Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 3, Number 2, pp. 291–296 (2008)
http://campus.mst.edu/adsa

A Note on Retarded Ouyang Integral Inequalities

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Abstract

In this note, we generalize two retarded Ouyang integral inequalities. One of these inequalities says: under suitable assumptions of functions w, α, h, f, g and p on $[0, \infty)$, if

$$w^2(t) \leq h^2(t) + 2\int_0^{\alpha(t)} \Big\{ f(s)w(s)\Big[w(s) + \int_0^s g(r)w(r)dr\Big] + p(s)w(s)\Big\}ds,$$

then

$$w(t) \leq \Big[h(t) + \int_0^{\alpha(t)} p(s)ds\Big] \exp\Big\{\int_0^{\alpha(t)} \Big[f(s) + \Big(\int_0^s g(r)dr\Big)ds\Big]\Big\}, \quad t \geq 0.$$

AMS Subject Classifications: 26D15.
Keywords: Ouyang integral inequality, retarded.

1 Introduction and Preliminaries

This paper is inspired by a paper of Xu and Xia [2]. In [2], Xu and Xia established two retarded Ouyang integral inequalities. In this note, we generalize these two results to more general cases. For other related result, we refer to [1].

For convenience, we assume throughout this paper that the following conditions hold:

Received November 29, 2007; Accepted March 15, 2008
Communicated by Martin Bohner

(a) $w, f, h, g, p \in C([0,\infty),[0,\infty))$ with h increasing;

(b) $\alpha \in C^1([0,\infty),[0,\infty))$, $\alpha(t) \le t$ and $\alpha'(t) \ge 0$ on $[0,\infty)$.

In order to discuss our main results, we need the following two lemmas.

Lemma 1.1. *If*

$$w(t) \le h(t) + \int_0^{\alpha(t)} f(s)w(s)ds, \quad t \in [0,\infty), \tag{1.1}$$

then

$$w(t) \le h(t)\exp\int_0^{\alpha(t)} f(s)ds, \quad t \in [0,\infty). \tag{1.2}$$

Proof. For any $\varepsilon > 0$ and any fixed $T > 0$, it follows from (1.1) that

$$w(t) \le h(T) + \varepsilon + \int_0^{\alpha(t)} f(s)w(s)ds =: k(t), \quad 0 \le t \le T.$$

Clearly, $k(t)$ is increasing and

$$\begin{aligned} k'(t) &= \alpha'(t)f(\alpha(t))w(\alpha(t)) \\ &\le \alpha'(t)f(\alpha(t))k(\alpha(t)) \le \alpha'(t)f(\alpha(t))k(t), \end{aligned}$$

$0 \le t \le T$, which implies

$$\frac{k'(t)}{k(t)} \le \alpha'(t)f(\alpha(t)), \quad 0 \le t \le T.$$

Integrating it from 0 to $t \in [0,T]$, we obtain

$$\ln\frac{k(t)}{h(T)+\varepsilon} \le \int_0^t \alpha'(s)f(\alpha(s))ds = \int_0^{\alpha(t)} f(u)du,$$

which implies

$$k(t) \le [\varepsilon + h(T)]\exp\int_0^{\alpha(t)} f(u)du.$$

Taking $t = T$, we get

$$k(T) \le [\varepsilon + h(T)]\exp\int_0^{\alpha(T)} f(u)du.$$

Letting $\varepsilon \to 0^+$ and noting T arbitrary, we obtain the desired result (1.2). □

Lemma 1.2. *If*

$$w(t) \leq h(t) + \int_0^{\alpha(t)} f(s) \int_0^s g(r)w(r)drds, \quad t \geq 0, \tag{1.3}$$

then

$$w(t) \leq h(t) \exp \int_0^{\alpha(t)} f(s) \int_0^s g(r)drds, \quad t \geq 0. \tag{1.4}$$

Proof. For any $\varepsilon > 0$ and any fixed $T > 0$, it follows from (1.3) that

$$w(t) \leq h(T) + \varepsilon + \int_0^{\alpha(t)} f(s) \int_0^s g(r)w(r)drds =: k(t), \quad 0 \leq t \leq T.$$

Clearly, $k(t)$ is increasing and

$$\begin{aligned} k'(t) &= \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(r)w(r)dr \\ &\leq \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(r)k(r)dr \\ &\leq k(t)\alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(r)dr, \end{aligned}$$

$0 \leq t \leq T$, which implies

$$\frac{k'(t)}{k(t)} \leq \alpha'(t)f(\alpha(t)) \int_0^{\alpha(t)} g(r)dr, \quad 0 \leq t \leq T.$$

Integrating it from 0 to $t \in [0, T]$, we obtain

$$\ln \frac{k(t)}{k(0)} = \ln \frac{k(t)}{\varepsilon + h(T)} \leq \int_0^{\alpha(t)} f(s) \int_0^s g(r)drds.$$

Letting $t = T$, we get

$$k(T) \leq (\varepsilon + h(T)) \exp \int_0^{\alpha(T)} f(s) \int_0^s g(r)drds.$$

Letting $\varepsilon \to 0^+$ and noting $T > 0$ arbitrary, we obtain the desired result (1.4). □

2 Main Results

We now can state and prove our main results.

Theorem 2.1. *If*

$$w^2(t) \le h^2(t) + 2\int_0^{\alpha(t)} \Big[f(s)w(s)\int_0^s g(r)w(r)dr + p(s)w(s)\Big]ds, \tag{2.1}$$

then

$$w(t) \le \Big[h(t) + \int_0^{\alpha(t)} p(s)ds\Big] \exp\Big[\int_0^{\alpha(t)} f(s)\Big(\int_0^s g(r)dr\Big)ds\Big] \tag{2.2}$$

for $t \in [0,\infty)$.

Proof. For any $\varepsilon > 0$ and any fixed $T > 0$, it follows from (2.1) that, for $0 \le t \le T$,

$$w^2(t) \le h^2(t) + \varepsilon + 2\int_0^{\alpha(t)} \Big[f(s)w(s)\int_0^s g(r)w(r)dr + p(s)w(s)\Big]ds =: k(t).$$

Clearly, $w(t) \le \sqrt{k(t)}$, $k(t)$ is increasing and

$$\begin{aligned} k'(t) &= 2\alpha'(t)\Big[f(\alpha(t))w(\alpha(t))\int_0^{\alpha(t)} g(r)w(r)dr + p(\alpha(t))w(\alpha(t))\Big] \\ &\le 2\sqrt{k(t)}\alpha'(t)\Big[f(\alpha(t))\int_0^{\alpha(t)} g(r)w(r)dr + p(\alpha(t))\Big] \end{aligned}$$

for $0 \le t \le T$. This implies

$$\sqrt{k(t)} \le \sqrt{k(0)} + \int_0^{\alpha(t)} f(s)\int_0^s g(r)w(r)dr + \int_0^{\alpha(t)} p(s)ds.$$

This and Lemma 1.2 imply

$$\sqrt{k(t)} \le \Big[\sqrt{k(0)} + \int_0^{\alpha(t)} p(s)ds\Big] \exp\Big[\int_0^{\alpha(t)} f(s)\int_0^s g(r)drds\Big]$$

for $0 \le t \le T$. Letting $t = T$, we obtain

$$\sqrt{k(T)} \le \Big[\sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(T)} p(s)ds\Big] \exp\Big[\int_0^{\alpha(T)} f(s)\int_0^s g(r)drds\Big].$$

Letting $\varepsilon \to 0^+$ and noting $T > 0$ arbitrary, we obtain the desired result (2.2). $\square$

Taking $f(s) = 0$ or $g(s) = 0$ in (2.1), we have the following.

Corollary 2.2. *If*

$$w^2(t) \le h^2(t) + 2\int_0^{\alpha(t)} p(s)w(s)ds, \quad t \ge 0,$$

then

$$w(t) \le h(t) + \int_0^{\alpha(t)} p(s)ds, \quad t \ge 0.$$

Theorem 2.3. *If*

$$w^2(t) \le h^2(t) + 2\int_0^{\alpha(t)} \Big\{ f(s)w(s)\Big[w(s) + \int_0^s g(r)w(r)dr\Big] + p(s)w(s)\Big\}ds, \quad (2.3)$$

then

$$w(t) \le \Big[h(t) + \int_0^{\alpha(t)} p(s)ds\Big] \exp\Big\{ \int_0^{\alpha(t)} \Big[f(s) + \Big(\int_0^s g(r)dr\Big)ds\Big]\Big\}, \quad t \ge 0. \quad (2.4)$$

Proof. For any $\varepsilon > 0$ and any fixed $T > 0$, it follows from (2.3) that

$$w^2(t) \le h^2(T) + \varepsilon + 2\int_0^{\alpha(t)} \Big\{ f(s)w(s)\Big[w(s) + \int_0^s g(r)w(r)dr\Big] + p(s)w(s)\Big\}ds$$
$$=: k(t), \quad 0 \le t \le T.$$

Clearly, $k(t)$ is increasing, $k(t) > 0$ and $w(t) \le \sqrt{k(t)}$ on $[0, T]$. Differentiating $k(t)$ with respect to t and using $\alpha(t) \le t$, we obtain

$$\begin{aligned} k'(t) &= 2\alpha'(t)\Big\{f(\alpha(t))w(\alpha(t))\Big[w(\alpha(t)) + \int_0^{\alpha(t)} g(r)w(r)dr\Big] + p(\alpha(t))w(\alpha(t))\Big\} \\ &\le 2\sqrt{k(t)}\alpha'(t)\Big\{f(\alpha(t))\Big[w(\alpha(t)) + \int_0^{\alpha(t)} g(r)w(r)dr\Big] + p(\alpha(t))\Big\}, \end{aligned}$$

which implies

$$\begin{aligned} \sqrt{k(t)} &\le \sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} p(s)ds + \int_0^{\alpha(t)} f(s)\Big[w(s) + \int_0^s g(r)w(r)dr\Big]ds \\ &\le \sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} p(s)ds + \int_0^{\alpha(t)} \Big[f(s) + \int_0^s g(r)dr\Big]\sqrt{k(s)}ds, \end{aligned}$$

$0 \le t \le T$. This and Lemma 1.2 imply for $0 \le t \le T$,

$$w(t) \le \sqrt{k(t)} \le \Big[\sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} p(s)ds\Big] \exp\Big\{ \int_0^{\alpha(t)} \Big[f(s) + \int_0^s g(r)dr\Big]ds\Big\}.$$

Taking $t = T$, we get

$$w(T) \le \Big[\sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(T)} p(s)ds\Big] \exp\Big\{ \int_0^{\alpha(T)} \Big[f(s) + \int_0^s g(r)dr\Big]ds\Big\}.$$

Letting $\varepsilon \to 0^+$ and noting $T > 0$ arbitrary, we obtain the desired result (2.4). □

Taking $g(t) = 0$ in Theorem 2.3, we obtain the following.

Corollary 2.4. *If*

$$w^2(t) \leq h^2(t) + 2\int_0^{\alpha(t)} \Big[f(s)w^2(s) + p(s)w(s)\Big]ds, \quad t \geq 0,$$

then

$$w(t) \leq \left\{h(t) + \int_0^{\alpha(t)} p(s)ds\right\} \exp\left\{\int_0^{\alpha(t)} f(s)ds\right\}, \quad t \geq 0.$$

Remark 2.5. Taking $h(t) = c$ (constant) in Theorems 2.1 and 2.3, we obtain the results in [2].

References

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