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Motivation for Introducing q-Complex Numbers

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Abstract

We give a conceptual introduction to the q-umbral calculus of the author, a motivation for the q-complex numbers and a historical background to the development of formal computations, umbral calculus and calculus. The connection to trigonometry, prosthaphaeresis, logarithms and Cardan's formula followed by the notational inventions by Viéte and Descartes is briefly described. One other keypoint is the presumed precursor of the q-integral used by Euclid, Archimedes, (Pascal) and Fermat. This q-integral from q-calculus is also seen in time scales.

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1. Introduction

In another paper [22] the author has introduced so-called q-complex numbers. Since this subject is somewhat involved, it should be appropriate to give a historical background and motivation. This should also include a historical introduction to the logarithmic q-umbral calculus of the author, which follows here. In Section 2 we outline the early history of logarithms and its affinity to trigonometric functions, and prosthaphaeresis (the predecessor of logarithms). The early imaginary numbers arising from the Cardan *casus irreducibilis* are introduced. The story continues with the Viète introduction of mathematical variables, and the logarithms of Napier, Bürgi, and Kepler. In Section 3 the development of calculus from Fermat, Pascal and Descartes to Newton, Barrow, Wallis and Leibniz is treated. The early Fermat q-integral is also mentioned. The formal

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computations and its evolution from the Hindenburg combinatorial school as well as the development of abstract algebra and umbral calculus in England after 1830 are treated in Section 4. These formal computations are important in the q-umbral calculus of the author, since often formal results are obtained for functions of n complex variables. The validity of such a formula will then sometimes be restricted to a subset of $\mathbb{C}^n$. Four characteristics of this q-umbral calculus are the logarithmic behaviour, the formal power series, the formal computations, and the frequent use of the infinity symbol.

In Section 5 a brief survey of arguments for and against discrete and continuous calculus, with a lot of physics is given. In the final section, some famous struggles relevant to the text are recited.

We start with a brief survey of the author's q-umbral calculus [19–21].

Definition 1.1. The power function is defined by $q^a \equiv e^{a \log(q)}$. We always use the principal branch of the logarithm. The variables

$$a, b, c, a_1, a_2, \dots, b_1, b_2, \dots \in \mathbb{C}$$

denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q}, \; q \in \mathbb{C}\setminus\{1\}, \tag{1.1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q, \; \{0\}_q! \equiv 1, \; q \in \mathbb{C}, \tag{1.2}$$

Let the q-shifted factorial be given by

$$\langle a; q\rangle_n = \begin{cases} 1, & n = 0; \\ \displaystyle\prod_{m=0}^{n-1}(1-q^{a+m}) & n = 1, 2, \dots. \end{cases} \tag{1.3}$$

Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1; q\rangle_n}{\langle 1; q\rangle_k \langle 1; q\rangle_{n-k}}, \quad k = 0, 1, \dots, n. \tag{1.4}$$

The q-multinomial coefficient is defined by

$$\binom{n}{\vec{k}}_q \equiv \frac{\langle 1; q\rangle_n}{\prod_{i=1}^{\infty}\langle 1; q\rangle_{k_i}}, \quad \sum_{i=1}^{\infty} k_i = n. \tag{1.5}$$

Definition 1.2. The Nalli–Ward–AlSalam q-addition (NWA), compare [3, p. 240], [46, p. 345], [50, p. 256] is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \quad n = 0, 1, 2, \ldots. \tag{1.6}$$

Furthermore, we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k}, \quad n = 0, 1, 2, \ldots. \tag{1.7}$$

Definition 1.3. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod $\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. The operator

$$\widetilde{\ }: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{1.8}$$

Furthermore we define

$$\widetilde{\langle a; q\rangle_n} \equiv \langle \widetilde{a}; q\rangle_n. \tag{1.9}$$

The following simple rules follow from (1.8). Clearly the first two equations are applicable to q-exponents. Compare [48, p. 110].

$$\widetilde{a} \pm b \equiv \widetilde{a \pm b} \bmod \frac{2\pi i}{\log q}, \tag{1.10}$$

$$\widetilde{a} \pm \widetilde{b} \equiv a \pm b \bmod \frac{2\pi i}{\log q}, \tag{1.11}$$

$$q^{\widetilde{a}} = -q^a, \tag{1.12}$$

where the second equation is a consequence of the fact that we work mod $\frac{2\pi i}{\log q}$.

If we want to be formal, we could introduce a symbol ∞_{H}, with property

$$\langle \infty_{\mathrm{H}}; q\rangle_n = \langle \infty_{\mathrm{H}} + \alpha; q\rangle_n = 1, \ \alpha \in \mathbb{C}, \ 0 < |q| < 1. \tag{1.13}$$

The symbol ∞_{H} corresponds to parameter 0 in [25, p. 4]. We will denote ∞_{H} by ∞ in the rest of the paper.

Definition 1.4. We shall define a q-hypergeometric series by (compare [25, p. 4], [29, p. 345]):

$$
\begin{aligned}
{}_p\phi_r(\hat{a}_1,\ldots,\hat{a}_p;\hat{b}_1,\ldots,\hat{b}_r|q,z) &\equiv {}_p\phi_r\left[\begin{matrix}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{matrix}|q,z\right]\\
&=\sum_{n=0}^{\infty}\frac{\langle\hat{a}_1,\ldots,\hat{a}_p;q\rangle_n}{\langle 1,\hat{b}_1,\ldots,\hat{b}_r;q\rangle_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} z^n,
\end{aligned}
\tag{1.14}
$$

where $q \neq 0$ when $p > r+1$, and

$$
\widehat{a} = \begin{cases} a, & \text{if no tilde is involved}\\ \widetilde{a} & \text{otherwise.}\end{cases} \tag{1.15}
$$

Definition 1.5. The series

$$
{}_{r+1}\phi_r(a_1,\ldots,a_{r+1};b_1,\ldots,b_r|q,z) \tag{1.16}
$$

is called *k-balanced* if $b_1+\ldots+b_r = k+a_1+\ldots+a_{r+1}$; and a 1-balanced series is called balanced (or Saalschützian). Analogous to the hypergeometric case, we shall call the q-hypergeometric series (1.16) *well poised* if its parameters satisfy the relations

$$
1+a_1 = a_2+b_1 = a_3+b_2 = \ldots = a_{r+1}+b_r. \tag{1.17}
$$

If the series (1.16) is well poised and if, in addition

$$
a_2 = 1+\tfrac{1}{2}a_1,\ a_3 = \widetilde{1+\tfrac{1}{2}a_1}, \tag{1.18}
$$

then it is called a very-well-poised series.

There are several advantages with this new notation:

1. The theory of hypergeometric series and the theory of q-hypergeometric series will be united.

2. We work on a logarithmic scale; i.e., we only have to add and subtract exponents in the calculations. Compare with the ‘index calculus’ from [4].

3. The conditions for k-balanced hypergeometric series and for k-balanced q-hypergeometric series are the same.

4. The conditions for well-poised and nearly-poised hypergeometric series and for well-poised and nearly-poised q-hypergeometric series are the same. Furthermore the conditions for almost poised q-hypergeometric series are expressed similarly.

5. The conditions for very-well-poised hypergeometric series and for very-well-poised q-hypergeometric series are similar. In fact, the extra condition for a very-well-poised hypergeometric series is $a_2 = 1 + \frac{1}{2}a_1$, and the extra conditions for a very-well-poised q-hypergeometric series are $a_2 = 1 + \frac{1}{2}a_1$ and $a_3 = \widetilde{1 + \frac{1}{2}a_1}$.
6. We do not have to distinguish between the notation for integers and nonintegers in the q-case anymore.
7. It is easy to translate to the work of Cigler [16] for q-Laguerre-polynomials.

An example of how the Ward–AlSalam q-addition fits into the new method is the following q-analogue of a result of Gauss given by

$$(1 \oplus_q x)^n + (1 \ominus_q x)^n = 2\,{}_4\phi_1\left(-\frac{n}{2}, \frac{1-n}{2}, \infty, \infty; \frac{1}{2}\middle| q^2, x^2 q^{2n-1}\right). \tag{1.19}$$

2. Trigonometry, Prosthaphaeresis, Logarithms, ...

There has always been a strong connection between mathematics and physics. Textbooks on mathematics in the late eighteenth century contained a variety of subjects like mechanics, optics and astronomy. One example is the book *die Elemente der Mathematik* by Johann Friedrich Lorentz from 1797, which contained such diverse subjects as refraction, parallax, geography, the atmosphere of the moon. Elementary trigonometric formulas were given, and these trigonometric functions were used to treat the physics involved. This was natural since many earlier mathematicians, like the Bernoullis, had been both physicians and physicists.

We will start with a brief history about trigonometry and its relationship to spherical triangles and astronomy. The Egyptian mathematician and astronomer Ibn Yunus (950–1009) demonstrated the product formula for cos and made many astronomical observations. During the doldrums of the dark ages not much happened in trigonometry until the renaissance with the rediscovery of the old Arabic culture. Prosthaphaeresis (the Greek word for addition and subtraction) is a technique for computing products quickly using trigonometric identities, which predated logarithms. Myriads of books have been written on trigonometry in Latin before the modern notations sin and cos were introduced by Leonard Euler (1707–1783). The Greek and Alexandrian mathematicians were prominent in proof theory and geometry, including conic sections. Certainly these ancient scientists had some notation for trigonometric functions, and some of this great research has survived in terms of Latin translations during the renaissance. Since we are going to introduce q-complex numbers here, a brief sketch of the discovery of imaginary (and complex numbers) will now be given. After the Greeks found solutions of quadratic equations, the world had to wait till the sixteenth century, when the cubic equation was solved by Italian mathematicians 1515 and 1545. The solution of the cubic equation

$$x^3 + px + q = 0 \tag{2.1}$$

can be written in the form

$$x = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}. \tag{2.2}$$

The case $\frac{q^2}{4} + \frac{p^3}{27} < 0$ caused much confusion for centuries. It corresponds to three real roots.

In the following we will use the letter i for the imaginary unit, also for references which predate Euler's invention of this symbol 1748 [23, p. 147]. Rafael Bombelli (1526–1572) in his *Algebra* [10, p. 169,190] defined complex numbers as linear combinations with positive coefficients of four basis elements piu (+1), meno (-1), piu de meno (+i), and meno de meno (-i). He states explicitly that piu and piu de meno cannot be added, a first appearance of the notion of linear independence.

Bombelli wrote

$$(6 \pm \sqrt{-121})^{\frac{1}{3}} = 2 \pm i. \tag{2.3}$$

In his book Bombelli showed the usual sign rules for multiplication of imaginary numbers; Bombelli also showed that the Cardan–Tartaglia formula gives correct values also for the above case. He went on to develop some rules for complex numbers. He also worked with examples involving addition and multiplication of complex numbers. René Descartes (1596–1650) invented the name imaginary for these numbers and guessed that every algebraic equation has as many roots (including complex numbers) as its degree. When in Paris 1673, Gottfried Wilhelm Leibniz (1646–1716) studied Bombelli's Algebra and told his friend Christiaan Huygens (1629–95) about this [32], [45, p. 24]. Leibniz and Johann Bernoulli (1667–1748) used imaginary numbers in their development of calculus.

Probably the first western European work for systematic computations in plane and spherical triangles was written 1579 by François Viète, (1540–1603), called 'the father of modern algebraic notation'. Viète introduced letters as symbols for known and unknown quantities. The convention that letters in the beginning of the alphabet denote known quantities and letters in the end of the alphabet denote unknown quantities was introduced somewhat later by Descartes; it is still used today. Descartes published his geometry in 1637, where the well-known Cartesian coordinates were introduced. Jean Beaugrand (1590–1640) claimed that Descartes had got his results from research done by Thomas Harriot (1560–1621). It was not customary at the time to report the sources of your ideas, so Beaugrand's claim might be partly true, though it seems unlikely.

This is expressed somewhat differently in [33, p. 145]: Many years before the appearance of the *Geometrie* Harriot had worked on transformations of quadratic equations, and as Wallis (for once) acknowledges in his *algebra*, Descartes certainly contrived his own procedure quite independently of Harriot.

In fact Harriot's main work *Artis* was published posthumously 1631, and according to Cantor clearly shows that Harriot was a pupil to Viète as far as notation is concerned. However we know that Harriot adopted small letters instead of the capitals used by Viète. We also know today that the Stirling numbers, which are treated in another paper of the author, were found in a handwriting of Harriot kept in a museum in London.

Viète used a certain notation for multiplication and found [26] formulas almost equivalent to multiplication and division for complex numbers as well as the De Moivre theorem. Joost Bürgi (1552–1632) was also an advocate for prosthaphaeresis. At about the same time as Viète's book appeared, he invented the logarithms, but unfortunately he did not dare to publish his invention until 1620 [15, p. 165]. Bürgi, who was a Swiss clockmaker was also a member of the so-called *Rosenkreuzer* society, which we will come back to later.

Two years later, James VI of Scotland, who was on a journey to Norway in the year 1590 together with his entourage, including Dr. John Craig, visited the island Ven on his return to Scotland. There Tycho Brahe (1546–1601) had constructed a big machine for prosthaphaeresis computations to ease the burden of calculation. Craig told his friend John Napier (1550–1617) about the visit to the famous astronomer, and that inspired Napier to the development of his logarithms and the generation of his tables, a work to which he dedicated his remaining 25 years.

Johannes Kepler (1571–1630), who was collecting Tycho Brahe's immense data, read Napier's book on logarithms in 1616; he found that he could describe his laws for orbital periods and semi-major axis for planetary ellipses as a straight line in a log-log diagram. Thanks to these laws Isaac Newton (1642–1727) was able to discover the gravitation law. At the request of Kepler, Bürgi finally brought himself to publish his important book on logarithms in 1620. Logarithms have been in great use ever since, even physicians and nurses have employed these tables for various long computations.

Another example of the profitable interdependence between scientists is the so-called Snell's law. Willebrord Snell van Rojen (Snellius in Latin) (1580–1626) was born in Leiden. Already in the year 1600 he was able to lecture on mathematics in Leiden. He traveled to different cities, among others to Prague, where he met Tycho Brahe and Kepler, and published works on mathematics and astronomy by classic and contemporary authors. He is most famous for his formulation of the refraction law, which he discovered through studying Kepler's works on optics. But he did not publish his discovery. Descartes learned it, however, from Snellius's lectures and gave it a somewhat different formulation in his work *La Dioptrique* in 1637 without stating the source. Snell's law follows from Fermat's principle of least time, which in turn follows from the propagation of light as waves. This general principle was first stated by Fermat in a letter dated January 1st, 1662, to Cureau de la Chambre.

Pierre de Fermat (1601–1665) was a famous mathematician who founded modern number theory, analytic geometry (together with Descartes), and introduced the precursor of the q-integral.

Fermat followed in the footsteps of Heron in Alexandria (10–75), who introduced the concept of minimal path to derive certain optical laws. Fermat was also very much influenced by Apollonius, Pappus, and Diophantos [42, p.27]. Around this time one of

Descartes' students was collecting his correspondence. This led Fermat to look again at the argument he had used 20 years ago in his discussions about the law of refraction.

Conic sections have been known ever since the old masters in Greece and Alexandria. Its algebraic descendant the parabola describes free fall studied by Galilei, Fermat and Newton. In Richelieu's France the work of the old masters was not very widespread, but Girard Desargues (1591–1661) published a book about projective geometry in the year 1639 [41, p. 16]. This work was highly praised by Fermat [14, p. 1618], but sadly enough it was destroyed after a few years due to political reasons. Only one person managed to complete Desargues's geometrical achievements. In the year 1640, Blaise Pascal (1623–62) wrote a paper on conic sections, which Leibniz reviewed in a letter to Pascal's nephew 1676. Pascal's work was so precocious that Descartes, when shown the manuscript, refused to believe [8, p. 100] that the composition was not by the elder Pascal, Étienne Pascal (1588–1651).

3. The Development of Calculus

The first calculating machine was built by the German astronomer Wilhelm Schickard (1592–1635) in 1623 [41, p. 48], and was designed for Kepler. The Schickard calculator could add, subtract, multiply and divide, but remained unknown for 300 years.

In 1642, Pascal constructed a mechanical calculator, capable of addition and subtraction, called Pascal's calculator or the *Pascaline*, in order to help his father with his calculations of taxes [41, p. 16]. One of his calculators was exhibited in museums both in Paris and Dresden, but it failed to be a commercial success. Although Pascal made further improvements and built fifty machines, the *Pascaline* became little more than a toy and status symbol for the very rich families in Europe, since it was extremely expensive. Also people feared it might create unemployment, since it could do the work of six accountants.

Gottfried Wilhelm Leibniz (1646–1716) made two tries to build a calculating machine before he succeeded in 1673; it could do addition, subtraction, multiplication, and possibly division [33, p.79].

Since the q-difference operator is fundamental for our treatment, we will go through the historical development of the calculus in some detail. Euclid computed the volume of a pyramid by a geometric series in Elements XII 3/5 [7, p. 48]. Archimedes used a geometric series to do the quadrature of the parabola. In the 1600:s each mathematician did his own proofs in calculus. Fermat, Gilles Roberval (1602–75) and Evangelista Toricelli (1608–47) had great success in the theory of integration. All three, independent of each other, found the integral and derivative of power functions, but in a geometric way. Roberval kept his post as professor in Paris by winning every contest that was set up. Thus he couldn't publish his discoveries since then he would reveal the secrets of his methods [49, p. 21]. Roberval's win of the competition in 1634 was probably because of his knowledge of indivisibels [49, p. 21]. Between 1628 and 1634 Roberval invented his method of infinitesimals [49, p. 59]. Roberval plotted graphs of trigonometric functions before 1637 in connection with a volume calculation [49, p. 67]. Roberval was also the first to compute certain trigonometric integrals [49, p. 72].

In 1635 Bonaventura Cavalieri (1598–1647) was the first to publish integrals of power functions x^n in his book *Geometria indivisibilus continuorum*; but he proved it explicitly only for the first few cases, including $n = 4$, while, as he stated, the general proof which he published was communicated to him by a French mathematician Jean Beaugrand (1590–1640), who quite probably had got it from Fermat. Beaugrand made a trip to Italy in 1635 to tell Cavalieri about Fermat's achievements [42, p. 51]. Cavalieri's method was much like Roberval's, but mathematically inferior [49, p. 21].

At this time, father Marin Mersenne (1588–1648) kept track of science in France, and knew about all important discoveries, a kind of human internet. Mersenne had made the work of Fermat, Descartes and Roberval known in Italy, both through correspondence with Galileo Galilei (1564–1642) dating from 1635, and during a pilgrimage to Rome in 1644.

Fermat's contribution became known through a translation of *Diophantus Arithmetica* by Claude Gaspard de Bachet (1591–1639) in 1621. Fermat adhered to the algebraic notation of Viète, and relied heavily on Pappus in his development of calculus. Like Kepler, Fermat uses the fact that extreme-values of polynomials are characterized by multiple roots of the function put equal to zero [34, p. 63].

Originally Fermat put $f(x + h) = f(x - h)$ for extreme values [35], developed the expression in terms of powers of h, then finally decided the type of the extreme value from the sign. Later in 1643–44 he even talked of letting $h \to 0$ [34, p. 63]. The leading mathematician of the first part of the seventeenth century was Fermat, who was very talented in languages and handwritings [34, p. 62]. Few results in the history of science have been so closely examined as Fermat's method of maxima and minima [27, p. 24].

Pierre-Simon Laplace (1749–1827) acclaimed Fermat as the discoverer of the differential calculus, as well as a co-discoverer of analytic geometry. Fermat was the first to apply analytic geometry to $\mathbb{R}^3$. Descartes contended himself with $\mathbb{R}^2$ [8, p. 83]. According to Cantor [14, p. 800], Descartes and Fermat were the greatest mathematicians mentioned here.

At that time Fermat, and then also Toricelli, had already generalized the power formula to rational exponents $n \neq -1$. Fermat determined areas under curves which he called general parabolas and general hyperbolas. This was equivalent to calculating integrals for fractional powers of x. Vincenzo Viviani (1622–1703), another prominent pupil of Galilei, determined the tangent to the cycloid.

In 1665, the first two scientific journals were published, Philosophical Transactions of the Royal Society in England, and *Journal des savants* in France. The idea of private, for-profit, journal publishing was already established during this time. Fermat however, chose not to publish, maybe for political reasons. It is also possible that Fermat was influenced by the old Greek habit not to publish one's own proofs [42, p. 31].

Roberval was the first to (q-)integrate certain trigonometric functions. At about the same time important contributions were made by the English mathematician John Wallis (1616–1703) in Oxford, who in his book *the Arithmetic of Infinitesimals* 1656 stressed the notion of the limit. It was in mathematics that Wallis became an outstanding scientist in his country, although he engaged himself in a wide range of interests.

Blaise Pascal associated himself with his contemporaries in Paris, like Roberval and Mersenne. He learned a method similar to q-integration from his masters, and also corresponded with Fermat. Their short correspondence in 1654 [8, p. 107] set up the theory of probability. During several months from 1658 to 1659 Pascal summed infinite series, found derivatives of the trigonometric functions geometrically and calculated power series for sine and cosine. Trigonometric tables were published in great amount in the early 17th century; e.g., Mathias Bernegger published such tables 1612 and 1619 in Strassburg. However in Pascal's time there were no signs for sine and cosine. During these last years of his life Pascal also published the philosophical work *Lettres provinciales* under the pseudonym Louis de Montalte. The power series for sin and arcsin were communicated by Oldenburg to Leibniz in 1675 [2]. These series were also known to Georg Mohr (1640–1697) and Collins.

We will now outline the development of calculus and discover that it was actually preceded by the q-integral in geometric disguise.

Isaac Barrow (1630–77) who was Isaac Newton's teacher in Cambridge [8, p. 117], published his geometrical lectures in 1664. Barrow was familiar to the concept of drawing tangents and curves, probably from the works of Cavalieri and Pascal. Barrow developed a kind of calculus in a geometrical way, but did not have a suitable algebraic notation for it. Child claims that Barrow was the first to give a rigorous demonstration of the derivative for fractional powers of x. Barrow also touched upon logarithmic differentiation.

Leibniz was a child prodigy at Leipzig, where he learned Latin and Greek by studying books in his father's library. At the age of 15 Leibniz explained the theorems of Euclidean geometry to his fellow students at the university. Leibniz also studied philosophy and was fascinated by Descartes's ideas. Leibniz had found remarks on combinations of letters in a book by Clavius [33, p. 3]. Leibniz could decipher both letter- and number codes, and in 1666 published a thesis [40], where the mathematical foundations of combinations were given.

He was also interested in alchemy and became a member of an alchemical society in Leipzig [1]. Despite his outstanding qualifications in law, Leibniz was not given his doctor's degree in Leipzig, so he turned to another city. He stayed in Nuremberg for several months 1667 to learn the secrets of the *Rosenkreuzer* and their scientific books. Because of his outstanding knowledge in alchemy, he was elected as secretary of the society [39, p. 109]. He remembers that he had Cavalieri's book about indivisibles in his hands during this stay [33, p. 5]. During this time, Leibniz was completely under the spell of the concept of indivisibles, and had no clear idea of the real nature of infinitesimal calculus [33, p. 8].

Bored by the trivialities of the alchemists, and realizing that the world's scientific center was in France, Leibniz then entered the diplomatic service for several German royal families. As a diplomat, Leibniz first went to Paris in 1672, where he mixed in scientific and mathematical circles for four years. He was advised by the Dutch mathematician Huygens to read Pascal's work of 1659 *A Treatise on the Sines of a Quadrant of a Circle*.

In January and February 1673 he visited Royal society and was elected to membership [51, p. 260]. He wished to display one of the first models of his calculating machine [33, p. 24]. An English calculating machine was also shown to him, which used Napier's bones [33, p. 25]. Heinrich Oldenburg (1626–1678) mailed him on 6 April 1673 a long report, which John Collins (1625–1683) had drawn up for him, on the status of British mathematics. James Gregory (1638–1675) and Newton dominated the report, which included a number of series expansions, although no suggestion of the method of proof was given [51, p. 260]. At the end of 1675, Leibniz had received only some of Newton's results, and these results had been confined to infinite series [51, p. 262]. At Leibniz' second visit to London, Collins expressed fear that Leibniz' method would prove more general left him wholly unmoved. Even before Leibniz' visit, Collins had been impressed enough to urge Newton anew to publish his method. But since Newton at the time was engrossed in other interests, he did not respond to Collin's suggestion, and Oldenburg, who was engaged in Newton's research, unfortunately died two years later [51, p. 264].

Leibniz rewrote Pascal's proof of $\sin' x = \cos x$ in terms of increment in y/increment in x, using finite differences. The idea of a limit in the definition of the derivative was introduced by Jean D'Alembert (1717–83) and Augustin Louis Cauchy (1789–1857) in 1821.

Earlier, there was a suspicion that Leibniz got many of his ideas from the unpublished works of Newton, but nowadays it is evident that Leibniz and Newton have arrived at their results independently. They have both contributed successfully to the development of calculus; Leibniz was the one who started with integration and Newton with differentiation.

It was Leibniz, however, who named the new discipline. It was actually the Leibniz symbolism that built up European mathematics. Leibniz kept an important correspondence with Newton, where he introduced and forcefully emphasized his own ideas on the subject of tangents and curves. Newton says explicitly that he got the hint of the method of the differential calculus from Fermat's method of drawing tangents [8, p. 83].

Leibniz also continued the use of infinity that had been used by Kepler and Fermat. In fact Kepler started with an early calculus for the purpose of calculating the perimeter of an ellipse and the optimal shape of wine-casks. When we, according to Leibniz, speak of infinity great or infinitely small quantities, we mean quantities that are infinity great or infinitely small, i.e., as great or small as you please. Leibniz said: It will be sufficient if, when we speak of infinity great (or more strictly unlimited), or of infinitely small quantities, it is understood that we mean quantities that are indefinitely great or indefinitely small, i.e., as great as you please, or as small as you please. These notions were then further used in the works of Euler, Carl Friedrich Gauss (1777–1855), Eduard Heine (1821–81), and the present author.

Newton called his calculus the "the science of fluxions". Newton used the notation $\dot{x}$ for his fluxions, this symbol is much used in mechanics today. He wrote his *Methodus fluxionum et serium infinitorum* in 1670–1671; but it was not published until 1736, nine years after his death. It was the astronomer Edmond Halley (1656–1742) who payed the printing of the masterpiece *Principia* in 1687, where the first rigorous theory of mechanics and gravitation was given. At this time there was only one journal in England where Newton could publish, Philosophical Transactions of the Royal society, dating back to 1665. Newton's theory of light was published here 1672. The Transactions of the Cambridge Philosophical Society did not appear until 1821–1928.

On the other hand, the infinity symbol was often used in the circle around Leibniz. In 1682 the first scientific journal of the German lands, *Acta Eruditorum* in Leipzig was founded, the editor was Otto Mencke (1644–1707). Leibniz published his first paper on calculus in this journal in 1684. The Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748), soon also started to publish in *Acta Eruditorum*. The result was that almost all of the elementary calculus that we now know was published here before the end of the eighteenth century. For instance, the well-known Taylor formula was published in a different form by Johann Bernoulli 1694 [24].

We will briefly touch another aspect of logarithms, namely the one associated with congruences. Congruences have been known since the invention of the Euclidean algorithm and diophantic equations. We start with a congruence $a \equiv b \mod m$. Now pick out a primitive root r for m. According to the so-called index calculus, this is equivalent to studying the congruence $i(a) \equiv i(b) \mod \varphi(m)$, where $i(a)$ and $i(b)$ are the so-called indices of a and b for the root r. The concept of indices was introduced by Euler, but the first systematic treatment of the subject was given by Gauss in the third section of his Disquisitiones.

We will stop from time to time and note some connections to motivate the q-umbral calculus introduced here. Our first concern is of computational nature. Today there are several programs used by scientists for computations. The q-calculus computations are especially hard because of their symbolic nature; there are only two programs which can do the job well. We are only going to grade these two programs on a relative scale, since an absolute evaluation is not possible; we will take the ability of drawing colour graphs into consideration. To describe the relative grades we write a table which compares Maple and Mathematica on the one hand, and prosthaphaeresis and logarithms on the other hand. The astute reader immediately observes that logarithms has the higher grade, since it shortens down the calculations considerably. The present paper continues the logarithmic method for q-calculus which enables additions instead of multiplications in computations. The resemblance to hypergeometric formulas is also appealing. The q-addition corresponds to the Viète formula for cos and to the De Moivre theorem. We try to use a uniform notation for q-special functions, like the Eulerian notations sin and cos which are now generally accepted. All this is called renaissance [19–21] in the table.

For each row, the relative merit is higher to the right.

Viète	Descartes
prosthaphaeresis	logarithms
congruences	index calculus
Pascaline	Babbage computer
Basic	Fortran
Maple	Mathematica
Gasper/Rahman [25]	Renaissance

4. The Hindenburg School and Babbage in England

Descartes had traveled a lot in his youth and met Johann Faulhaber (1580–1635) in the year 1620 in Ulm [54, p. 223]. Faulhaber was an excellent mathematician, who among other things published the first 16 Bernoulli numbers in 1631 [18, p. 128]. Faulhaber was also a member of the *Rosenkreuzer*, a secret society named after Christian Rosenkreuz (1378–1484), and founded in the seventeenth century. The aim of this society was to further knowledge, in particular mathematics and alchemy. The *Rosenkreuzer* liked to use unusual signs in their texts; e.g., Faulhaber showed the astrological sign of Jupiter to Descartes. The *Rosenkreuzer* liked to collect scientific books; Faulhaber had collected several books about algebra and geometry in his home. According to [44, p. 50], Descartes attempted in vain to contact the brotherhood during his travels in Germany. Around this time mathematics in France was not prosperous due to the dominance of the catholic church (Richelieu, Mazarin), as we have seen the most talented scientists had to keep a secret correspondence; they knew what had happened to Galilei. In Germany the situation was different, some states had changed to protestantism. The *Rosenkreuzer* were opposed to the catholic church and preferred a reform of the religious system in continental Europe. The brothers disliked the opposition of the catholic church to scientific ideas and wanted a change. This could have been the most important reason why the *Rosenkreuzer* brotherhood was a secret society. There were also English mathematicians, like John Dee (1527–1609), who were *Rosenkreuzer*. The famous scientist Robert Hooke (1635–1703), contemporary of Newton, wrote in codes, and Newton was a dedicated alchemist who virtually gave up science during the last third of his life [6, p. 196]. When Descartes returned to Paris in October 1628, there were rumours that he had become a *Rosenkreuzer*. According to [1], Descartes started to keep a secret notebook with signs used by the *Rosenkreuzer*. Then at the end of 1628, Descartes definitely moved to Holland, where he stayed for 21 years.

The infinite series was introduced by Newton. The formal power series was conceptually introduced by Euler [23]. We will see that this concept prevailed for quite a while until the introduction of modern analysis by Cauchy 1821.

Leibniz did not have many pupils in Germany, since he worked much as a librarian and traveled a lot. Whereas in France mathematical physics was flourishing (Laplace, Lagrange, Legendre, Biot), German mathematics had a weak scientific position in the time between Frederick the great (1712–1786) [12, p. 256], and the Humbold education reform [36, p. 173]. One exception was Georg Simon Klügel (1739–1812) who introduced a relatively modern concept of trigonometric function in [37] 1770.

Combinatorial notions such as permutation and combination had been introduced by Pascal and by Jacob Bernoulli (1654–1705) in [9].

In the footsteps of Leibniz, Carl Friedrich Hindenburg (1741–1808), a professor of physics and philosophy at Leipzig founded the first modern school of combinatorics with the intention that this subject should occupy a major position in mathematics. One of the ways to achieve this aim was through journals. The *Acta Eruditorum* continued until 1782, then Hindenburg and Johann Bernoulli (III) (1744–1807) edited *Leipziger Magazin für reine und angewandte Mathematik* 1786–1789. This was followed by *Archiv* *der reinen und angewandten Mathematik* 1794–1799 with Hindenburg as sole editor. Hindenburg used certain complicated notations for binomial coefficients and powers; one can feel the influence of the *Rosenkreuzer* secret codes here. The main advantages of the Hindenburg combinatorial school was the use of combinatorics in power series and the partial transition from the Latin language of Euler. The disadvantages were the limitation to formal computations and the old-fashioned notation.

A partial improvement was made by Bernhard Friedrich Thibaut (1775–1832) [36, p, 193], [47], who expressed the multinomial theorem, a central formula in the Hindenburg school, in a slightly more modern way. Thibaut also made a strict distinction between formal power series equality and equality in finite formulas [36, p, 196], a central theme in q-calculus. A similar discussion about formal power series was made by Christof Gudermann (1798–1852) in 1825 [28].

The Hindenburg combinatorial school can be divided into two phases [36, p. 171].

1. 1780–1808,
2. 1808–1840.

The first important Hindenburg book [31] began with a long quotation from Leibniz [40], this quotation from Leibniz would be normal in Hindenburg combinatorial school publications until 1801 [36, p. 178].

One of the main reasons for this is the so-called multinomial expansion theorem, a central formula in the Hindenburg combinatorial school, which was first mentioned in a letter 1695 from Leibniz to Johann Bernoulli, who proved it.

Two natural q-analogues are given next.

Definition 4.1. If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th NWA-power is given by

$$(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q. \tag{4.1}$$

Definition 4.2. If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th JHC-power is given by

$$(\boxplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \boxplus_q a_1 x \boxplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q q^{\binom{\vec{n}}{2}}, \tag{4.2}$$

where $\vec{n} = (m_2, \ldots, m_n)$.

Hindenburg had high hopes for his combinatorial school, and as a result Heinrich August Rothe (1773–1842) and Franz Ferdinand Schweins (1780–1856) formulated the

q-binomial theorem, but without proof. Rothe introduced a sign for sums, which was used by Gudermann. In 1793 Rothe found a formula for the inversion of a formal power series, improving on a formula found without proof by Hieronymus Eschenbach (1764–1797) in 1789 [36, p. 200]. This invention gave the combinatorial school a rise in Germany, as can be seen from the list of its ensuing publications [36, p. 201].

In 1798 Christian Kramp (1760–1826) invented the *factoriellen*, a side-track of the Γ-function. Kramp also introduced the notation $n!$. Louis Arbogast (1759–1803), also from Strasbourg, suggested [5] to substitute a capital D for the little $\frac{dy}{dx}$ of Leibniz to simplify the computations [5]. Arbogast [5, p. v] writes: The Leibniz theory of combinatorics has been improved by Hindenburg, who has given the development of functions of one variable and the multinomial theorem. The procedure and notation of Hindenburg is not familiar. I do not know his writings except for the title; I have followed my proper ideas. The procedure I will give is very analytical. In [5, p. 127] Arbogast gives Taylor's formula for a function of two variables. The use of signs for sums would significantly increase the clarity of this long formula. Anyway this had a long-lasting influence on the development of calculus in Germany and in England. In 1812, in the footsteps of Arbogast another Frenchman Jacques–Fréderic Français [38, p. 163] wrote E for the forward shift operator and reproved the Lagrange formula from 1772

$$E = e^D. \tag{4.3}$$

A decade later Cauchy in his *Exercices de mathématiques* for the first time found operational formulas like [38, p. 163]

$$D(e^{rx} f(x)) = e^{rx}(r + x) f(x). \tag{4.4}$$

Arbogast and Fourier regarded this umbral method as an elegant way of discovering, expressing, or verifying theorems, rather than as a valid method of proof [38, p. 172]. Cauchy had similar suspicions. We will see that this sometimes also obtains for the method of the author. From about this time, one can say that special function theory started to develop after Euler. This development was made parallel to the theory of Bernoulli numbers and Stirling numbers. The Bessel functions introduced by Friedrich Wilhelm Bessel (1784–1846) in 1824, had previously been investigated by Jacob Bernoulli, Daniel Bernoulli, Euler and Lagrange. There is also a connection to differential equations here, the Bessel differential equation is related to the Riccati differential equation, introduced by Jacopo Francesco Riccati (1676–1754) in 1724. This made the way for C.-F. Gauss (1777–1855) and Johann Friedrich Pfaff (1765–1825), who developed the hypergeometric function, the corner-stone of special functions, and the prerequisite for q-hypergeometric functions. Andreas von Ettingshausen (1796–1878) introduced $\binom{m}{n}$ for binomial coefficients and used the so-called Stirling numbers in a book about combinatorics.

We are going to quote the Elaine Koppelman article [38] many times in the sequel. The fluxion concept was prevalent in England until 1820, when Robert Woodhouse

(1773–1827), George Peacock (1791–1858), Charles Babbage (1791–1871), and John Herschel (1792–1871), all from Cambridge [38, p. 156] managed to recognize Leibniz' and Arbogast's notation in England. This soon led to the introduction of the calculus of operations [38, p. 156] or umbral calculus.

Woodhouse discussed at length the importance of a good notation [38, p. 177], since the development of calculus in Cambridge had been slow until 1820 [38, p. 155]. His conclusion is that the Arbogast notation is by far superior. A similar conclusion is drawn by Cajori [11]. Woodhouse's book *Analytical calculation* had a long-lasting influence on Babbage [38, p. 178]. In the Babbage, Peacock, and Herschel translation of Lacroix into English (1816) it is claimed in the preface that calculus was discovered by Fermat, made analytical by Newton and enriched with a powerful and comprehensive notation by Leibniz [38, p. 181]. Before Babbage dropped this subject he once again stressed the importance of a good notation for calculus [38, p. 184]. E. T. Bell (adviser of Morgan Ward) wrote that operational mathematics, which was developed in England 1835–1860, despite its obvious utility, was scarcely reputable mathematics, because no validity condition or validity region accompanied the formulas obtained [38, p. 188].

In the year 1837 the *Cambridge mathematical journal* was founded by among others Duncan Gregory (1813–1844), to provide a place for publication of short mathematical research papers, and thus encourage young researchers. In a letter to John Herschel 1845 [38, p. 189], DeMorgan described the journal and Gregory's contributions as full of very original communications, very full of symbols. In Gregory's first paper on the separation of symbols, the linear differential equation with constant coefficients was treated. Similar studies had already been published by Cauchy in France; Gregory was familiar with Cauchy's and Brisson's works on this subject [38, p. 190]. As pointed out by DeMorgan 1840 [38, p. 234], the symbolic algebra method gives a strong presumption of truth, not a method of proof. Gregory correctly claimed that the operations multiplication and function differentiation obey the same laws [38, p. 192]. Gregory's methods only applied to differential operators with constant coefficients. A generalization to non-commutative operators was given in 1837 [38, p. 195] by Robert Murphy (1806–1843). The studies of non-commutative operators were continued by among others George Boole (1815–1864), William Donkin (1814–1869), and Charles Graves (1812–1899). More general functional operators were studied by W. H. L. Russell [38, p. 204], William Spottiswoode (1825–1883), William Hamilton (1805–1865) and William Clifford (1845–1879).

Let us summarize: the calculus of operations, imported from France and extended by Babbage and Herschel, was an important mathematical research area in England between 1835 and 1865, [38, p. 213]. Most of these articles were published in *Cambridge mathematical journal* and its successor, *Cambridge and Dublin mathematical journal* (1845–1854).

From about 1860 the calculus of operations split into different areas, some of which are:

1. umbral calculus,

2. q-calculus,
3. theory of linear operators [38, p. 214],
4. algebra.

These different subjects are however far from disjoint.

The theory of formal power series within the Hindenburg combinatorial school continued under Martin Ohm, who influenced by Cauchy's *Cours d'Analyse* put up convergence criteria for the known elementary transcendental functions in certain regions. Ohm defined mathematical analysis from seven basic operations in a rather involved theory, which clearly was a forerunner of later attempts to base all of mathematics on the integers [38, p. 226]. In 1833 Hamilton read a paper expressing complex numbers as algebraic couples, and in 1837 presented an article on the arithmetization of analysis [43]. This was a careful, detailed and logical criticism of the foundations of algebra, which has been an important step in the development of modern abstract algebra. (C. C. Macduffee 1945) [38, p. 222].

In 1846 Hamilton published a series of papers with the title *on symbolic geometry*. Again he cited Peacock and Martin Ohm as the authors who had inspired him to a deeper appreciation of the new school of algebra [38, p. 222]. Like DeMorgan, Hamilton wanted, at first, a system which would form an associative, commutative, division algebra over the reals [38, p. 228]. Out of this grew finally the famous quatenions.

The *Quarterly Journal of Pure and Applied Mathematics* (QJPAM) rose from the ashes of the CDMJ in April 1855 [17]. The first two British editors were Ferrers and Sylvester. It was here the first umbral calculus was written by John-Charles Blissard (1835–1904) [17]. F.H. Jackson (1870–1960), the first master of q-calculus, who was also a priest like Blissard, published many of his papers here.

Hindenburg thought that a mechanical calculation would be an important aim for his school [36, p. 222], and this is exactly what the present author has achieved in his papers. The halting-places are the elementary formulas for q-shifted factorials, which have similarities with formulas for the *factoriellen*. The general formulas, or expansions can very often be expressed by q-binomial coefficients like (4.2).

5. Infinity, the Discrete and Continuous

We now make a further leap forward in time and move to Göttingen and Zurich and the time immediately after world war 1. David Hilbert had a talented student named Hermann Weyl. In 1918 Weyl initiated a program for the arithmetical foundations of mathematics in his monograph *das Kontinuum* [52]. In [52, p. 66] Weyl discusses the coordinates of a mass point as a function of time. He then proceeds to claims that rational functions could certainly approximate the real function values involved. In a later work [53] Weyl tries to approximate intervals with geometric series just as in the q-integral. Weyl writes: Auf jeden Fall aber ist es unsinnig, das Kontinuum als ein Fertig-Seiendes zu betrachten. Or in English translation: In any case, it makes no sense to regard the continuum as something complete.

Hilbert writes in [30], just after the invention of quantum mechanics: Throughout mathematical literature, when scrutinized, there are running strong streams of absurdity and thoughtlessness that have in most cases given rise to wrongs in questions of the infinite. When we now turn to the task of analysing the nature of the infinite, we must give a very short explanation of what the real meaning of the infinite is; first we will see what we can learn about this from physics. The first naive impression of nature and matter is constancy and continuity. If we have a piece of metal or a volume of liquid, the idea presents itself inevitably that they are infinitely divisible, that a part of them, though ever so small, will always retain the same qualities. But everywhere, when you sufficiently refine the methods of research in the physics of matter, you come across limits to the divisibility that do not lie in the inaccessibility of our experiments, but in the nature of matter, so that you could simply regard the tendency of modern science as a liberation from the concept of the infinitely small and that instead of the old leading principle: natura non facit saltus, you could maintain the opposite, that nature does take leaps. As is well-known, all matter is composed of small building stones, the atoms, through the combinations and connections of which all the manifold variety of the macroscopic matter arises. But the atomic theory of matter did not bring physics to a standstill. Parallel to this there appeared towards the end of the last century the atomic theory of electricity, which above all had a much stranger effect. While electricity was until then regarded as a fluid and a prototype of a continuously working force, now it, too, turned out to be built of positive and negative electrons. Besides matter and electricity there is in physics yet another reality, for which the maintenance of this law is also valid, namely energy. Now not even energy will quite easily and unrestrictedly admit unlimited division; Planck discovered the energy quanta. The final result is in all these cases that a homogeneous continuance that would admit a continuous divisibility and implement infinity in the smallest parts, will in reality be found nowhere. The unlimited divisibility of a continuum is only a process in the mind, only an idea, that will be confuted by our observations of nature and the discoveries in physics and chemistry.

We are going to describe a few twin models in physics, one discrete and one continuous. We start with statistical mechanics and thermodynamics. The latter is much older and is intimately connected to chemistry. The earliest propagators for thermodynamics were among others Pascal (again!) and Newtons colleague Robert Boyle (1627–91). In his thesis 1811 Amadeo Avogadro (1776–1856) formulated that all gases of a given volume have the same number of molecules, regardless of pressure or temperature. The Avogadro number, one mole equal to 6.023×10^{23} was found much later. In Germany, this number is called the Loschmidt constant, named after a colleague of Boltzmann's. In 1859 James Clerk Maxwell (1831–79) discovers his kinetic theory of gases, and revives an old idea of Daniel Bernoulli (1700–82) from 1738. In 1872 Ludwig Boltzmann (1844–1906) derived his entropy-theorem, and introduced a number of mathematical innovations, including a technique of discretizing the allowed energy levels for a molecule. In 1884 Boltzmann succeeded in theoretically deriving the radiation law found by Stephan. In the same year Willard Gibbs (1839–1903) coined the term "statistical mechanics" for the kinetic theory's treatment of thermodynamic issues.

This duality between the discrete (particle model) and the continuous (wave theory) has continued up to the present day. Louis de Broglie (1892–1987) studied relativity carefully and was able to find linear connections between the particle quantities E, $\vec{p}$ and the wave quantities ω, $\vec{k}$.

We have thus found that particle physics is discrete, and this gives one reason why there are two competing models for hadrons, "Quantum Chromodynamics" (QCD), and the Santilli hadronic mechanics. Both these models are connected to q-calculus.

We can make some further comparisons with the current paper.

1. The q-complex numbers introduced here are in principle a continuation of the Hindenburg combinatorial school, which guarantees the beauty of the formulas. We conjecture that the convergence radius for functions $f \in \mathbb{C}[[x]]$ increases with the following sequence of function values:real numbers, complex numbers, and finally q-complex numbers.

2. There are at least two examples of atomic behaviour in the q-umbral method of the author which should be mentioned: The NWA q-addition and the discrepancy of the shape of summation formulas for q-hypergeometric series for integer values of the parameters and for other complex values. This is of course directly connected to the Γ_q-function.

6. Famous Controversies in Calculus

Fermat received a copy of Descartes' *La Dioptrique* in 1639 from mathematician Beaugrand, but paid it little attention since he was in the middle of a correspondence with Roberval and Étienne Pascal over methods of integration and using them to find centers of gravity. Mersenne asked him for his opinion on *La Dioptrique* which Fermat did describing it as

> "groping about in the shadows."

He claimed that Descartes had not correctly deduced his law of refraction since it was inherent in his assumptions. Descartes was furious, because he felt Fermat's work on maxima, minima and tangents reduced the importance of his own work *La Geometrie* which Descartes was most proud of and which he sought to show that his *Discours de la méthode* alone could give.

He attacked Fermat's method of maxima, minima and tangents. Roberval and Étienne Pascal became involved in the argument and eventually so did Girard Desargues who Descartes asked to act as a referee. Fermat proved correct and eventually Descartes admitted this, writing:

... seeing the last method that you use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would not have contradicted it at all.

Descartes now tried to damage Fermat's reputation. He wrote to Fermat praising his work on determining the tangent to a cycloid (which is indeed correct), meanwhile writing to Marin Mersenne (1588–1648) claiming that it was incorrect and calling Fermat an inadequate mathematician and thinker. Descartes was important and respected and was able to severely damage Fermat's reputation.

A priority dispute between Leibniz and Newton followed which had far-reaching consequences. This was also one of the first great disputes between mathematics and mathematical physics. Ultimately Leibniz became involved in a dispute which was concerned less with matters of science and learning than with personal touchiness and questions of national vanity [33, p. 10]. The result was that mathematical physics remained very popular in England up to the present day. One further consequence was the habit among some mathematical physicists to put $c = 1$ in formulas of general relativity. This inappropriate habit has however been amended in the recent excellent textbook about this subject by Callahan [13].

We now move to Lübeck 1895 and an intense discussion between Wilhelm Oswald (1845–1915) and the atomists represented by Boltzmann and Felix Klein (1849–1925). This time the atomists won. Already in 1891 Boltzmann had said that he does not see why also the energy should not be atomically distributed. 14 years later, in 1905, Einstein used a similar reasoning when he investigated Brownian motion, convincing many colleagues of the atomic hypothesis. The final breakthrough for the atomic hypothesis was given in the Millikan experiments with the photoelectric effect 1916, and the Compton effect 1923. In one of Boltzmann's derivations for the partition function, a logarithmization transformed a product to a sum and enabled two rows to be equal to each other, the proportionality constant k was found by Planck. Interestingly enough, Planck had originally been one of the opponents to the atomist theory; opinions change. The atomistic theory has been prevalent ever since. In 1924 Satyendranath Bose (1894–1974) sends Einstein a copy of his paper, containing a new derivation of Planck's radiation law based purely on photon statistics, after it was rejected by the Philosophical Magazine. Einstein translates it into German and submits it to the Zeitschrift für Physik for him with a recommendation.

Since then particle physics has been a part of the Boltzmann theory, and this also marked the start for advanced quantum mechanics. On Boltzmann's gravestone the formula

$$S = k \log W \tag{6.1}$$

is written, a token for the discrete behaviour of matter.

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