorem, the following lemma will be useful. It describes the monotonicity of the function $r\mapsto r^\eta u(r)$ on $(0,\dfrac{1}{2})$ for some $\eta>0$.

Lemma 3.5. *If u is a nonnegative function in $C^2(0,1)$ satisfying*

$$-\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)'\geq 0\quad\textit{and}\quad u'\leq 0\quad\forall r\in(0,1),$$

then there exists a positive exponent $\eta=\eta_{N,p}$ such that the function $r\mapsto r^\eta u(r)$ is nondecreasing on $(0,\dfrac{1}{2})$.

Proof. Consider a nonnegative function u such that

$$-\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)' \geq 0 \quad \text{and} \quad u'(r) \leq 0 \quad \forall r \in (0,1) \tag{3.13}$$

and let $r \in (0, \dfrac{1}{2})$ and $s \in [r, 2r]$. Integrating (3.13) on $[r, s]$, we get

$$-s^{N-1}|u'(s)|^{p-2}u'(s) \geq -r^{N-1}|u'(r)|^{p-2}u'(r),$$

and since u' is nonpositive on $(0,1]$, we derive

$$-u'(s) \geq \left(r^{N-1}|u'(r)|^{p-1}\right)^{\frac{1}{p-1}} s^{\frac{1-N}{p-1}}$$

which we integrate once more on $[r, 2r]$ to obtain

$$u(r) - u(2r) \geq \left(\frac{p-N}{p-1}\right)^{-1} \left(r^{N-1}|u'(r)|^{p-1}\right)^{\frac{1}{p-1}} \left[(2r)^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}\right].$$

Thus

$$u(r) \geq \left(\frac{N-p}{p-1}\right)^{-1} \left[1 - 2^{\frac{p-N}{p-1}}\right] r|u'(r)|.$$

In this way we get

$$(r^\eta u(r))' \geq 0 \quad \forall r \in \left(0, \frac{1}{2}\right),$$

where $\eta = \left(\dfrac{N-p}{p-1}\right)\left[1 - 2^{\frac{p-N}{p-1}}\right]^{-1}$. Therefore, the proof is complete. □

Proof of Theorem 3.3. If the problem (2.1) has a solution (u, v) satisfying

$$\lim_{r \to 0^+} r^{\frac{N-p}{p-1}} u(r) < \infty,$$

then we have for all $r \in (0,1]$

$$u(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left(\lambda_1 + \int_0^s t^{N-1}\left(a(t)u^\alpha v^\beta + h_1(t)\right) dt\right)^{\frac{1}{p-1}} ds \tag{3.14}$$

$$v(r) = \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \int_s^1 t^{N-1}\left(b(t)u^\gamma v^\delta + h_2(t)\right) dt\right)^{\frac{1}{q-1}} ds, \tag{3.15}$$

where λ_1, λ_2 are nonnegative numbers. Since the function $r \mapsto a(r)$ does not vanish near zero and if we choose $0 < r < \dfrac{1}{4}$, then we obtain from (3.14) that

$$u(r) \geq K \int_r^{2r} s^{\frac{1-N}{p-1}} \left(\int_{\frac{s}{2}}^s t^{N-1} u^\alpha(t) v^\beta(t) dt\right)^{\frac{1}{p-1}} ds$$

with $K > 0$, and since v is nonincreasing, we get

$$
\begin{aligned}
u(r) &\geq K \int_r^{2r} s^{\frac{1-N}{p-1}} u^{\frac{\alpha}{p-1}}(s) v^{\frac{\beta}{p-1}}(s) \left(\int_{\frac{s}{2}}^{s} t^{N-1} dt \right)^{\frac{1}{p-1}} ds \\
&\geq K u^{\frac{\alpha}{p-1}}(2r) v^{\frac{\beta}{p-1}}(2r) \int_r^{2r} s^{\frac{1-N}{p-1}} \left(\int_{\frac{s}{2}}^{s} t^{N-1} dt \right)^{\frac{1}{p-1}} ds.
\end{aligned}
$$

Thus

$$
u(r) \geq K r^{\frac{p}{p-1}} u^{\frac{\alpha}{p-1}}(2r) v^{\frac{\beta}{p-1}}(2r). \tag{3.16}
$$

Multiplying (3.16) by $r^{\frac{N-p}{p-1}}$, we obtain

$$
r^{\frac{N-p}{p-1}} u(r) \geq K r^{(p-1)^{-1}\left(N - \frac{N-p}{p-1}\alpha\right)} \left(r^{\frac{N-p}{p-1}} u(2r) \right)^{\frac{\alpha}{p-1}} v^{\frac{\beta}{p-1}}(2r),
$$

and using Lemma 3.5, we get

$$
r^{\frac{N-p}{p-1}-\eta} \left[(2r)^{\eta} u(2r) \right] \geq K r^{(p-1)^{-1}\left(N - \frac{N-p}{p-1}\alpha\right)} \left(r^{\frac{N-p}{p-1}} u(2r) \right)^{\frac{\alpha}{p-1}} v^{\frac{\beta}{p-1}}(2r),
$$

where η is the positive number which has been constructed in Lemma 3.5. Thus

$$
2^{\eta} \left(r^{\frac{N-p}{p-1}} u(2r) \right)^{1-\frac{\alpha}{p-1}} \geq K r^{(p-1)^{-1}\left(N - \frac{N-p}{p-1}\alpha\right)} v^{\frac{\beta}{p-1}}(2r), \tag{3.17}
$$

where $K > 0$. Using $\alpha < p-1$ and (3.11), this becomes

$$
v(r) \leq K r^{\frac{1}{\beta}\left(\frac{N-p}{p-1}\alpha - N\right)}, \quad 0 < r < \frac{1}{2}. \tag{3.18}
$$

From another side, since the function $r \mapsto -r^{N-1}|u'|^{p-2}u'$ is nondecreasing and u' is nonpositive,

$$
u(r) \leq K_0 r^{\frac{p-N}{p-1}} \quad \forall r \in (0,1], \tag{3.19}
$$

with $K_0 = -u'(1)\dfrac{p-1}{N-p} > 0$. Now, we use the expression (3.15) of v and we apply the L'Hôpital rule to obtain

$$
\begin{aligned}
\lim_{r \to 0^+} r^{\frac{N-q}{q-1}} v(r) &= \left(\frac{N-q}{q-1} \right)^{-1} \lim_{r \to 0^+} \frac{-v'(r)}{r^{\frac{1-N}{q-1}}} \\
&\leq \left(\frac{N-q}{q-1} \right)^{-1} \left(\lambda_2 + \int_0^1 t^{N-1} \left(c u^{\gamma} v^{\delta} + h_2 \right) dt \right)^{\frac{1}{q-1}}.
\end{aligned} \tag{3.20}
$$

Consequently, by using inequalities (3.18) and (3.19) in (3.20), we have

$$\begin{aligned}\lim_{r\to 0^+} r^{\frac{N-q}{q-1}} v(r) &\le \left(\frac{N-q}{q-1}\right)^{-1} \left(\lambda_2' + cK\int_0^{\frac{1}{2}} t^{N-\frac{N-p}{p-1}\gamma-\left(N-\frac{N-p}{p-1}\alpha\right)\frac{\delta}{\beta}-1} dt + \frac{h_2}{N}\right)^{\frac{1}{q-1}} \\ &= \left(\frac{N-q}{q-1}\right)^{-1} \left(\lambda_2' + \frac{cK}{N-\frac{N-p}{p-1}\gamma-\left(N-\frac{N-p}{p-1}\alpha\right)\frac{\delta}{\beta}} + \frac{h_2}{N}\right)^{\frac{1}{q-1}} \\ &< \infty\end{aligned}$$

because (3.12) holds. Therefore, the solution (u, v) of the system (2.1) is fundamentally singular. $\square$

4 Nonexistence Results

In this section, we prove nonexistence of nonnegative nontrivial radially solutions to the system

$$\begin{cases} -\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}\left(a(r)u^\alpha v^\beta + h_1(r)\right), & \text{in } (0,1) \\ \left(r^{N-1}|v'|^{q-2}v'\right)' = r^{N-1}\left(b(r)u^\gamma v^\delta + h_2(r)\right), & \text{in } (0,1) \\ u(1) = v(1) = 0, \end{cases} \tag{4.1}$$

where a, b, h_1, h_2 are nonnegative continuous functions defined on $[0,1]$ in $\mathbb{R}$. The following is the main result of this section.

Theorem 4.1. *Consider that $r \mapsto a(r)$ does not vanish near zero, $\inf\limits_{r\in[0,1]} b(r) > 0$, and suppose that*

$$\sup_{r\in[0,1]} h_2(r) > 0, \tag{4.2}$$

$$\alpha < p-1 \quad \textit{and} \quad \frac{N-p}{N(p-1)}\alpha + \frac{N-q}{N(q-1)}\beta > 1. \tag{4.3}$$

Then the problem (4.1) *has no radial positive solutions such that*

$$\lim_{r\to 0^+} r^{\frac{N-p}{p-1}} u(r) < \infty. \tag{4.4}$$

Proof. Suppose that the system (4.1) has a solution (u, v) satisfying the condition (4.4). Then we obtain for all $r \in (0,1]$

$$u(r) = \int_r^1 s^{\frac{1-N}{p-1}} \left(\lambda_1 + \int_0^s t^{N-1}\left(a(t)u^\alpha v^\beta + h_1(t)\right) dt\right)^{\frac{1}{p-1}} ds \tag{4.5}$$

$$v(r) = \int_r^1 s^{\frac{1-N}{q-1}} \left(\lambda_2 + \int_s^1 t^{N-1}\left(b(t)u^\gamma v^\delta + h_2(t)\right) dt\right)^{\frac{1}{q-1}} ds, \tag{4.6}$$

where λ_1, λ_2 are nonnegative numbers. Since the function $r \mapsto a(r)$ does not vanish near zero, we obtain from (4.5)

$$u(r) \geq K \int_r^{2r} s^{\frac{1-N}{p-1}} \left(\int_{\frac{s}{2}}^{s} t^{N-1} u^{\alpha}(t) v^{\beta}(t) dt \right)^{\frac{1}{p-1}} ds$$

with $K > 0$. Since v is nonincreasing, we get

$$\begin{aligned} u(r) &\geq K \int_r^{2r} s^{\frac{1-N}{p-1}} u^{\frac{\alpha}{p-1}}(s) v^{\frac{\beta}{p-1}}(s) \left(\int_{\frac{s}{2}}^{s} t^{N-1} dt \right)^{\frac{1}{p-1}} ds \\ &\geq K u^{\frac{\alpha}{p-1}}(2r) v^{\frac{\beta}{p-1}}(2r) \int_r^{2r} s^{\frac{1-N}{p-1}} \left(\int_{\frac{s}{2}}^{s} t^{N-1} dt \right)^{\frac{1}{p-1}} ds \end{aligned}$$

and

$$u(r) \geq K r^{\frac{p}{p-1}} u^{\frac{\alpha}{p-1}}(2r) v^{\frac{\beta}{p-1}}(2r).$$

By multiplying the last inequality by $r^{\frac{N-p}{p-1}}$, we obtain

$$r^{\frac{N-p}{p-1}} u(r) \geq K r^{(p-1)^{-1}\left(N - \frac{N-p}{p-1}\alpha - \frac{N-q}{q-1}\beta\right)} \left(r^{\frac{N-p}{p-1}} u(2r) \right)^{\frac{\alpha}{p-1}} \left(r^{\frac{N-q}{q-1}} v(2r) \right)^{\frac{\beta}{p-1}},$$

and using Lemma 3.5, we get

$$r^{\frac{N-p}{p-1} - \eta} \left((2r)^{\eta} u(2r) \right) \geq K r^{(p-1)^{-1}\left(N - \frac{N-p}{p-1}\alpha - \frac{N-q}{q-1}\beta\right)} \left(r^{\frac{N-p}{p-1}} u(2r) \right)^{\frac{\alpha}{p-1}} \left(r^{\frac{N-q}{q-1}} v(2r) \right)^{\frac{\beta}{p-1}},$$

where η is the positive number which has been constructed in Lemma 3.5. Thus

$$2^{\eta} \left(r^{\frac{N-p}{p-1}} u(2r) \right)^{1 - \frac{\alpha}{p-1}} \geq K r^{(p-1)^{-1}\left(N - \frac{N-p}{p-1}\alpha - \frac{N-q}{q-1}\beta\right)} \left(r^{\frac{N-q}{q-1}} v(2r) \right)^{\frac{\beta}{p-1}} \tag{4.7}$$

with $K > 0$. Since (4.3) and (4.4) hold, we get from (4.7) that

$$\lim_{r \to 0^+} r^{\frac{N-q}{q-1}} v(r) = 0.$$

Now, we use expression (4.6) of v to apply the L'Hôpital rule

$$\begin{aligned} 0 &= \lim_{r \to 0^+} r^{\frac{N-q}{q-1}} v(r) \\ &= \left(\frac{N-q}{q-1} \right)^{-1} \lim_{r \to 0^+} \frac{-v'(r)}{r^{\frac{1-N}{q-1}}} \\ &= \left(\frac{N-q}{q-1} \right)^{-1} \left(\lambda_2 + \int_0^1 t^{N-1} \left(b(t) u^{\gamma} v^{\delta} + h_2(t) \right) dt \right)^{\frac{1}{q-1}} \end{aligned} \tag{4.8}$$

which contradicts the hypothesis (4.2). Therefore, the conclusion follows. □

Remark 4.2. If we suppose that $h_2 \equiv 0$, then we may deduce from (4.6) and (4.8) that $v \equiv 0$. So the expressions (4.5) and (4.8) imply that the system (4.1) has only two solutions: $(0,0)$ and $(u,0)$, where $u \geq 0$ on B', and thus the singularity is of fundamental type.

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