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Positive Solutions to a Singular Second Order Boundary Value Problem

L. Erbe and A. Peterson

Department of Mathematics, University of Nebraska–Lincoln,
Lincoln, NE 68588-0130, USA
E-mail: lerbe2@math.unl.edu, apeterso1@math.unl.edu

J. Weiss

Department of Mathematics and Statistics,
University of Nebraska–Kearney, Kearney, NE 68847, USA
E-mail: weissjj@unk.edu

Abstract

In this paper, we establish some criteria for the existence of positive solutions for certain two point boundary value problems for the singular nonlinear second order equation

$$-(ru^{\Delta})^{\Delta} + qu^{\sigma} = \lambda f(t, u^{\sigma})$$

on a time scale $\mathbb{T}$. As a special case when $\mathbb{T} = \mathbb{R}$, our results include those of Erbe and Mathsen [11]. Our results are new in a general time scale setting and can be applied to difference and q-difference equations.

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1. Introduction

In this paper we consider the dynamic equation

$$Lu = \lambda f(t, u^\sigma) \tag{1.1}$$

with the boundary conditions

$$\begin{cases} \alpha u(a) - \beta u^\Delta(a) & = 0 \\ \gamma u(\sigma^2(b)) + \delta u^\Delta(\sigma(b)) & = 0, \end{cases} \tag{1.2}$$

where the operator L is defined by $Lu := -(ru^\Delta)^\Delta + qu^\sigma$. The domain $\mathcal{D}$ of L is the set of functions $u : \mathbb{T} \to \mathbb{R}$ such that u is continuous on $[a, \sigma^2(b)]$, u^Δ is continuous on $[a, \sigma(b)]$, and $(ru^\Delta)^\Delta$ is rd-continuous on $[a, b]$. Since we shall be interested in the case when f may have singularities at one or both of the endpoints, we shall assume, either f is continuous on $(a, b) \times \mathbb{R}$, if f is singular at b or f is continuous on $(a, b] \times \mathbb{R}$ if f is not singular at b. In addition, we assume $r > 0$ is rd-continuous, $q \geq 0$ is rd-continuous, $\alpha, \beta, \gamma, \delta \geq 0$ and $\rho := \gamma\beta(\sigma^2(b) - a) + \alpha\gamma + \alpha\delta > 0$. Special cases of this problem have been studied by many authors (see [8, 11] and the references therein). Our concern is to find conditions on f so that (1.1)-(1.2) has a positive solution for some $\lambda > 0$. If f has a singularity at b, then $\rho(b) = b$ and in this case we assume thoughout this paper that $\sigma(b) = \sigma^2(b) = b$.

Green's function for (1.1)–(1.2) is

$$G(t, s) = \frac{1}{c} \begin{cases} \phi(t)\psi^\sigma(s) & t \leq s \\ \phi^\sigma(s)\psi(t) & \sigma(s) \leq t \end{cases} \tag{1.3}$$

where ϕ and ψ are the solutions of

$$\begin{aligned} L\phi &= 0, \qquad & \phi(a) &= \beta, \quad \phi^\Delta(a) = \alpha \\ L\psi &= 0, \qquad & \psi(\sigma^2(b)) &= \delta, \quad \psi^\Delta(\sigma(b)) = -\gamma \end{aligned}$$

respectively.

Lemma 1.1. We have $\phi(t) > 0$ for all $t \in (a, \sigma^2(b)]$ and $\phi^\Delta(t) \geq 0$ for all $t \in [a, \sigma(b)]$. We have $\psi(t) > 0$ and $\psi^\Delta(t) \leq 0$ for all $t \in [a, \sigma^2(b))$.

Proof. First, notice that

$$\begin{aligned} -(r(t)\phi^\Delta(t))^\Delta + q(t)\phi^\sigma(t) &= 0 \\ (r(t)\phi^\Delta(t))^\Delta &= q(t)\phi^\sigma(t), \\ r(t)\phi^\Delta(t) - r(a)\phi^\Delta(a) &= \int_a^t q(s)\phi^\sigma(s)\Delta s \\ \phi^\Delta(t) &= \frac{1}{r(t)}\left[r(a)\alpha + \int_a^t q(s)\phi^\sigma(s)\Delta s\right]. \end{aligned}$$

Notice that $\phi(a) = \beta \geq 0$. We claim that $\phi(t) > 0$ on $(a, \sigma^2(b)]$. Now suppose for the purpose of contradiction that $\phi(t) \leq 0$ for some $t \in (a, \sigma^2(b)]$.

Case 1: $\alpha > 0$: If $\alpha > 0$, then $\phi^\Delta(a) = \alpha > 0$, so there is $k \in \mathbb{T}$, $k > a$ such that $\phi(t) > \phi(a)$ for all $t \in (a, k]$. Suppose that $t_0 > k$ is the smallest point bigger than a in $\mathbb{T}$ such that $\phi(t_0) \leq \phi(a)$.

If t_0 is left-scattered, then $\phi^\Delta(\rho(t_0)) < 0$, and hence $\int_a^{\rho(t_0)} q(s)\phi^\sigma(s)\Delta s < 0$. But then there is a point $t_1 \in [k, \rho(t_0))$ such that $\phi^\sigma(t_1) < 0$. That means that there is $t_2 \in [k, t_0)$ such that $\phi(t_2) < 0$. This contradicts the fact that $\phi(t) > 0$ on $[k, t_0)$.

If t_0 is left-dense, then there is a $t_1 < t_0$ with $\phi^\Delta(t_1) < 0$. Then $\int_a^{t_0} q(s)\phi^\sigma(s)\Delta s < 0$, and so $\phi^\sigma(t_2) < 0$ for some $t_2 \in [a, t_1]$. But then as $\sigma(t_1) < t_0$, we get that $\phi(t_3) < 0$ for some $t_3 \in [\sigma(a), t_0)$. This is a contradiction.

Case 2: $\alpha = 0$: In this case, $\phi(a) = \beta > 0$. Let k be such that $\phi(t) > 0$ on $[a, k]$. Let t_0 be the smallest point in $\mathbb{T}$ greater than a such that $\phi(t_0) \leq 0$. Similar to the proof of Case 1, we get a contradiction.

Hence, $\phi(t) > 0$ on $(a, \sigma^2(b)]$, and thus $\phi^\Delta(t) > 0$ on $(a, \sigma^2(b)]$. The second statement in this lemma can be proven similarly. ■

It follows from Lemma 1.1 that

$$c := -r(t)\left[\phi(t)\psi^\Delta(t) - \phi^\Delta(t)\psi(t)\right] > 0.$$

The BVP (1.1)–(1.2) is equivalent to the integral equation

$$u(t) = \lambda \int_a^{\sigma(b)} G(t, s) f(s, u^\sigma(s))\Delta s,$$

and so we look for positive fixed points of the operator T_λ defined by:

$$(T_\lambda u)(t) = \lambda \int_a^{\sigma(b)} G(t, s) f(s, u^\sigma(s))\Delta s$$

for $t \in [a, \sigma^2(b)]$.

We shall impose some of the following conditions on f (if f is not singular at b, then we assume (a, b) is replaced by $(a, b]$ in each of the following):

(H_1) $f(t, u) \geq 0$ for $(t, u) \in (a, b) \times [0, \infty)$.

(H_2) For each $M > 0$ there exists a function $g_M : (a, b) \to \mathbb{R}^+$ such that $f(t, z) \leq g_M(t)$ for $(t, z) \in (a, b) \times [0, M]$ and $\int_a^{\sigma(b)} G(\sigma(s), s) g_M(s)\Delta s < \infty$.

(H_3) For each $t \in (a, b)$, $f(t, u)$ is nondecreasing in u.

(H_4) There exists a continuous function $p_1 \geq 0$, $p_1 \not\equiv 0$ on any time scale subinterval of (a, b), such that for each constant $d > 0$, there is a $R_d > 0$ with $f(t, z) \geq dp_1(t)z$ for $(t, z) \in (a, b) \times (0, R_d]$.

(H_5) There exists $e > 0$ and a continuous function $p_2 : (a, b) \to \mathbb{R}^+$ such that $f(t, z) \geq ep_2(t)z$ for $(t, z) \in (a, b) \times [0, \infty)$ and $\int_a^{\sigma(b)} G(\sigma(s), s)p_2(s)\Delta s < \infty$.

2. Preliminary Lemmas

First, let E be the Banach space $E := \{u : [a, \sigma^2(b)] \to \mathbb{R} : u \text{ is continuous}\}$, where the norm is the sup norm. We define the cone P by $P := \{u \in E : u(t) \geq 0 \text{ for all } t \in [a, \sigma^2(b)]\}$. Now we state some preliminary lemmas.

Lemma 2.1. Assume (H_1) and (H_2) hold. Then $T_\lambda : P \to P$ is compact.

Proof. (H_1) and (H_2) imply that T_λ exists for $u \in P$ and $T_\lambda u(t) \geq 0$. If $\sigma(a) = a$, we define

$$t_n := \sup\left\{t \in \mathbb{T} : a \leq t \leq a + \frac{1}{n}\right\}.$$

Note that $t_n > a$. If $\rho(b) = b$, we define

$$\tau_n := \inf\left\{t \in \mathbb{T} : b - \frac{1}{n} \leq t \leq b\right\}.$$

Note that $\tau_n < b$. In the case where $\sigma(a) = a$ and $\rho(b) = b$, define

$$f_n(t, x) = \begin{cases} \min\{f(t, x), f(t_n, x)\} & a < t \leq a + \frac{1}{n} \\ f(t, x) & a + \frac{1}{n} \leq t \leq b - \frac{1}{n} \\ \min\{f(t, x), f(\tau_n, x)\} & b - \frac{1}{n} \leq t < b. \end{cases}$$

For $\sigma(a) = a$, and $\rho(b) < b$, define

$$f_n(t, x) = \begin{cases} \min\{f(t, x), f(t_n, x)\} & a < t \leq a + \frac{1}{n} \\ f(t, x) & a + \frac{1}{n} \leq t < b. \end{cases}$$

And finally, for $\sigma(a) > a$ and $\rho(b) = b$, define

$$f_n(t, x) = \begin{cases} f(t, x) & a < t \leq b - \frac{1}{n} \\ \min\{f(t, x), f(\tau_n, x)\} & b - \frac{1}{n} \leq t < b. \end{cases}$$

For $u \in E$, $f_n(t, u^\sigma(t))$ is nonnegative and rd-continuous on (a, b). For each $\lambda > 0$, define an operator T_n by

$$(T_n u)(t) = \lambda \int_a^{\sigma(b)} G(t, s) f_n(s, u^\sigma(s)) \Delta s,$$

for $t \in [a, \sigma^2(b)]$. Then $(T_n u)(t) \geq 0$ on $[a, \sigma^2(b)]$ by (H_1). If $t, t_0 \in [a, \sigma^2(b)]$, then

$$|T_n u(t) - T_n u(t_0)| \leq \lambda \int_a^{\sigma(b)} |G(t, s) - G(t_0, s)| f_n(s, u^\sigma(s)) \Delta s$$

which goes to 0 as $t \to t_0$ by continuity of G. Hence $T_n u \in E$. Let $u \in E$ and $\|u\| \leq M$. Then

$$(T_n u)^\Delta(t) = \frac{\lambda}{c} \psi^\Delta(t) \int_a^t \phi^\sigma(s) f_n(s, u^\sigma(s)) \Delta s + \frac{\lambda}{c} \phi^\Delta(t) \int_t^{\sigma(b)} \psi^\sigma(s) f_n(s, u^\sigma(s)) \Delta s.$$

Hence, $(T_n u)^\Delta(t)$ is continuous and uniformly bounded for $\|u\| \leq M$. Thus, T_n is a compact operator on K by the Ascoli–Arzela theorem. Moreover, $T_n \to T_\lambda$ uniformly as $n \to \infty$ on any bounded subset of K. We show this only for the case $\sigma(a) = a$ and $\rho(b) = b$.

$$\begin{aligned}
|T_\lambda u(t) - T_n u(t)| &\leq \lambda \int_a^{t_n} G(t, s) |f(s, u^\sigma(s)) - f(t_n, u^\sigma(s))| \Delta s \\
&\quad + \lambda \int_{\tau_n}^{\sigma(b)} G(t, s) |f(s, u^\sigma(s)) - f(t_n, u^\sigma(s))| \Delta s \\
&\leq 2\lambda \left(\int_a^{t_n} G(t, s) g_M(s) \Delta s + \int_{\tau_n}^{\sigma(b)} G(t, s) g_M(s) \Delta s \right) \\
&\leq 2\lambda \left(\int_a^{t_n} G(\sigma(s), s) g_M(s) \Delta s + \int_{\tau_n}^{\sigma(b)} G(\sigma(s), s) g_M(s) \Delta s \right) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

(note here we use the fact that we assumed when $\rho(b) = b$ it follows that $\sigma(b) = b$). This implies T_λ is compact on K (see [5, Proposition 4.2]). ■

Lemma 2.2. Let $\lambda > 0$ and (H_1)–(H_3) hold. Suppose further that there exist $w, v \in \mathcal{D}$ with $0 \leq w(t) \leq v(t)$, $Lw(t) \leq \lambda f(t, w^\sigma(t))$, $Lv(t) \geq \lambda f(t, v^\sigma(t))$ on (a, b) (on $(a, b]$ if f is not singular at b) and

$$\alpha w(a) - \beta w^\Delta(a) \leq 0, \qquad \gamma w(\sigma^2(b)) + \delta w^\Delta(\sigma(b)) \leq 0 \tag{2.1}$$

$$\alpha v(a) - \beta v^\Delta(a) \geq 0, \qquad \gamma v(\sigma^2(b)) + \delta v^\Delta(\sigma(b)) \geq 0. \tag{2.2}$$

Then (1.1)–(1.2) has a solution $z \in \mathcal{D}$ with $w(t) \le z(t) \le v(t)$ on $[a, \sigma^2(b)]$.

Proof. Condition (H_3) implies that T_λ is monotone nondecreasing with respect to the cone K. Suppose first that equality holds in all four equations in (2.1)–(2.2). Let $w_0 = w$ and let $w_{n+1} = T_\lambda w_n$. Then

$$\begin{aligned} w_0(t) = w(t) &= \int_a^{\sigma(b)} G(t,s) L w(s) \Delta s \\ &\le \lambda \int_a^{\sigma(b)} G(t,s) f(s, w^\sigma(s)) \Delta s \\ &= T_\lambda w(t) = w_1(t) \end{aligned}$$

for $t \in [a, \sigma^2(b)]$. By induction, the fact that T_λ is monotone nondecreasing gives us that $w_n(t) \le w_{n+1}(t)$. Similarly, $v_{n+1} := T_\lambda v_n \le v_n$ where $v_0(t) = v(t)$. Then since $w \le v$, we get

$$w = w_0 \le w_1 \le \cdots \le w_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0 = v.$$

Now, if equality does not hold, we may modify w to a function $\widetilde{w}$ as follows: let $r_1 := -(\alpha w(a) - \beta w^\Delta(a)) \ge 0$ and $r_2 := -(\gamma w(\sigma^2(b)) + \delta w^\Delta(\sigma(b))) \ge 0$, and define $\widetilde{w} := w(t) + c\phi(t) + d\psi(t)$ where c and d are chosen below and ϕ and ψ are defined earlier. Note that $L\widetilde{w}(t) = Lw(t)$. Since we have

$$\alpha \widetilde{w}(a) - \beta \widetilde{w}^\Delta(a) = -r_1 + d(\alpha \psi(a) - \beta \psi^\Delta(a)),$$

if we set $d := \dfrac{r_1}{\alpha\psi(a) - \beta\psi^\Delta(a)} \ge 0$, then $\alpha\widetilde{w}(a) - \beta\widetilde{w}^\Delta(a) = 0$. Similarly, set $c := \dfrac{r_2}{\gamma\phi(\sigma^2(b)) + \delta\phi^\Delta(\sigma(b))} \ge 0$, then $\gamma\widetilde{w}(\sigma^2(b)) + \delta\widetilde{w}^\Delta(\sigma(b)) = 0$. Note,

$$\begin{aligned} \widetilde{w}(t) &= w(t) + c\phi(t) + d\psi(t) \\ &= \int_a^{\sigma(b)} G(t,s) L\widetilde{w}(s) \Delta s \\ &= \int_a^{\sigma(b)} G(t,s) L w(s) \Delta s \\ &\le \lambda \int_a^{\sigma(b)} G(t,s) f(s, w^\sigma(s)) \Delta s \\ &= T_\lambda w(t) = w_1(t). \end{aligned}$$

Since $c\phi + d\psi \ge 0$, we will generate the same set of iterates as above. Likewise, with

$L\widetilde{v} = Lv$ and $\widetilde{v}$ modified to satisfy the homogeneous boundary conditions, we get

$$\begin{aligned}\widetilde{v}(t) &= v(t) + \tilde{c}\phi(t) + \tilde{d}\psi(t)\\ &= \int_a^{\sigma(b)} G(t,s)L\widetilde{v}(s)\Delta s\\ &= \int_a^{\sigma(b)} G(t,s)Lv(s)\Delta s\\ &\geq \lambda \int_a^{\sigma(b)} G(t,s)f(s,v^{\sigma}(s))\Delta s\\ &= T_\lambda v(t) =: v_1(t).\end{aligned}$$

This also gives the same sequence of iterates. Now let $n \to \infty$ to get

$$w(t) \leq w^*(t) := \lim_{n\to\infty} w_n(t) \leq v^*(t) := \lim_{n\to\infty} v_n(t) \leq v(t).$$

By the dominated convergence theorem (see Peterson and Thompson [19] for a more general case), we have $w^* = T_\lambda w^*$ and $v^* = T_\lambda v^*$, so both w^* and v^* are solutions to (1.1)–(1.2), with possibly $w^* = v^*$. ■

Lemma 2.3. Suppose (H_1) and (H_2) hold. Let $p \geq 0$ be a continuous function with $p(t) \not\equiv 0$ on (a,b) (on $(a,b]$ if f is not singular at b) and $\int_a^{\sigma(b)} p(s)\Delta s < \infty$. Then the BVP

$$Ly(t) = p(t) \tag{2.3}$$

$$\begin{cases}\alpha y(a) - \beta y^{\Delta}(a) & = 0\\ \gamma y(\sigma^2(b)) + \delta y^{\Delta}(\sigma(b)) & = 0,\end{cases} \tag{2.4}$$

has a unique solution y satisfying

$$c_1 y(t) \leq \psi(t)\phi(t) \leq c_2 y(t)$$

for $a \leq t \leq \sigma^2(b)$, where c_1 and c_2 are positive constants.

Proof. Note that $y(t) = \int_a^{\sigma(b)} G(t,s)p(s)\Delta s$ solves the BVP (2.3)–(2.4) and is the unique solution, since the homogeneous equation has only the trivial solution. Pick

$t_0 \in (a, b)$ such that $p(t_0) > 0$. Then for $a < t \leq t_0$ we have

$$\begin{aligned} y(t) &= \int_a^{\sigma(b)} G(t,s)p(s)\Delta s \\ &= \frac{1}{c}\int_a^t \phi^\sigma(s)\psi(t)p(s)\Delta s + \frac{1}{c}\int_t^{\sigma(b)} \phi(t)\psi^\sigma(s)p(s)\Delta s \\ &\geq \frac{\phi(t)}{c}\int_t^{\sigma(b)} \psi^\sigma(s)p(s)\Delta s \\ &\geq \frac{\phi(t)\psi(t)}{c\psi(a)}\int_{t_0}^{\sigma(b)} \psi^\sigma(s)p(s)\Delta s \end{aligned}$$

since ψ is monotone decreasing. Similarly, for $t_0 \leq t < \sigma^2(b)$,

$$\begin{aligned} y(t) &= \int_a^{\sigma(b)} G(t,s)p(s)\Delta s \\ &= \frac{1}{c}\int_a^t \phi^\sigma(s)\psi(t)p(s)\Delta s + \frac{1}{c}\int_t^{\sigma(b)} \phi(t)\psi^\sigma(s)p(s)\Delta s \\ &\geq \frac{\psi(t)}{c}\int_a^t \phi^\sigma(s)p(s)\Delta s \\ &\geq \frac{\phi(t)\psi(t)}{c\phi(\sigma^2(b))}\int_a^{t_0} \phi^\sigma(s)p(s)\Delta s \end{aligned}$$

since ϕ is monotone increasing. Define c_2 by

$$\frac{1}{c_2} := \min\left\{\frac{1}{c\phi(\sigma^2(b))}\int_a^{t_0} \phi^\sigma(s)p(s)\Delta s, \frac{1}{c\psi(a)}\int_{t_0}^{\sigma(b)} \psi^\sigma(s)p(s)\Delta s\right\}.$$

Then $y(t) \geq \dfrac{\psi(t)\phi(t)}{c_2}$. Also,

$$\begin{aligned} y(t) &= \frac{1}{c}\int_a^t \phi^\sigma(s)\psi(t)p(s)\Delta s + \frac{1}{c}\int_t^{\sigma(b)} \phi(t)\psi^\sigma(s)p(s)\Delta s \\ &\leq \frac{1}{c}\psi(t)\int_a^t \phi^\sigma(s)p(s)\Delta s + \frac{1}{c}\phi(t)\psi^\sigma(t)\int_t^{\sigma(b)} p(s)\Delta s \\ &\leq \frac{1}{c}\phi(t)\psi(t)\int_a^t p(s)\Delta s + \frac{1}{c}\phi(t)\psi(t)\int_t^{\sigma(b)} p(s)\Delta s \\ &= \frac{\phi(t)\psi(t)}{c_1}, \end{aligned}$$

where $c_1 := \dfrac{c}{\int_a^{\sigma(b)} p(s)\Delta s}$. Hence the lemma holds. ■

Lemma 2.4. $0 \le G(t,s) \le G(\sigma(s),s)$ for all $a \le t \le \sigma^2(b)$, $a \le s \le b$.

Proof. From equation (1.3), we have

$$G(t,s) = \frac{1}{c}\begin{cases} \phi(t)\psi^{\sigma}(s) & t \le s \\ \phi^{\sigma}(s)\psi(t) & \sigma(s) \le t. \end{cases}$$

We have shown $\phi, \psi \ge 0$ in Lemma 1.1, therefore we have $G(t,s) \ge 0$ for all $a \le t \le \sigma^2(b)$, $a \le s \le b$. Now,

$$G(\sigma(s),s) = \frac{1}{c}\phi(\sigma(s))\psi(\sigma(s)).$$

Thus, if $t \le s$, we have

$$G(t,s) = \frac{1}{c}\phi(t)\psi(\sigma(s)) \le \frac{1}{c}\phi(\sigma(s))\psi(\sigma(s)) = G(\sigma(s),s).$$

If $t \ge \sigma(s)$, we have

$$G(t,s) = \frac{1}{c}\phi(\sigma(s))\psi(t) \le \frac{1}{c}\phi(\sigma(s))\psi(\sigma(s)) = G(\sigma(s),s).$$

Hence, the statement of the lemma holds. ■

Fix $0 < \xi < 1$ and choose $t_1, t_2 \in \mathbb{T}$ with $a < t_1 < t_2 \le \sigma(t_2) < \sigma^2(b)$ such that

$$G(t,s) \ge \xi G(\sigma(s),s) \text{ for } t \in [t_1, \sigma(t_2)]. \tag{2.5}$$

We define the interval $I := [t_1, t_2]$ and a cone $K \subseteq C[a, \sigma^2(b)]$ by

$$K := \left\{ u \in C[a,\sigma^2(b)] : u(t) \ge 0, \text{ and } \min_{t\in I} u^{\sigma}(t) \ge \xi\|u\| \right\},$$

where $\|\cdot\|$ is the sup norm on $C[a,\sigma^2(b)]$.

Lemma 2.5. If (H_2) holds, then $T_\lambda(K) \subseteq K$.

Proof. Note that as $G(t,s) \ge 0$ for all $t \in [a,\sigma^2(b)]$ and $s \in [a,b]$, and $f(t,u) \ge 0$ for $t \in (a,b)$ and $u \ge 0$, we get $(T_\lambda u)(t) \ge 0$ for all $t \in [a,\sigma^2(b)]$. Let $t \in I$ and $u \in K$,

then

$$\begin{aligned}(T_\lambda u)^\sigma(t) &= \lambda \int_a^{\sigma(b)} G(\sigma(t), s) f(s, u^\sigma(s)) \Delta s \\ &\geq \xi \lambda \int_a^{\sigma(b)} G(\sigma(s), s) f(s, u^\sigma(s)) \Delta s \\ &= \xi \max_{t \in [a, \sigma^2(b)]} \lambda \int_a^{\sigma(b)} G(t, s) f(s, u^\sigma(s)) \Delta s \\ &= \xi \|T_\lambda u\|.\end{aligned}$$

The proof is complete. ■

We now state the following fixed-point theorem of cone compression/expansion type due to Krasnolselskii.

Lemma 2.6. Let E be a Banach space, $K \subseteq E$ a cone and assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(i) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$ or

(ii) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7. Assume (H_1), (H_2) and (H_4) hold. Then there exists $\lambda_0 > 0$ such that (1.1)–(1.2) has a positive solution z for $\lambda = \lambda_0$.

Proof. Let $R > 0$ be fixed. Choose λ_0 such that $\lambda_0 \int_a^{\sigma(b)} G(\sigma(s), s) g_R(s) \Delta s \leq R$. Then for $z \in \partial K_R$,

$$\begin{aligned}\|T_{\lambda_0} z\| &= \lambda_0 \int_a^{\sigma(b)} G(\sigma(s), s) f(s, z^\sigma(s)) \Delta s \\ &\leq \lambda_0 \int_a^{\sigma(b)} G(\sigma(s), s) g_R(s) \Delta s \\ &\leq R = \|z\|.\end{aligned}$$

Now, fix $\tau_0 \in I$, and pick $M > 0$ sufficiently large so that

$$\xi \lambda_0 M \int_I G(\tau_0, s) p_1(s) \Delta s \geq 1,$$

where p_1 is the function in (H_4). Now let $r = R_M$ in (H_4). So for $0 \le u \le r$, we have $f(t,u) \ge Mp_1(t)u$. Then for $z \in \partial K_r$,

$$\begin{aligned}(T_{\lambda_0}z)(\tau_0) &= \lambda_0 \int_a^{\sigma(b)} G(\tau_0, s) f(s, z^\sigma(s))\Delta s \\ &\ge \lambda_0 M \int_I G(\tau_0, s) p_1(s) z^\sigma(s)\Delta s \\ &\ge \lambda_0 M \xi \|z\| \int_I G(\tau_0, s) p_1(s)\Delta s \\ &\ge \|z\|.\end{aligned}$$

The proof is complete. ■

3. Main Results

Theorem 3.1. Assume (H_1)–(H_4) hold. Then there exists $\lambda_0 > 0$ such that (1.1)–(1.2) has at least one positive solution for $0 < \lambda < \lambda_0$.

Proof. By the previous lemma, there exists $\lambda_0 > 0$ such that (1.1)–(1.2) has a positive solution z for $\lambda = \lambda_0$. Now suppose $0 < \lambda < \lambda_0$. Then $Lz(t) = \lambda_0 f(t, z^\sigma(t)) \ge \lambda f(t, z^\sigma(t))$. Also, $w(t) \equiv 0$ solves $Lw = 0 \le \lambda f(t, 0)$. Also, both z and $w = 0$ satisfy the boundary conditions. So by Lemma 2.2 there is a nonnegative solution z^* to (1.1)–(1.2). Now, if $\alpha > 0$ or $\beta > 0$, then $z^* > 0$ on (a, t_0).

Claim: $t_0 = \sigma^2(b)$.

If not, $z^*(t_0) = 0 = \lambda \int_a^{\sigma(b)} G(t_0, s) f(s, z^{*\sigma}(s))\Delta s$. But by ($H_4$), there is $\eta > 0$ such that $f(t, w) \ge p_1(t)w$ for $0 < w \le \eta$, $t \in (a, b)$. Define $f_\eta(t, u) := \min\{f(t, u), f(t, \eta)\}$. Then, $f_\eta(t, u) \ge p_1(s)\min\{u, \eta\}$. Let $I = [t_1, t_2]$ so that (2.5) holds. If $\min_{s\in I} z^{*\sigma}(s) < \eta$, we have

$$\begin{aligned}0 = z^*(t_0) &= \lambda \int_a^{\sigma(b)} G(t_0, s) f(s, z^{*\sigma}(s))\Delta s \\ &\ge \lambda \int_I G(t_0, s) f(s, z^{*\sigma}(s))\Delta s \\ &\ge \lambda \int_I G(t_0, s) f_\eta(s, z^{*\sigma}(s))\Delta s \\ &\ge \lambda \int_I G(t_0, s) p_1(s) \min_{t\in I} z^{*\sigma}(t)\Delta s\end{aligned}$$

$$\geq \lambda\xi\|z^*\|\int_I G(t_0,s)p_1(s)\Delta s$$
$$> 0$$

which is a contradiction. If $\min_{s\in I} z^{*\sigma}(s) \geq \eta$, we have

$$\begin{aligned}
0 = z^*(t_0) &= \lambda\int_a^{\sigma(b)} G(t_0,s)f(s,z^{*\sigma}(s))\Delta s \\
&\geq \lambda\int_I G(t_0,s)f(s,z^{*\sigma}(s))\Delta s \\
&\geq \lambda\int_I G(t_0,s)f_\eta(s,z^{*\sigma}(s))\Delta s \\
&\geq \lambda\int_I G(t_0,s)p_1(s)\eta\Delta s \\
&\geq \lambda\eta\int_I G(t_0,s)p_1(s)\Delta s \\
&> 0
\end{aligned}$$

which is also a contradiction. Hence $z^*(t) > 0$ on $(a,\sigma^2(b))$. ■

Theorem 3.2. Assume (H_1)–(H_5) hold. Then there exists $\lambda^* > 0$ such that the BVP (1.1)–(1.2) has at least one positive solution for $0 < \lambda < \lambda^*$ and no positive solution for $\lambda > \lambda^*$.

Proof. Let B be the set of all $\lambda > 0$ for which the BVP (1.1)–(1.2) has a positive solution in $\mathcal{D}$. Then by Lemma 2.7, we have that $B \neq \emptyset$ and Theorem 3.1 shows that if $\lambda_0 \in B$, then $(0,\lambda_0] \subseteq B$. To show B is bounded, let $\lambda \in B$ and let u_λ be the corresponding positive solution. By (H_5), there exists $e > 0$ and a continuous function p_2 such that $f(t,z) \geq ep_2(t)z$ for all $(t,z) \in (a,\sigma^2(b)) \times [0,\infty)$ and $\int_a^{\sigma(b)} G(\sigma(s),s)p_2(s)\Delta s < \infty$. Now let y be the solution to the BVP (2.3)–(2.4) with $p(t) = \phi^\sigma(t)\psi^\sigma(t)p_2(t)$. Then by Lemma 2.3, we have $\phi(t)\psi(t) \leq c_2 y(t)$ and

$$\begin{aligned}
\int_a^{\sigma(b)} y^\sigma(t)Lu_\lambda(t)\Delta t &= \lambda\int_a^{\sigma(b)} y^\sigma(t)f(t,u_\lambda^\sigma(t))\Delta t \\
&\geq e\lambda\int_a^{\sigma(b)} y^\sigma(t)u_\lambda^\sigma(t)p_2(t)\Delta t.
\end{aligned}$$

On the other hand, by Green's formula ([3, Theorem 4.94]), we have

$$\begin{aligned}\int_a^{\sigma(b)} y^\sigma(t)Lu_\lambda(t)\Delta t &= \int_a^{\sigma(b)} u_\lambda^\sigma(t)Ly(t)\Delta t \\ &= \int_a^{\sigma(b)} u_\lambda^\sigma(t)\phi^\sigma(t)\psi^\sigma(t)p_2(t)\Delta t \\ &\le c_2\int_a^{\sigma(b)} y^\sigma(t)u_\lambda^\sigma(t)p_2(t)\Delta t.\end{aligned}$$

So $e\lambda \le c_2$. Hence $\lambda \le \dfrac{c_2}{e}$. Thus, B is bounded. Setting $\lambda^* = \sup B$, we get that for $0 < \lambda < \lambda^*$ the BVP (1.1)–(1.2) has a positive solution, and for $\lambda > \lambda^*$ there is no positive solution. ■

4. Example

Suppose that $q > 1$ and $\mathbb{T} = \left\{\dfrac{1}{q^n} : n \in \mathbb{N}_0\right\} \cup \{0\}$. The singular BVP

$$-u^{\Delta\Delta} = \lambda f(t, u^\sigma), \qquad 0 < t < 1/q^2 \tag{4.1}$$

$$u(0) = 0 = u(1) \tag{4.2}$$

satisfies Theorems 3.1 and 3.2, for

$$f(t, z) = (t(1-t))^{-2+1/(z+1)}h(z),$$

(note that f is singular at $t = 0$ and $t = 1$), where

$$h(z) = \begin{cases}\sqrt{z} & 0 \le z \le 1 \\ z^2 & z \ge 1.\end{cases}$$

First, note that $a = 0$, $\sigma^2(b) = 1$, $\alpha = \gamma = 1$ and $\beta = \delta = 0$. Then $\rho = 1 > 0$, and Green's function for (4.1)–(4.2) is

$$G(t, s) = \begin{cases}t(1-\sigma(s)) & t \le s \\ \sigma(s)(1-t) & \sigma(s) \le t.\end{cases}$$

Note that $f(t, z) \ge 0$ on $(0, 1) \times [0, \infty)$, and is nondecreasing in z. Thus (H_1) and (H_3) are satisfied. Also,

$$f(t, z) \le g_M(t) := M^2(t(1-t))^{-2+1/(M+1)}$$

for $(t, z) \in (0, 1) \times [0, M]$. Also,

$$\begin{aligned}
&\int_a^{\sigma(b)} G(\sigma(s), s) g_M(s) \Delta s \\
&= M^2 \int_0^{1/q} G(\sigma(s), s)[s(1-s)]^{-2+1/(M+1)} \Delta s \\
&= M^2 \int_0^{1/q} \sigma(s)(1-\sigma(s))[s(1-s)]^{-2+1/(M+1)} \Delta s \\
&= M^2 \sum_{k=2}^{\infty} \frac{1}{q^{k-1}} \left(1 - \frac{1}{q^{k-1}}\right) \left(\frac{1}{q^k}\left(1 - \frac{1}{q^k}\right)\right)^{-2+1/(M+1)} \left(\frac{q-1}{q^k}\right) \\
&= M^2 \sum_{k=2}^{\infty} \left(\frac{q^{k-1}-1}{q^{2k-2}}\right) \left(\frac{q^k-1}{q^{2k}}\right)^{-2+1/(M+1)} \left(\frac{q-1}{q^k}\right) \\
&= M^2 q^2 \sum_{k=2}^{\infty} \left(\frac{q^{k-1}-1}{q^{2k}}\right) \left(\frac{q-1}{q^k}\right) \left(\frac{q^{2k}}{q^k-1}\right)^{2-1/(M+1)} \\
&= M^2 q^2 \sum_{k=2}^{\infty} \left(\frac{q^{k-1}-1}{q^{2k}}\right) \left(\frac{q-1}{q^k}\right) \left(\frac{q^{2k}}{q^k-1}\right)^{2} \left(\frac{q^k-1}{q^{2k}}\right)^{1/(M+1)} \\
&\le M^2 q^2 \sum_{k=2}^{\infty} \left(\frac{q^{k-1}}{q^{2k}}\right) \left(\frac{q}{q^k}\right) \left(\frac{q^k}{q^k-1}\right)^{2} q^{2k} \left(\frac{q^k}{q^{2k}}\right)^{1/(M+1)} \\
&= M^2 q^2 \sum_{k=2}^{\infty} \left(\frac{q^k}{q^k-1}\right)^{2} \left(\frac{1}{q^k}\right)^{1/(M+1)} .
\end{aligned}$$

Now, as the term $\left(\frac{q^k}{q^k-1}\right)^2$ approaches 1 as $k \to \infty$, there is K such that for $k \ge K$, we have $\left(\frac{q^k}{q^k-1}\right)^2 < 2$. Therefore, we have:

$$\begin{aligned}
\int_0^{1/q} G(\sigma(s), s) g_M(s) \Delta s &\le M^2 q^2 \sum_{k=2}^{K} \left(\frac{q^k}{q^k-1}\right)^2 \left(\frac{1}{q^k}\right)^{1/(M+1)} \\
&\quad + 2M^2 q^2 \sum_{k=K+1}^{\infty} (q^{-1/(M+1)})^k \\
&< \infty
\end{aligned}$$

since the last term is a convergent geometric series. Thus, (H_2) is satisfied.

Next, we show (H_4) holds. To see this, let $d > 0$ be given and let $0 < z \leq R_d$, where $R_d := \dfrac{1}{d^2}$. It follows that $z = \dfrac{1}{c^2}$ for some $c \geq d$. Then, for $t \in (0, 1) \cap \mathbb{T}$,

$$\begin{aligned}
f(t, z) &= (t(1-t))^{-2+1/(z+1)} h(z) \\
&= (t(1-t))^{-1} (t(1-t))^{-z/(z+1)} h(z) \\
&\geq \frac{1}{t(1-t)} h(z) \\
&\geq \frac{1}{t(1-t)} \sqrt{z} \\
&= \frac{1}{t(1-t)} \frac{1}{c} \\
&= c \frac{1}{t(1-t)} z \\
&\geq d \frac{1}{t(1-t)} z \\
&\geq d p_1(t) z,
\end{aligned}$$

where $p_1(t) := \dfrac{1}{t(1-t)}$. So ($H_4$) holds.

Finally, we show (H_5) holds. First note that

$$f(t, z) \geq (t(1-t))^{-1} h(z) \geq (t(1-t))^{-1} z \ = p_2(t) z,$$

where $p_2(t) = \dfrac{1}{t(1-t)}$. Also

$$\begin{aligned}
\int_a^{\sigma(b)} G(\sigma(s), s) p_2(s) \Delta s &= \int_0^{1/q} \sigma(s)(1-\sigma(s))(s(1-s))^{-1} \Delta s \\
&= \sum_{k=2}^{\infty} \frac{1}{q^{k-1}} \left(1 - \frac{1}{q^{k-1}}\right) \left(\frac{1}{q^k}\left(1 - \frac{1}{q^k}\right)\right)^{-1} \frac{q-1}{q^k} \\
&= \sum_{k=2}^{\infty} \frac{q^k (1 - 1/q^{k-1})}{q^{k-1}(1 - 1/q^k)} \left(\frac{q-1}{q^k}\right)
\end{aligned}$$

$$
\begin{aligned}
&= q(q-1)\sum_{k=2}^{\infty}\frac{q^{k-1}-1}{q^k-1}\frac{q^k}{q^{k-1}}\frac{1}{q^k}\\
&= q(q-1)\sum_{k=2}^{\infty}\frac{q^k-q}{q^k-1}\frac{1}{q^k}\\
&\le q(q-1)\sum_{k=2}^{\infty}q^{-k}\\
&= 1 < \infty.
\end{aligned}
$$

Thus, (H_5) holds.

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