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Nonhyperbolic Dynamics for Competitive Systems in the Plane and Global Period-doubling Bifurcations

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Abstract

We investigate the global period-doubling bifurcations of solutions of the equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

where the function f satisfies certain monotonicity conditions. We also obtain a global asymptotic result for competitive systems of difference equations in the plane in the nonhyperbolic case when the considered system has an infinite number of equilibrium points located along the graph of a nonincreasing function.

AMS Subject Classifications: 37E30, 37G99, 39A10, 39A11.
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1 Introduction and Preliminaries

The *period-two trichotomy* of difference equations was discovered in [3] in the study of the equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n} \quad n = 0, 1, \ldots\ , \tag{1.1}$$

where $p > 0$ and $x_{-1}, x_0 > 0$, and can be stated as the following result:

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Theorem 1.1. *The following period-two trichotomy result holds for Eq.* (1.1)

$p < 1 \;\Rightarrow\;$ *there exist unbounded solutions*
$p = 1 \;\Rightarrow\;$ *every solution converges to a period-two solution*
$p > 1 \;\Rightarrow\;$ *every solution converges to the equilibrium.*

Recently, this result, which is not global, has been improved in the sense that the statement "there exist unbounded solutions" was replaced by the statement that "every solution in the complement of the global stable manifold of the unique equilibrium is unbounded", see [18]. A similar phenomenon was discovered for some special cases of second-order linear fractional difference equations of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}, \quad n = 0, 1, \ldots \tag{1.2}$$

where the parameters and the initial conditions are nonnegative, in [8] and [9]. Precisely, it has been observed for some special cases of Eq. (1.2) that for the values on one side of the critical curve the positive equilibrium is a global attractor, while on the opposite side of the critical curve all solutions that start in the complement of the global stable manifold of the positive equilibrium are unbounded, while for the values on the critical curve there is an infinite number of period-two solutions and every solution converges to the period-two solution, see [3, 8, 9, 14].

Theorem 1.1, with the above mentioned globalization has a flavor of a bifurcation result. In this paper we will show that Theorem 1.1 is indeed a special case of a global bifurcation result, actually period-doubling bifurcation for the general difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{1.3}$$

where the function f satisfies certain monotonicity conditions.

Related nonlinear, second order, rational difference equations were investigated in numerous papers and in the monographs [12] and [13]. The study of these equations is quite challenging and is in rapid development. The only bifurcation results obtained for Eq. (1.3) are period-doubling bifurcation of Selgrade and Roberds [22] and Naimark–Sacker bifurcation [10, 16]. Both results are local as they guarantee the existence and stability of a bifurcating periodic solution in a neighborhood of the critical value(s) of the parameter(s). In particular, a period-doubling bifurcation which is discussed in this paper, typically occurs in a system of nonlinear difference equations when varying the parameter(s) causes an eigenvalue of the characteristic equation of a linearized equation evaluated at an equilibrium to pass through -1. In that case, typically the equilibrium loses stability and a stable cycle of period two appears. Continued parameter changes may result in a cascade of period-doubling bifurcations and the onset of chaos. Hale and Koçak [10] describe such cascades for the Henon map leading to the Henon strange attractor. See also [2] and [21]. In [22] a general bifurcation theorem which may be used to verify the occurrence of period-doubling and to determine the direction of bifurcation for any differentiable one parameter family of two-dimensional difference equations, has been established. This result is a local result and gives the existence and stability of prime period-two solution in some neighborhood of critical value of parameter.

In this paper we will obtain the global bifurcation result for Eq. (1.3) where f satisfies certain monotonicity conditions, which will guarantee the existence and stability of prime period-two solution at a critical value of the parameter. In order to achieve this goal we obtain some global asymptotic result for the competitive system of difference equations in the plane in the nonhyperbolic case when the considered system has an infinite number of the equilibrium points located along the graph of nonincreasing function. Our bifurcation results are motivated by the above mentioned period-two trichotomy results.

The rest of Section 1 introduces some preliminary results. Section 2 gives the conditions for the existence of unbounded solutions of Eq. (1.3). Section 3 presents some new results for nonhyperbolic dynamics of competitive discrete dynamical systems in a plane. Section 4 presents our major result on the global period-doubling bifurcation of Eq. (1.3).

Let I be an interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\bar{x} \in I$ be an equilibrium point of the difference equation (1.3), that is, $\bar{x} = f(\bar{x}, \bar{x})$.

Let

$$s = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) \quad \text{and} \quad t = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\bar{x}$ of Eq. (1.3). Then the equation

$$y_{n+1} = sy_n + ty_{n-1}, \quad n = 0, 1, \dots \tag{1.4}$$

is called the **linearized equation** associated with Eq. (1.3) about the equilibrium point $\bar{x}$.

Theorem 1.2. (Linearized Stability)

(a) If both roots of the quadratic equation

$$\lambda^2 - s\lambda - t = 0 \tag{1.5}$$

lie in the open unit disk $|\lambda| < 1$*, then the equilibrium* $\bar{x}$ *of Eq.* (1.3) *is locally asymptotically stable.*

(b) If at least one of the roots of Eq. (1.5) *has absolute value greater than one, then the equilibrium* $\bar{x}$ *of Eq.* (1.3) *is unstable.*

(c) A necessary and sufficient condition for both roots of Eq. (1.5) *to lie in the open unit disk* $|\lambda| < 1$*, is*

$$|s| < 1 - t < 2. \tag{1.6}$$

In this case the locally asymptotically stable equilibrium $\bar{x}$ *is also called a* **sink**.

(d) *A necessary and sufficient condition for both roots of Eq.* (1.5) *to have absolute value greater than one is*

$$|t| > 1 \quad \textit{and} \quad |s| < |1-t|\,.$$

In this case $\bar{x}$ is called a **repeller**.

(e) *A necessary and sufficient condition for one root of Eq.* (1.5) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$s^2 + 4t > 0 \quad \textit{and} \quad |s| > |1-t|\,.$$

In this case the unstable equilibrium $\bar{x}$ is called a **saddle point**.

Definition 1.3. ([2]) Let $\mathcal{T}$ be a map on $\mathbb{R}^2$ and let $\mathbf{p}$ be an equilibrium point or periodic point for $\mathcal{T}$. The orbit of a map $\mathcal{T}$ that starts at $\mathbf{x}_0$ is the set $\{\mathcal{T}^n(\mathbf{x}_0)\}_{n=0}^{\infty}$, where $\mathcal{T}^n$ is the n-th iterate of $\mathcal{T}$. The **basin of attraction** of $\mathbf{p}$, denoted as $\mathcal{B}_{\mathbf{p}}$, is the set of points $\mathbf{x} \in R^2$ such that $|\mathcal{T}^k(\mathbf{x}) - \mathcal{T}^k(\mathbf{p})| \to \mathbf{0}$, as $k \to \infty$, that is,

$$\mathcal{B}_{\mathbf{p}} = \{\mathbf{x} \in \mathbb{R}^2 : \ |\mathcal{T}^k(\mathbf{x}) - \mathcal{T}^k(\mathbf{p})| \to \mathbf{0}, \quad \text{as} \ \ k \to \infty\},$$

where $| \quad |$ denotes any norm in $\mathbb{R}^2$.

Definition 1.4. ([16]) Consider the difference equation

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n), \quad n = 0, 1, \ldots, \tag{1.7}$$

where $\mathbf{x}_n$ is in R^k and $\mathbf{f} : D \to D$ is continuous, where $D \subset R^k$. We call a nonconstant continuous function $I : R^k \to R$ an invariant for the system (1.7) if

$$I(\mathbf{x}_{n+1}) = I(\mathbf{f}(\mathbf{x}_n)) = I(\mathbf{x}_n), \quad \text{for every } n = 0, 1, \ldots \ .$$

The first result is an important characterization of the global behavior of solutions of Eq. (1.3) when f satisfies specific monotonicity conditions, which was established recently in [4, 5].

Theorem 1.5. *Consider Eq.* (1.3) *and assume that $f : I \times I \to I$, $I \subset R$ is a function which is decreasing in first variable and increasing in second variable. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.* (1.3) *the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd indexed terms of the solution do exactly one of the following:*

(i) *They are both monotonically increasing;*

(ii) *They are both monotonically decreasing;*

(iii) *Eventually (that is for $n \geq N$), one of them is monotonically increasing and the other is monotonically decreasing.*

An immediate consequence of Theorem 1.5 is that every bounded solution of (1.3), where f satisfies the monotonicity conditions of theorem, converges to either an equilibrium or period-two solution. In the case where all solutions of Eq. (1.3) are bounded the most important problem is to find the basins of attraction of the equilibrium and the period-two solutions. This problem was answered in a satisfactory way for some special cases of second-order linear fractional difference equation (1.2) in a series of papers [14, 17, 18]. In these papers it was also observed that there exist the critical values of the parameters involved which belong to the critical curve such that for the values on one side of the critical curve the positive equilibrium is a global attractor while on the opposite side of the critical curve all solutions are attracted to the pair of a period-two solutions with the exception of the global stable manifold of the positive equilibrium. In other words we have observed certain period-two bifurcation. This phenomenon was explained for Eq. (1.3) when f decreases in first and increases in second variable, see [15].

Next we present a result on the convergence to the equilibrium of Eq. (1.3).

Theorem 1.6. *Consider Eq.* (1.3) *and assume that f is a continuous function which is nonincreasing in first variable and nondecreasing in second variable. Assume there exist numbers $0 \le L < U$ such that*

$$f(U, L) \ge L \tag{1.8}$$

and

$$f(L, U) \le U \tag{1.9}$$

are satisfied. Then $[L, U]$ is an invariant interval for solutions of Eq. (1.3)*, that is, if $x_{-1}, x_0 \in [L, U]$, then $x_n, \in [L, U]$ for all $n \ge 0$. If, in addition, the only solution of the system*

$$f(m, M) = m \quad \textit{and} \quad f(M, m) = M, \tag{1.10}$$

is $m = M$, then every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1.3) *which eventually enters $[L, U]$, satisfies*

$$\lim_{n\to\infty} x_n = \overline{x},$$

where $\overline{x}$ is a unique equilibrium of Eq. (1.3) *in $[L, U]$.*

Proof. Take $x_{-1}, x_0 \in [L, U]$. By using the monotonicity of f and (1.8) and (1.9) we obtain

$$x_1 = f(x_0, x_{-1}) \le f(L, U) \le U$$

and

$$x_1 = f(x_0, x_{-1}) \ge f(U, L) \ge L.$$

An induction argument implies that $x_n \in [L, U]$ for all $n \ge 0$. Thus the function f satisfies all conditions of Theorem 1.4.5 in [13] which completes the proof of theorem. $\square$

2 Existence of Unbounded Solutions

In this section we present some results on the existence of unbounded solutions of Eq. (1.3).

Theorem 2.1. *Assume that $\overline{x}$ is the unique equilibrium of Eq.* (1.3). *Assume that*

$$f : I \times I \to I$$

is a continuous function which is nonincreasing in first variable and nondecreasing in second variable, where $I \subset R$ is an interval. Assume there exist numbers $L, U \in I$ such that $0 < L < \overline{x} < U$ such that

$$f(U, L) \leq L \tag{2.1}$$

and

$$f(L, U) \geq U \tag{2.2}$$

are satisfied, where at least one inequality is strict. If $x_{-1} \leq L$ and $x_0 \geq U$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_{2n-1} \leq L \quad \text{and} \quad x_{2n} \geq U \quad n = 0, 1, \ldots\ .$$

If, in addition, Eq. (1.3) *has no prime period-two solution then*

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Proof. Assume that $x_{-1} \leq L$ and $x_0 \geq L$. Then by using the monotonicity of f and conditions (2.1) and (2.2) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(x_0, L) \leq f(U, L) \leq L$$

and

$$x_2 = f(x_1, x_0) \geq f(x_1, U) \geq f(L, U) \geq U.$$

By using induction we complete the proof of first statement of the theorem. If, in addition, we assume that Eq. (1.3) has no prime period-two solution then both subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ must be either increasing or $\{x_{2n}\}_{n=0}^{\infty}$ is eventually an increasing sequence and $\{x_{2n-1}\}_{n=0}^{\infty}$ is eventually a decreasing sequence. Otherwise, in view of Theorem 1.5, we conclude that either $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are both decreasing or that $\{x_{2n}\}_{n=0}^{\infty}$ is eventually a decreasing sequence and $\{x_{2n-1}\}_{n=0}^{\infty}$ is eventually an increasing sequence. In this case both subsequences will have finite limits and the solution will converge to prime period-two solution, which is a contradiction.

Consequently,

$$\lim_{n\to\infty} x_{2n-1} = \ell \geq 0 \quad \text{and} \quad \lim_{n\to\infty} x_{2n} = P \leq \infty.$$

Clearly, $P = \infty$ because otherwise $\{x_n\}_{n=-1}^{\infty}$ will converge to prime period-two solution, which is a contradiction. $\square$

Now we present two applications of Theorems 1.6 and 2.1 to two special cases of Eq. (1.2) for which the period-two trichotomy was discovered.

Example 2.2. Consider Eq. (1.1) ([3, 13]) where $p > 0$ and $x_{-1}, x_0 > 0$. Conditions (2.1) and (2.2) become

$$f(U,L) = p + \frac{L}{U} \leq L \quad \text{and} \quad f(L,U) = p + \frac{U}{L} \geq U.$$

respectively. These conditions are satisfied if we choose $L = 1$ and $U = \dfrac{1}{1-p}$ where $p < 1$. In addition, as is well-known Eq. (1.1) has a prime period-two solution if and only if $p = 1$.

□

Example 2.3. Consider the equation ([8, 13])

$$x_{n+1} = \frac{p + qx_{n-1}}{1 + x_n} \quad n = 0, 1, \dots\ , \tag{2.3}$$

where $p, q > 0$ and $x_{-1}, x_0 \geq 0$. Conditions (2.1) and (2.2) become

$$f(U,L) = \frac{p + qL}{1 + U} \leq L \quad \text{and} \quad f(L,U) = \frac{p + qU}{1 + L} \geq U,$$

respectively. These conditions are satisfied if we choose $L = q-1$ and $U = q-1+\dfrac{p}{q-1}$ where $q > 1$. In addition, as is well-known Eq. (2.3) has a prime period-two solution if and only if $q = 1$.

□

Example 2.4. Consider Eq. (1.1) where $p > 1$. Conditions (1.8) and (1.9) become

$$f(U,L) = p + \frac{L}{U} \geq L \quad \text{and} \quad f(L,U) = p + \frac{U}{L} \leq U$$

respectively. These conditions are satisfied if we choose $L = p$ and $U \geq \dfrac{p^2}{p-1}$. System (1.10) is clearly satisfied and so every solution that starts in $[L, U]$ converges to the equilibrium $p + 1$. It has been also shown in [3] that this interval is also an attractive set which means that every solution enters this interval in finite number of steps. □

Example 2.5. Consider Eq. (2.3) where $q < 1$. Conditions (1.8) and (1.9) become

$$f(U,L) = \frac{p + qL}{1 + U} \geq L \quad \text{and} \quad f(L,U) = \frac{p + qU}{1 + L} \leq U$$

respectively. These conditions are satisfied for $L = 0$ and $U = \dfrac{p}{1-q}$. System (1.10) is clearly satisfied and so every solution that starts in $[L, U]$ converges to the equilibrium $\overline{y}$. It has been also shown in [8] that this interval is also an attractive set which means that every solution enters this interval in finite number of steps. □

Remark 2.6. Observe that the conditions (1.8) and (1.9) are almost opposite of conditions (2.1) and (2.2). The borderline case is

$$f(L,U)=U, \quad f(U,L)=L \tag{2.4}$$

which is equivalent to the existence of a prime period-two solution.

In fact, one can show that in the special case of Eq. (1.2), the conditions (1.8) and (1.9) are exactly opposite of the conditions (2.1) and (2.2) with the condition (2.4) as the borderline case. An interesting feature of Eq. (1.2) is that whenever (1.8) and (1.9) are separated from (2.1) and (2.2) with the condition (2.4), Eq. (1.2) possesses an infinite number of period-two solutions. In addition, all special cases of Eq. (1.2) with this property generate maps for which the second iterate is competitive. In the next section we will address the behavior of competitive systems in the plane with an infinite number of the equilibrium points.

Thus, we conclude that in the case of Eq. (1.2) the borderline case between the global attractivity of an equilibrium and the existence of unbounded solutions is the existence of period-two solution(s). This fact is illustrated by the well-known examples of equations (1.1) and (2.3), which exhibit the period-two trichotomy, see [3, 8, 14, 17].

3 Nonhyperbolic Dynamics of Competitive Discrete Dynamical Systems

In this section we present the attractivity results for a competitive discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n), \quad n = 0, 1, \dots \end{aligned} \tag{3.1}$$

where f and g are continuous functions and $f(x,y)$ is nondecreasing in x and nonincreasing in y and $g(x,y)$ is nonincreasing in x and nondecreasing in y in some domain A. Along with system (3.1) we consider the corresponding map $\mathcal{T}$ defined as

$$\mathcal{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

Here we give some basic notions about competitive maps in plane.

Competitive systems of the form (3.1) were studied by many authors such as Clark and Kulenović [6], Hirsch and Smith [11], Kulenović and Merino [17], Kulenović and Nurkanović [20], Smith [23–26] and others. All known results, with the exception of [6], are dealing with hyperbolic dynamics. The results presented here are results that hold in nonhyperbolic case.

We define a partial order $\preceq$ on $\mathbb{R}^2$ so that the positive cone is the fourth quadrant, i.e. this partial order is defined by:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Leftrightarrow \begin{cases} x^1 \leqslant x^2 \\ y^1 \geqslant y^2. \end{cases} \tag{3.2}$$

Two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ are called *related* if $\mathbf{x} \preceq \mathbf{y}$ or $\mathbf{y} \preceq \mathbf{x}$. Otherwise they are called *unrelated*. A *linearly ordered set* is a set where each two points are related.

A map $\mathcal{T}$ is called *competitive* if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow \mathcal{T}\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq \mathcal{T}\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{3.3}$$

A map $\mathcal{T}$ *strongly competitive* if $\mathcal{T}(x^1, y^1) - \mathcal{T}(x^2, y^2)$ is in the interior of the fourth quadrant whenever $(x^1, y^1) \preceq (x^2, y^2)$.

For each $\mathbf{v} = (v^1, v^2) \in \mathbb{R}^2_+$, define $\mathcal{Q}_i(\mathbf{v})$ for $i = 1, .., 4$ to be the usual four quadrants based at $\mathbf{v}$ and numbered in a counterclockwise direction, e.g., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2_+ : v^1 \leq x, v^2 \leq y\}$. For $S \subset \mathbb{R}^2_+$ let S° denote the interior of S. For standard definitions of attracting fixed point, saddle point, stable manifold, see [16] and [21].

Theorem 3.1. *Consider a competitive map $\mathcal{T} : \mathcal{R} \to \mathcal{R}, \mathcal{R} \subseteq \mathbb{R}^2$, where $\mathcal{R}$ has a nonempty interior. Assume that the equilibrium points of $\mathcal{T}$ form a linearly-ordered set $E = \{(h_1(t), h_2(t)) : t \in I \subset \mathbb{R}\}$, where h_1 and h_2 are continuous functions on the interval I. If E is a bounded set and B_E is the smallest rectangle that contains E, then B_E is an invariant and attracting set for $\mathcal{T}$. Then the trajectory of a point $x_0 \notin E$ consists of unrelated points.*

Proof. We will use the "triangulation principle" visualized in Figure 3.1. Take two points on E such that P_x is intersection of vertical line trough P and E, P_y is intersection of horizontal line through P and E. Then

$$P_y = \mathcal{T}(P_y) \preceq \mathcal{T}(P) \preceq \mathcal{T}(P_x) = P_x.$$

Continuing this process we obtain that

$$P_y \preceq \mathcal{T}^n(P) \preceq P_x, \quad \text{for all } n \geq 0.$$

Furthermore, we can prove that

$$\mathcal{T}(P) \leq_{SW} P,$$

where $A \leq_{SW} B$ means that the point A is southwest of point B. Indeed, since

$$P_y \preceq \mathcal{T}(P) \preceq P_x$$

Figure 3.1:

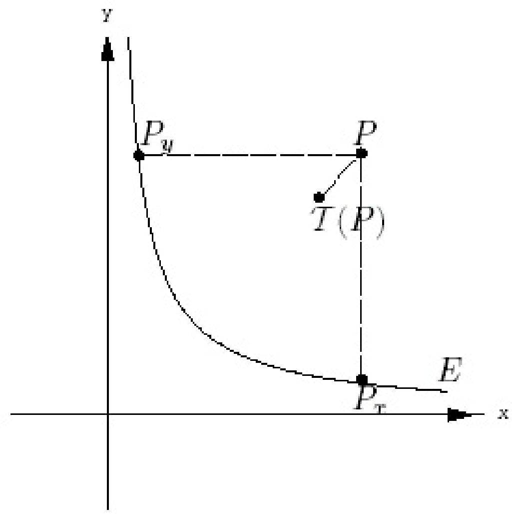
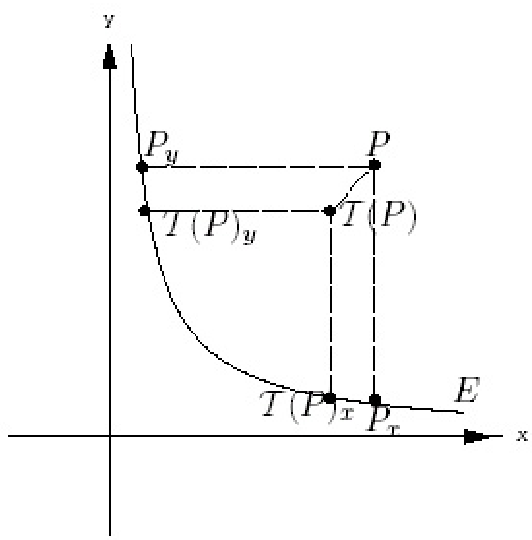


Figure 3.2:



we must have

$$P_y \le \mathcal{T}(P)_y \quad \text{and} \quad \mathcal{T}(P)_x \le P_x$$

with at least one of these inequalities strict. See Figure 3.2. Thus

$$[\mathcal{T}(P)_x, \mathcal{T}(P)_y] \subset [P_x, P_y],$$

where

$$[A, B] = \{\mathbf{x} \in R^2 : A \preceq \mathbf{x} \preceq B\}$$

denotes the interval between A and B with respect to the ordering $\preceq$. Geometrically, $[A, B]$ is a box with north-west vertex at A and south-east vertex at B.

Similarly, we can prove that $\mathcal{T}^2(P) \le_{SW} \mathcal{T}(P)$, and thus the orbit $\{\mathcal{T}^n(P)\}$ is linearly ordered set with $\le_{SW}$ and so is convergent to $P^* \in E$. $\square$

If we additionally assume that two regions separated by E are invariant, then we can get the stronger conclusion and define the map that assign to each point outside E its limiting value in E.

Corollary 3.2. *If we additionally assume that the regions $H_u = \{(x, y) : x \ge h_1(t), y \ge h_2(t)\} \setminus E$ and $H_l = \{(x, y) : x \le h_1(t), y \le h_2(t)\} \setminus E$ are invariant, then*

$$\lim_{k\to\infty} \mathcal{T}^k(P) = \mathcal{T}^*(P), \quad P \in \mathcal{R} \setminus E.$$

Thus, we define a map

$$\mathcal{T}^* : \mathcal{R} \setminus E \to E$$

where

$$\mathcal{T}^*(P) = \lim_{k\to\infty} \mathcal{T}^k(P).$$

Remark 3.3. Denote by H_p the following set

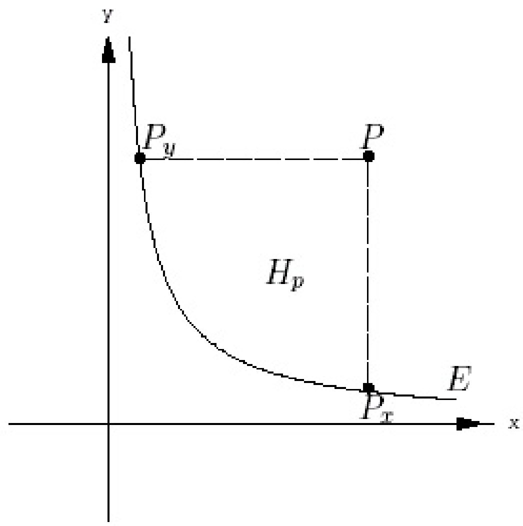
$$H_p = H_u \cap [P_y, P_x].$$

See Figure 3.3. Then the proof of Theorem 3.1 can be visualized as follows: the sequence $H_{\mathcal{T}^n(P)}$ is a sequence of nested compact sets which diameter approaches zero. Thus by principle of nested sets the limiting set is a point.

Theorem 3.4. *Consider a strongly competitive continuous map $\mathcal{T} : \mathcal{R} \to \mathcal{R}, \mathcal{R} \subseteq \mathbb{R}^2$, where $\mathcal{R}$ has a nonempty interior. Assume that H_u and H_l are invariant. Then the limit $\mathcal{T}^*$ is a continuous function of initial point $P \in \mathcal{R} \setminus E$.*

Proof. If $\mathcal{T}^*$ is not continuous at $P \in \mathcal{R} \setminus E$, then there exists a sequence of points $\{P_k\}$ that converges to P such that $\{\mathcal{T}^*(P_k)\}$ does not converge to $\mathcal{T}^*(P)$. By passing to a subsequence if necessary, we may assume that (a) $\{P_k\}$ is a subset of one of the four quadrants $\mathcal{Q}_\ell(P)$, $\ell = 1, \dots, 4$, and (b) $\{P_k\}$ is monotone with respect to $\preceq$ if $\{P_k\} \subset \mathcal{Q}_2 \cup \mathcal{Q}_4$ or $\{P_k\}$ is monotone with respect to $\le_{SW}$ if $\{P_k\} \subset \mathcal{Q}_1 \cup \mathcal{Q}_3$.

Figure 3.3:



Consider first the case when $\{P_k\} \subset \mathcal{Q}_2(P)$ and $\{P_k\}$ is monotone with respect to $\preceq$. The sequence $\{\mathcal{T}(P_k)\}$ converges to a point Q since it is bounded above by $\mathcal{T}^*(P)$, where $Q \preceq \mathcal{T}^*(P)$. In view of Theorem 3.1 and [19] there exists an invariant manifold $\mathcal{W}(Q)$ that is an increasing curve that starts at the point Q, and which for $k = 1, 2, \ldots$ is located between the invariant manifold $\mathcal{W}(\mathcal{T}^*(P_k))$ and $\mathcal{W}(\mathcal{T}^*(P))$. In particular, P_k is above $\mathcal{W}(Q)$ for $k = 1, 2, \ldots$ and P is below $\mathcal{W}(Q)$. Since $P_k \to P$, we conclude that $P \in \mathcal{W}(Q)$, which is impossible.

The case when $\{P_k\} \subset \mathcal{Q}_4(P)$ has a similar proof.

If now $\{P_k\} \subset \mathcal{Q}_1(P)$, consider the sequences $\{P_k^{(1)}\}$ and $\{P_k^{(2)}\}$ obtained by projecting the points of the sequence $\{P_k\}$ onto the horizontal and vertical lines through P. Clearly $\{P_k^{(1)}\}$ and $\{P_k^{(2)}\}$ converge to P, and by the first part of the proof, we know $\mathcal{T}^*(P_k^{(\ell)}) \to \mathcal{T}^*(P)$ as $k \to \infty$ for $\ell = 1, 2$. We also have that $T^j(P_k^{(2)}) \preceq T^j(P) \preceq T^j(P_k^{(1)})$. By taking limit as $k \to \infty$ we get $\mathcal{T}^j(P_k) \to \mathcal{T}^*(P)$. $\square$

Remark 3.5. Now we will prove the existence of a continuous invariant for a competitive map $\mathcal{T}$. We will use the technique of Clark, Thomas, and Wilken from [27]. The key fact in the application of this result to Eq. (1.3) is that the second iterate

$$\mathcal{T}^2(u, v) = (f(v, u),\ f(f(v, u), v))\,.$$

of the map

$$\mathcal{T}(u, v) = (v, f(v, u)) \tag{3.4}$$

that corresponds to Eq. (1.3) is a competitive map, see [18].

Theorem 3.6. *Suppose that $\mathcal{T}$ is a continuous map with domain $\mathcal{D} \subset \mathbb{R}^2$ such that $\mathcal{T}^2$ is a strongly competitive map. Assume that for each $P \in \mathcal{D}$, the orbit $\{(\mathcal{T}^2)^k(P)\}$ converges. Set*

$$P^* = \lim_{k\to\infty} (\mathcal{T}^2)^k(P)$$

and let $S = \{P^* : P \in \mathcal{D}\}$ be the set of all limiting points. Finally, suppose that the map $\mathcal{T}^* : \mathcal{D} \to S$ defined by $\mathcal{T}^*(P) = P^*$ is continuous.

Then each pair $\{P^*, T(P^*)\}$ constitutes a period-two solution for $\mathcal{T}$. Furthermore, if i is any continuous map from S to $\mathbb{R}$ which is constant on period-two solutions, then i extends to a continuous invariant I for both $\mathcal{T}^2$ and $\mathcal{T}$ defined simply by $I(P) := i(P)$.

Proof. The map $I = i \circ \mathcal{T}^*$ will be a continuous invariant for $\mathcal{T}^2$ and $\mathcal{T}$ provided it is continuous and constant on orbits of $\mathcal{T}$. Being the composition of continuous maps, I certainly satisfies the first requirement. To see that I is constant on orbits of $\mathcal{T}$, take any $P \in \mathcal{D}$ and any $P' \in O_{\mathcal{T}}$ (the orbit of $\mathcal{T}$). There is a positive integer m such that $P' = \mathcal{T}^m(P)$. Thus, we may express $I(P')$ as

$$I(P') = I(\mathcal{T}^m(P)) = (i \circ \mathcal{T}^*)(\mathcal{T}^m(P)) = i(\lim_{k\to\infty} \mathcal{T}^{2k+m}(P)).$$

Depending on the parity of m, $\lim\limits_{k\to\infty} \mathcal{T}^{2k+m}(P)$ is equal to either P^* (m even) or $\mathcal{T}(P^*)$ (m odd). Finally, $i(P^*) = i(\mathcal{T}(P^*))$ since i is constant on period-two solutions by hypothesis. Thus I is constant on orbits of $\mathcal{T}$, and consequently, on orbits of $\mathcal{T}^2$ since $O_{\mathcal{T}}(P) \supset O_{\mathcal{T}^2}(P)$. □

An application of Theorems 3.4 and 3.6 to Eq. (1.3) where f satisfies the monotonicity conditions of Theorem 1.6 leads to the following result.

Corollary 3.7. *Consider Eq.* (1.3) *and assume that f is a continuous function which is nonincreasing in first variable and nondecreasing in second variable. Assume that conditions of Theorem 3.6 are satisfied for the associated map* (3.4). *If Eq.* (1.3) *possesses an infinite number of period-two solutions which belong to the graph of a continuous nonincreasing function C, then every solution of Eq.* (1.3) *belongs to a continuous invariant curve and converges to the limiting period-two solution $P((x_{-1}, x_0)) = (\Phi((x_{-1}, x_0)), \Psi((x_{-1}, x_0)))$. In addition, the limiting period-two solution $P((x_{-1}, x_0))$ depends continuously on the initial point (x_{-1}, x_0).*

The next two examples illustrates Theorems 3.4 and 3.6.

Example 3.8. The following system was considered in [6]:

$$\begin{aligned} x_{n+1} &= \frac{x_n}{1+y_n} \\ y_{n+1} &= \frac{y_n}{1+x_n}, \quad n = 0, 1, \ldots \end{aligned} \tag{3.5}$$

where $x_0, y_0 \geq 0$. System (3.5) has an infinite number of equilibrium points. In fact, both positive semiaxes consist of equilibrium points which are all nonhyperbolic. Clearly, $x_{n+1} \leq x_n, y_{n+1} \leq y_n$ for every n and so both $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are convergent to some equilibrium point on one of the axis. As we have shown in [6] system (3.5), has two functionally dependent invariants:

$$I(x_n, y_n) = \frac{y_n + 1}{x_n + 1} \quad \text{and} \quad J(x_n, y_n) = y_n - I(x_0, y_0)x_n.$$

The invariant $J(x_n, y_n)$ is actually a line that intersects an axis at an equilibrium point. Indeed, $J(x_n, y_n) = J(x_0, y_0)$ implies that

$$y_n - y_0 = I(x_0, y_0)(x_n - x_0)$$

which is an equation of the line through (x_0, y_0). So in this case we have an explicit expression for the invariant whose existence was proved in Theorem 3.6. Furthermore, Theorem 3.4 implies that the limits of (x_n, y_n) are continuous functions of initial point (x_0, y_0). An additional feature of system (3.5) is the existence of explicit solution. Indeed by using the discrete Riccati equation, the explicit solution of (3.5) was found in [6] to be:

$$\begin{aligned} x_n &= \frac{1}{1/(A-1) + (1/x_0 - 1/(A-1))1/A^n} \quad \text{if} \quad A \neq 1 \\ y_n &= \frac{1}{n + 1/x_0}, \quad \text{if} \quad A = 1, \end{aligned} \tag{3.6}$$

where $A = 1/I(x_0, y_0) = \dfrac{x_0 + 1}{y_0 + 1}$. Thus, if $A < 1$, which is equivalent to $x_0 < y_0$, then $(x_n, y_n) \to (0, (y_0 - x_0)/(y_0 + 1))$ as $n \to \infty$. If $A > 1$, which is equivalent to $x_0 > y_0$, then $(x_n, y_n) \to ((x_0 - y_0)/(y_0 + 1), 0)$ as $n \to \infty$. Finally, if $A = 1$, which is equivalent to $x_0 = y_0$, then $(x_n, y_n) \to (0, 0)$ as $n \to \infty$. $\square$

Example 3.9. Consider the system of difference equations

$$x_{n+1} = \frac{b_1 x_n}{1 + x_n + c_1 y_n}, \quad y_{n+1} = \frac{b_2 y_n}{1 + c_2 x_n + y_n}, \quad n = 0, 1, \ldots \tag{3.7}$$

where the parameters b_1, b_2, c_1, and c_2 are positive real numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

Global behavior of (3.7) was considered in [7], and global behavior of related systems was considered in [20]. Here we give an application of Theorems 3.4, and 3.6. System (3.7) has always an equilibrium point $\mathbf{e}_0 = (0, 0)$ and it also has additional equilibrium points for values of parameters for which two lines $\ell_1 := x + c_1 y = b_1 - 1$ and $\ell_2 := c_2 x + y = b_2 - 1$ intersect each other or the boundary of the first quadrant.

The exceptional case when these lines coincide has not been addressed in the literature. Consider the case when ℓ_1 and ℓ_2 coincide, which is equivalent to the condition

$$c_1 c_2 = 1, \quad c_1 = \frac{b_1 - 1}{b_2 - 1}. \tag{3.8}$$

Under condition (3.8), system (3.7) has an infinite number of equilibrium points which are exactly all the points on the segment of the line ℓ_1 which belong to the closure of the first quadrant together with the origin. Furthermore, it can be shown that the square $[0, b_1 - 1] \times [0, b_2 - 1]$ is invariant and attracting set for the map generated by the system (3.7). When $b_1 > 1, b_2 > 1$ the zero equilibrium is a repeller and all conditions of Theorems 3.4, and 3.6 are satisfied and so every solution of (3.8) converges to its corresponding equilibrium point along the corresponding invariant set consisting of unrelated points. $\square$

Remark 3.10. Theorems 3.1, 3.4, and 3.6 can be applied in a similar fashion to systems of the form

$$\begin{aligned} x_{n+1} &= \frac{x_n}{1 + d_1(x_n, y_n)} \\ y_{n+1} &= \frac{y_n}{1 + d_2(x_n, y_n)}, \quad n = 0, 1, \ldots \end{aligned} \tag{3.9}$$

where the initial conditions are nonnegative numbers and the functions $d_i(u, v), i = 1, 2$ are defined for $u, v \geq 0$ and satisfy the following conditions:

$$d_1(u, v) \geq 0, d_1(u, v) = 0 \Longrightarrow v = 0,$$

$$d_1(u, v) \text{ is increasing in } v, \quad \frac{u}{1 + d_1(u, v)} \text{ is increasing in } u$$

and

$$d_2(u, v) \geq 0, d_2(u, v) = 0 \Longrightarrow u = 0,$$

$$d_2(u, v) \text{ is increasing in } u, \quad \frac{v}{1 + d_2(u, v)} \text{ is increasing in } v.$$

Another example where Theorems 3.4, and 3.6 apply is Eq. (1.1), with $p = 1$, which is discussed in great detail in [27]. Actually, Theorems 3.4, and 3.6 are motivated by results in [27]. Similar, application of Theorems 3.4, and 3.6 is Eq. (2.3) discussed in Example 2.3.

Example 3.11. Consider Eq. (2.3). Set

$$u_n = x_{n-1} \quad \text{and} \quad v_n = x_n \quad \text{for } n = 0, 1, \ldots$$

and write Eq. (2.3) as the first order system

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{p + q u_n}{1 + v_n}\,,\ n = 0, 1, \ldots. \end{aligned}$$

Let T be the function on $[0,\infty)\times[0,\infty)$ defined by:

$$T(u,v)=\left(v,\frac{p+qu}{v+1}\right).$$

The second iterate of T is given by

$$T^2\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\dfrac{p+qu}{1+v}\\[2ex]\dfrac{(p+qv)(1+v)}{1+p+qu+v}\end{pmatrix}.$$

Clearly, T^2 is a competitive map. The period-two solutions of Eq. (2.3) are the fixed points of T^2. By straightforward checking we can see that the fixed points of T^2 which are not fixed points of T exist if and only if $q=1$, in which case there exist an infinite number of fixed points that belong to the hyperbola $uv=p$ in the first quadrant. In this case all conditions of Theorems 3.1, 3.4, and 3.6 are satisfied and the conclusions of these results apply. This means that the prime period-two solutions of Eq. (2.3), which belong to the hyperbola $xy=p$ in the first quadrant, attract all initial points (x_{-1},x_0) in the plane of initial conditions, along some continuous invariant curves. In addition, the period-two solution $(\Phi((x_{-1},x_0)),\Psi((x_{-1},x_0)))$ depends continuously on the initial point (x_{-1},x_0). The same conclusion holds for Eq. (1.1), with $p=1$ and was obtained in [27]. In addition, Wilken, Thomas, and Clark showed in [28] that the invariants for Eq. (1.1), with $p=1$ can not be rational functions. □

Example 3.12. Consider the equation

$$x_{n+1}=\frac{px_{n-1}}{1+x_n+x_{n-1}},\quad n=0,1,\ldots \tag{3.10}$$

where $p>1$ and $x_{-1},x_0\geq 0$, which was studied in [13]. Eq. (3.10) has the zero equilibrium and the positive equilibrium $(p-1)/2$ as well as an infinite number of period-two solutions $\ldots,\Phi,\Psi,\Phi,\Psi,\ldots$ which satisfy $\Phi+\Psi=1$. Set

$$u_n=x_{n-1}\quad\text{and}\quad v_n=x_n\quad\text{for } n=0,1,\ldots$$

and write Eq. (3.10) as the first order system

$$u_{n+1}=v_n$$

$$v_{n+1}=\frac{pu_n}{1+u_n+v_n}\ ,\ n=0,1,\ldots.$$

Let T be the function on $[0,\infty)\times[0,\infty)$ defined by:

$$T(u,v)=\left(v,\frac{pu}{1+u+v}\right).$$

The second iterate of T is given by

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{pu}{1+u+v} \\ \\ \dfrac{(pv)(1+u+v)}{(1+u+v)(1+v)+pu} \end{pmatrix}.$$

Straightforward checking shows that T^2 is a competitive map. The period-two solutions of Eq. (3.10) are the fixed points of T^2. The fixed points of T^2 which are not fixed points of T exist if and only if $p > 1$, in which case there exist an infinite number of fixed points on the segment of line $u + v = p - 1$ that belongs to the closure of the first quadrant. Furthermore, an immediate checking shows that the square $[0, p-1]^2$ is invariant and attracting set for the map T. In this case all conditions of Theorems 3.1, 3.4, and 3.6 are satisfied and the conclusions of these theorems hold. This means that the prime period-two solutions of Eq. (3.10), which belong to the segment of the line $x + y = p - 1$ in the closure of the first quadrant, attract all initial points (x_{-1}, x_0) in the plane of initial conditions, along some continuous invariant curves. In addition, the period-two solution $(\Phi((x_{-1}, x_0)), \Psi((x_{-1}, x_0)))$ depends continuously on the initial point (x_{-1}, x_0). $\square$

Remark 3.13. The conclusion of Example 3.11 can be extended to the following equation

$$y_{n+1} = \frac{p + qy_n + ry_{n-1}}{1 + y_n}, \quad n = 0, 1, \dots, \tag{3.11}$$

where $p, q,$ and r are positive, and $r = q+1$. Related results about the global attractivity of period-two solutions of Eq. (1.3) based on hard analysis were obtained in [1]. These results are not applicable to competitive discrete dynamical systems but give the same conclusion in the case of three equations that exhibit period-two trichotomy (1.1), (2.3), and (3.11). The results from [1] however does not provide additional information about invariants and continuous dependence on initial conditions of the attracting period-two solutions of equations (1.1), (2.3), and (3.11). Furthermore, the global dynamics of equations (1.1), (2.3), and (3.11) in the case of existence of unbounded solutions, was described in [18].

4 Global Bifurcation Result

By combining Theorem 1.5 and Remark 3.5 with some results from [17] we have the following global period-doubling bifurcation result.

Theorem 4.1. *Let $\mathcal{I} = [a, \infty)$, and let $\mathcal{A}$ be a connected subset of $\mathbb{R}^2$. Given a family of difference equations*

$$x_{n+1} = f_\alpha(x_n, x_{n-1}), \quad x_{-1}, x_0 \in \mathcal{I}, \quad n = 0, 1, \dots \tag{4.1}$$

with $f_\alpha(x, y)$ continuous on $\mathcal{I} \times \mathcal{I}$, suppose that for each $\alpha \in \mathcal{A}$,

a_1. *$f_\alpha(x,y)$ is strictly decreasing in x and strictly increasing in y in the interior of $\mathcal{I}\times\mathcal{I}$.*

a_2. *$f_\alpha(x,y)$ is smooth in α and (x,y)*

a_3. *There is an interior equilibrium $\overline{x}_\alpha$ which varies continuously in α.*

a_4. *Let η_α and ν_α be the roots of the characteristic equation*

$$\lambda^2 - D_1 f_\alpha(\bar{x}_\alpha,\bar{x}_\alpha)\lambda - D_2 f_\alpha(\bar{x}_\alpha,\bar{x}_\alpha) = 0, \tag{4.2}$$

of (4.1) *at $\overline{x}_\alpha$, ordered as in $|\eta_\alpha| \leq |\nu_\alpha|$. There exists a continuous function $\Gamma : \mathcal{A} \to \mathbb{R}$ such that*

i. If $\Gamma(\alpha) < 0$, then $-1 < \nu_\alpha < 0 < \eta_\alpha < 1$.
ii. If $\Gamma(\alpha) = 0$, then $-1 = \nu_\alpha < 0 < \eta_\alpha < 1$.
iii. If $\Gamma(\alpha) > 0$, then $\nu_\alpha < -1 < 0 < \eta_\alpha < 1$.

a_5. *$\overline{x}_\alpha$ is the unique interior equilibrium of* (4.1)

a_6. *For α in the parametric region $\{\alpha : \Gamma(\alpha) < 0\} \cup \{\alpha : \Gamma(\alpha) > 0\}$ there are no prime period-two solutions. There exists a prime period-two solution for α in the parametric region $\{\alpha : \Gamma(\alpha) = 0\}$.*

a_7. *For α in the parametric region $\{\alpha : \Gamma(\alpha) < 0\}$ all solutions of Eq.* (4.1) *are bounded.*

Then the equilibrium $\overline{x}_\alpha$ is globally asymptotically stable for α in the parametric region $\{\alpha : \Gamma(\alpha) < 0\}$. For α in the parametric region $\{\alpha : \Gamma(\alpha) = 0\}$, every solution of Eq. (4.1) *converges to period-two solution (not necessarily prime) in the sense of Corollary 3.7. For α in the parametric region $\{\alpha : \Gamma(\alpha) > 0\}$, every solution of Eq.* (4.1) *is unbounded except for the solutions that belong to the closure of the global stable manifold of the equilibrium. The global stable manifold of the equilibrium is a curve which is the graph of a continuous and increasing function.*

Proof. The proof of Theorem follows from Theorem 1.5, Remark 3.5 and the result for competitive maps in a plane established in [17] and used in [14] in all special case of Eq. (1.2) that allow the existence of unbounded solutions. The new feature in this theorem is the convergence to period-two solution described in Corollary 3.7. □

Example 4.2. Combining Examples 2.2 and 2.4 with Theorem 4.1 we obtain the following global bifurcation result for Eq. (1.1):

$p < 1 \;\Rightarrow$ every solution off the global stable manifold of the equilibrium is unbounded

$p = 1 \;\Rightarrow$ every solution converges to a period-two solution

$p > 1 \;\Rightarrow$ every solution converges to the equilibrium.

Convergence to the period-two solution in Theorem 4.1 and for $p = 1$ in Example 4.2 is in the sense of Corollary 3.7. In this case $\Gamma(p) = 1-p$. A necessary and sufficient condition for the existence of a period-two solution is $p = 1$. This result was obtained in [3] and represents the first period-two trichotomy result. $\square$

Example 4.3. Combining Examples 2.3, 2.5, Theorem 4.1, and Corollary 3.7 we obtain the following global bifurcation result for Eq. (2.3):

$q < 1 \Rightarrow$ every solution off the global stable manifold of the equilibrium is unbounded

$q = 1 \Rightarrow$ every solution converges to a period-two solution

$q > 1 \Rightarrow$ every solution converges to the equilibrium.

Convergence to the period-two solution for $q = 1$ is in the sense of Corollary 3.7. In this case $\Gamma(q) = 1 - q$. A necessary and sufficient condition for the existence of a period-two solution of Eq. (2.3) is $q = 1$. This result was obtained in [8]. $\square$

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