Positive Solutions to a Singular Second Order Boundary Value Problem

L. Erbe and A. Peterson
Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, USA
E-mail: lerbe2@math.unl.edu, apetersol1@math.unl.edu

J. Weiss
Department of Mathematics and Statistics, University of Nebraska–Kearney, Kearney, NE 68847, USA
E-mail: weissjj@unk.edu

Abstract

In this paper, we establish some criteria for the existence of positive solutions for certain two point boundary value problems for the singular nonlinear second order equation

\[-(ru^{\Delta})^{\Delta} + qu^{\sigma} = \lambda f(t, u^{\sigma})\]

on a time scale \(\mathbb{T}\). As a special case when \(\mathbb{T} = \mathbb{R}\), our results include those of Erbe and Mathsen [11]. Our results are new in a general time scale setting and can be applied to difference and \(q\)-difference equations.

Keywords: Dynamic equation, time scale, positive solution.

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1. Introduction

In this paper we consider the dynamic equation

\[ Lu = \lambda f(t, u^\sigma) \]  

(1.1)

with the boundary conditions

\[
\begin{align*}
\alpha u(a) - \beta u(\Delta a) &= 0 \\
\gamma u(\sigma^2(b)) + \delta u(\Delta \sigma(b)) &= 0,
\end{align*}
\]

(1.2)

where the operator \( L \) is defined by

\[ Lu := -(ru/\Delta a)/\Delta a + qu^\sigma. \]

The domain \( D \) of \( L \) is the set of functions \( u : \mathbb{T} \rightarrow \mathbb{R} \) such that \( u \) is continuous on \([a, \sigma^2(b)]\), \( u^\Delta \) is continuous on \([a, \sigma(b)]\), and \((ru^\Delta)^\Delta\) is rd-continuous on \([a, b]\). Since we shall be interested in the case when \( f \) may have singularities at one or both of the endpoints, we shall assume, either \( f \) is continuous on \((a, b)\times\mathbb{R}\), if \( f \) is singular at \( b \) or \( f \) is continuous on \((a, b)\times\mathbb{R}\) if \( f \) is not singular at \( b \). In addition, we assume \( r > 0 \) is rd-continuous, \( q \geq 0 \) is rd-continuous, \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \rho := \gamma \beta (\sigma^2(b) - a) + \alpha \gamma + \alpha \delta > 0 \). Special cases of this problem have been studied by many authors (see [8,11] and the references therein). Our concern is to find conditions on \( f \) so that (1.1)-(1.2) has a positive solution for some \( \lambda > 0 \). If \( f \) has a singularity at \( b \), then \( \rho(b) = b \) and in this case we assume throughout this paper that \( \sigma(b) = \sigma^2(b) = b \).

Green’s function for (1.1)–(1.2) is

\[ G(t, s) = \frac{1}{c} \begin{cases} 
\phi(t)\psi^\sigma(s) & t \leq s \\
\phi^\sigma(s)\psi(t) & \sigma(s) \leq t
\end{cases} \]

(1.3)

where \( \phi \) and \( \psi \) are the solutions of

\[
\begin{align*}
L\phi &= 0, & \phi(a) = \beta, & \phi^\Delta(a) = \alpha \\
L\psi &= 0, & \psi(\sigma^2(b)) = \delta, & \psi^\Delta(\sigma(b)) = -\gamma
\end{align*}
\]

respectively.

Lemma 1.1. We have \( \phi(t) > 0 \) for all \( t \in (a, \sigma^2(b)) \) and \( \phi^\Delta(t) \geq 0 \) for all \( t \in [a, \sigma(b)] \). We have \( \psi(t) > 0 \) and \( \psi^\Delta(t) \leq 0 \) for all \( t \in [a, \sigma^2(b)] \).

Proof. First, notice that

\[
(r(t)\phi^\Delta(t))^\Delta + q(t)\phi^\sigma(t) = 0
\]

and

\[
(r(t)\phi^\Delta(t))^\Delta = q(t)\phi^\sigma(t),
\]

\[
r(t)\phi^\Delta(t) - r(a)\phi^\Delta(a) = \int_a^t q(s)\phi^\sigma(s)\Delta s
\]

\[
\phi^\Delta(t) = \frac{1}{r(t)} \left[ r(a)\alpha + \int_a^t q(s)\phi^\sigma(s)\Delta s \right].
\]
Notice that \( \phi(a) = \beta \geq 0 \). We claim that \( \phi(t) > 0 \) on \( (a, \sigma^2(b)) \). Now suppose for the purpose of contradiction that \( \phi(t) \leq 0 \) for some \( t \in (a, \sigma^2(b)) \).

**Case 1:** \( \alpha > 0 \): If \( \alpha > 0 \), then \( \phi'(a) = \alpha > 0 \), so there is \( k \in \mathbb{T}, k > a \) such that \( \phi(t) > \phi(a) \) for all \( t \in (a, k] \). Suppose that \( t_0 > k \) is the smallest point bigger than \( a \) in \( \mathbb{T} \) such that \( \phi(t_0) \leq \phi(a) \).

If \( t_0 \) is left-scattered, then \( \phi'(\rho(t_0)) < 0 \), and hence \( \int_a^{\rho(t_0)} q(s)\phi'(s)\Delta s < 0 \). But then there is a point \( t_1 \in [k, \rho(t_0)) \) such that \( \phi'(t_1) < 0 \). That means that there is \( t_2 \in [k, t_0) \) such that \( \phi(t_2) < 0 \). This contradicts the fact that \( \phi(t) > 0 \) on \( [k, t_0) \).

If \( t_0 \) is left-dense, then there is a \( t_1 < t_0 \) with \( \phi'(t_1) < 0 \). Then \( \int_a^{t_0} q(s)\phi'(s)\Delta s < 0 \), and so \( \phi'(t_2) < 0 \) for some \( t_2 \in [a, t_1] \). But then as \( \sigma(t_1) < t_0 \), we get that \( \phi(t_3) < 0 \) for some \( t_3 \in [\sigma(a), t_0) \). This is a contradiction.

**Case 2:** \( \alpha = 0 \): In this case, \( \phi(a) = \beta > 0 \). Let \( k \) be such that \( \phi(t) > 0 \) on \( [a, k] \). Let \( t_0 \) be the smallest point in \( \mathbb{T} \) greater than \( a \) such that \( \phi(t_0) \leq 0 \). Similar to the proof of Case 1, we get a contradiction.

Hence, \( \phi(t) > 0 \) on \( (a, \sigma^2(b)) \), and thus \( \phi'(t) > 0 \) on \( (a, \sigma^2(b)) \). The second statement in this lemma can be proven similarly.

It follows from Lemma 1.1 that

\[
 c := -r(t) \left[ \phi'(t)\psi'(t) - \phi'(t)\psi(t) \right] > 0.
\]

The BVP (1.1)–(1.2) is equivalent to the integral equation

\[
 u(t) = \lambda \int_a^{\sigma(b)} G(t, s) f(s, u(s)) \Delta s,
\]

and so we look for positive fixed points of the operator \( T_\lambda \) defined by:

\[
(T_\lambda u)(t) = \lambda \int_a^{\sigma(b)} G(t, s) f(s, u(s)) \Delta s
\]

for \( t \in [a, \sigma^2(b)] \).

We shall impose some of the following conditions on \( f \) (if \( f \) is not singular at \( b \), then we assume \( (a, b) \) is replaced by \( (a, b] \) in each of the following):

(H\(_1\)) \( f(t, u) \geq 0 \) for \( (t, u) \in (a, b) \times [0, \infty) \).

(H\(_2\)) For each \( M > 0 \) there exists a function \( g_M : (a, b) \to \mathbb{R}^+ \) such that \( f(t, z) \leq g_M(t) \) for \( (t, z) \in (a, b) \times [0, M] \) and \( \int_a^{\sigma(b)} G(s, s)g_M(s) \Delta s < \infty \).

(H\(_3\)) For each \( t \in (a, b) \), \( f(t, u) \) is nondecreasing in \( u \).
(H4) There exists a continuous function $p_1 \geq 0, p_1 \not\equiv 0$ on any time scale subinterval of $(a, b)$, such that for each constant $\delta > 0$, there is a $R_\delta > 0$ with $f(t, z) \geq dp_1(t)z$ for $(t, z) \in (a, b) \times (0, R_\delta]$.

(H5) There exists $e > 0$ and a continuous function $p_2 : (a, b) \to \mathbb{R}^+$ such that $f(t, z) \geq ep_2(t)z$ for $(t, z) \in (a, b) \times [0, \infty)$ and $\int_a^{\sigma(b)} G(\sigma(s), s)p_2(s)\Delta s < \infty$.

2. Preliminary Lemmas

First, let $E$ be the Banach space $E := \{ u : [a, \sigma^2(b)] \to \mathbb{R} : u$ is continuous$\}$, where the norm is the sup norm. We define the cone $P$ by $P := \{ u \in E : u(t) \geq 0 \text{ for all } t \in [a, \sigma^2(b)] \}$. Now we state some preliminary lemmas.

**Lemma 2.1.** Assume (H1) and (H2) hold. Then $T_\lambda : P \to P$ is compact.

**Proof.** (H1) and (H2) imply that $T_\lambda$ exists for $u \in P$ and $T_\lambda u(t) \geq 0$. If $\sigma(a) = a$, we define

$t_n := \sup \left\{ t \in \mathbb{T} : a \leq t \leq a + \frac{1}{n} \right\}$.

Note that $t_n > a$. If $\rho(b) = b$, we define

$\tau_n := \inf \left\{ t \in \mathbb{T} : b - \frac{1}{n} \leq t \leq b \right\}$.

Note that $\tau_n < b$. In the case where $\sigma(a) = a$ and $\rho(b) = b$, define

$$f_n(t, x) = \begin{cases} 
\min\{f(t, x), f(t_n, x)\} & a < t \leq a + \frac{1}{n} \\
f(t, x) & a + \frac{1}{n} \leq t \leq b - \frac{1}{n} \\
\min\{f(t, x), f(\tau_n, x)\} & b - \frac{1}{n} \leq t < b.
\end{cases}$$

For $\sigma(a) = a$, and $\rho(b) < b$, define

$$f_n(t, x) = \begin{cases} 
\min\{f(t, x), f(t_n, x)\} & a < t \leq a + \frac{1}{n} \\
f(t, x) & a + \frac{1}{n} \leq t < b.
\end{cases}$$

And finally, for $\sigma(a) > a$ and $\rho(b) = b$, define

$$f_n(t, x) = \begin{cases} 
f(t, x) & a < t \leq b - \frac{1}{n} \\
\min\{f(t, x), f(\tau_n, x)\} & b - \frac{1}{n} \leq t < b.
\end{cases}$$
For $u \in E$, $f_n(t, u^\sigma(t))$ is nonnegative and rd-continuous on $(a, b)$. For each $\lambda > 0$, define an operator $T_n$ by

$$(T_nu)(t) = \lambda \int_a^\sigma(b) G(t, s) f_n(s, u^\sigma(s)) \Delta s,$$

for $t \in [a, \sigma^2(b)]$. Then $(T_nu)(t) \geq 0$ on $[a, \sigma^2(b)]$ by $(H_1)$. If $t, t_0 \in [a, \sigma^2(b)]$, then

$$|T_nu(t) - T_nu(t_0)| \leq \lambda \int_a^\sigma(b) |G(t, s) - G(t_0, s)| f_n(s, u^\sigma(s)) \Delta s$$

which goes to 0 as $t \to t_0$ by continuity of $G$. Hence $T_nu \in E$. Let $u \in E$ and $\|u\| \leq M$. Then

$$(T_nu)^\Delta(t) = \frac{\lambda}{c} \psi^\Delta(t) \int_a^t \phi^\sigma(s) f_n(s, u^\sigma(s)) \Delta s + \frac{\lambda}{c} \phi^\Delta(t) \int_t^\sigma(b) \psi^\sigma(s) f_n(s, u^\sigma(s)) \Delta s.$$ 

Hence, $(T_nu)^\Delta(t)$ is continuous and uniformly bounded for $\|u\| \leq M$. Thus, $T_n$ is a compact operator on $K$ by the Ascoli–Arzela theorem. Moreover, $T_n \to T_\lambda$ uniformly as $n \to \infty$ on any bounded subset of $K$. We show this only for the case $\sigma(a) = a$ and $\rho(b) = b$.

$$(T_nu)^\Delta(t) - T_nu(t_0) \leq \lambda \int_a^{t_0} G(t, s)|f(s, u^\sigma(s)) - f(t_n, u^\sigma(s))| \Delta s
+ \lambda \int_{t_0}^{\sigma(b)} G(t, s)|f(s, u^\sigma(s)) - f(t_n, u^\sigma(s))| \Delta s
\leq 2\lambda \left( \int_a^{t_n} G(t, s)g_M(s) \Delta s + \int_{t_n}^{\sigma(b)} G(t, s)g_M(s) \Delta s \right)
\leq 2\lambda \left( \int_a^{t_n} G(\sigma(s), s)g_M(s) \Delta s + \int_{t_n}^{\sigma(b)} G(\sigma(s), s)g_M(s) \Delta s \right)
\to 0 \text{ as } n \to \infty$$

(note here we use the fact that we assumed when $\rho(b) = b$ it follows that $\sigma(b) = b$). This implies $T_\lambda$ is compact on $K$ (see [5, Proposition 4.2]).

**Lemma 2.2.** Let $\lambda > 0$ and $(H_1)$–$(H_3)$ hold. Suppose further that there exist $w, v \in D$ with $0 \leq w(t) \leq v(t), Lw(t) \leq \lambda f(t, w^\sigma(t)), Lv(t) \geq \lambda f(t, v^\sigma(t))$ on $(a, b)$ (on $(a, b]$ if $f$ is not singular at $b$) and

$$\alpha w(a) - \beta w^\Delta(a) \leq 0, \quad \gamma w(\sigma^2(b)) + \delta w^\Delta(\sigma(b)) \leq 0 \quad (2.1)$$

$$\alpha v(a) - \beta v^\Delta(a) \geq 0, \quad \gamma v(\sigma^2(b)) + \delta v^\Delta(\sigma(b)) \geq 0. \quad (2.2)$$
Then (1.1)–(1.2) has a solution \( z \in D \) with \( w(t) \leq z(t) \leq v(t) \) on \([a, \sigma^2(b)]\).

**Proof.** Condition (H3) implies that \( T_\lambda \) is monotone nondecreasing with respect to the cone \( K \). Suppose first that equality holds in all four equations in (2.1)–(2.2). Let \( w_0 = w \) and let \( w_{n+1} = T_\lambda w_n \). Then

\[
w_0(t) = w(t) = \int_a^{\sigma(b)} G(t, s)Lw(s)\Delta s \\
\leq \lambda \int_a^{\sigma(b)} G(t, s)f(s, w^\sigma(s))\Delta s \\
= T_\lambda w(t) = w_1(t)
\]

for \( t \in [a, \sigma^2(b)] \). By induction, the fact that \( T_\lambda \) is monotone nondecreasing gives us that \( w_n(t) \leq w_{n+1}(t) \). Similarly, \( v_{n+1} := T_\lambda v_n \leq v_n \) where \( v_0(t) = v(t) \). Then since \( w \leq v \), we get

\[ w = w_0 \leq w_1 \leq \cdots \leq w_n \leq \cdots \leq v \leq v_1 \leq v_0 = v. \]

Now, if equality does not hold, we may modify \( w \) to a function \( \tilde{w} \) as follows: let \( r_1 := -(\alpha w(a) - \beta w^\Delta(a)) \geq 0 \) and \( r_2 := -(\gamma w(\sigma^2(b)) + \delta w^\Delta(\sigma(b))) \geq 0 \), and define \( \tilde{w} := w(t) + c\phi(t) + d\psi(t) \) where \( c \) and \( d \) are chosen below and \( \phi \) and \( \psi \) are defined earlier. Note that \( L\tilde{w}(t) = Lw(t) \). Since we have

\[ \alpha \tilde{w}(a) - \beta \tilde{w}^\Delta(a) = -r_1 + d(\alpha\psi(a) - \beta\psi^\Delta(a)), \]

if we set \( d := \frac{r_1}{\alpha\psi(a) - \beta\psi^\Delta(a)} \geq 0 \), then \( \alpha \tilde{w}(a) - \beta \tilde{w}^\Delta(a) = 0 \). Similarly, set \( c := \frac{r_2}{\gamma\phi(\sigma^2(b)) + \delta\phi^\Delta(\sigma(b))} \geq 0 \), then \( \gamma \tilde{w}(\sigma^2(b)) + \delta \tilde{w}^\Delta(\sigma(b)) = 0 \). Note,

\[
\tilde{w}(t) = w(t) + c\phi(t) + d\psi(t) \\
= \int_a^{\sigma(b)} G(t, s)L\tilde{w}(s)\Delta s \\
= \int_a^{\sigma(b)} G(t, s)Lw(s)\Delta s \\
\leq \lambda \int_a^{\sigma(b)} G(t, s)f(s, w^\sigma(s))\Delta s \\
= T_\lambda w(t) = w_1(t).
\]

Since \( c\phi + d\psi \geq 0 \), we will generate the same set of iterates as above. Likewise, with
\[ L\tilde{v} = Lv \] and \( \tilde{v} \) modified to satisfy the homogeneous boundary conditions, we get

\[
\tilde{v}(t) = v(t) + \tilde{c}\phi(t) + \tilde{d}\psi(t)
\]
\[
= \int_a^{\sigma(b)} G(t, s)L\tilde{v}(s)\Delta s
\]
\[
= \int_a^{\sigma(b)} G(t, s)Lv(s)\Delta s
\]
\[
\geq \lambda \int_a^{\sigma(b)} G(t, s)f(s, v_{\sigma}(s))\Delta s
\]
\[
= T_{\lambda}v(t) =: v_1(t).
\]

This also gives the same sequence of iterates. Now let \( n \to \infty \) to get

\[
w(t) \leq w^*(t) := \lim_{n \to \infty} w_n(t) \leq v^*(t) := \lim_{n \to \infty} v_n(t) \leq v(t).
\]

By the dominated convergence theorem (see Peterson and Thompson [19] for a more general case), we have \( w^* = T_{\lambda}w^* \) and \( v^* = T_{\lambda}v^* \), so both \( w^* \) and \( v^* \) are solutions to (1.1)–(1.2), with possibly \( w^* = v^* \).

Lemma 2.3. Suppose (H1) and (H2) hold. Let \( p \geq 0 \) be a continuous function with \( p(t) \neq 0 \) on \( (a, b) \) (on \( (a, b] \) if \( f \) is not singular at \( b \)) and \( \int_a^{\sigma(b)} p(s)\Delta s < \infty \). Then the BVP

\[ Ly(t) = p(t) \tag{2.3} \]

\[
\begin{align*}
\alpha y(a) - \beta y_\Delta(a) &= 0 \\
yy(\sigma^2(b)) + \delta y_\Delta(\sigma(b)) &= 0,
\end{align*} \tag{2.4}
\]

has a unique solution \( y \) satisfying

\[
c_1y(t) \leq \psi(t)\phi(t) \leq c_2y(t)
\]

for \( a \leq t \leq \sigma^2(b) \), where \( c_1 \) and \( c_2 \) are positive constants.

Proof. Note that \( y(t) = \int_a^{\sigma(b)} G(t, s)p(s)\Delta s \) solves the BVP (2.3)–(2.4) and is the unique solution, since the homogeneous equation has only the trivial solution. Pick
$t_0 \in (a, b)$ such that $p(t_0) > 0$. Then for $a < t < t_0$ we have

$$y(t) = \int_a^{\sigma(b)} G(t, s)p(s)\Delta s$$

$$= \frac{1}{c} \int_a^t \phi^\sigma(s)\psi(t)p(s)\Delta s + \frac{1}{c} \int_t^{\sigma(b)} \phi(t)\psi^\sigma(s)p(s)\Delta s$$

$$\geq \frac{\psi(t)}{c} \int_a^t \phi^\sigma(s)p(s)\Delta s$$

$$\geq \frac{\phi(t)\psi(t)}{c\psi(a)} \int_{t_0}^{\sigma(b)} \psi^\sigma(s)p(s)\Delta s$$

since $\psi$ is monotone decreasing. Similarly, for $t_0 \leq t < \sigma^2(b),$

$$y(t) = \int_a^{\sigma(b)} G(t, s)p(s)\Delta s$$

$$= \frac{1}{c} \int_a^t \phi^\sigma(s)\psi(t)p(s)\Delta s + \frac{1}{c} \int_t^{\sigma(b)} \phi(t)\psi^\sigma(s)p(s)\Delta s$$

$$\geq \psi(t) \int_a^t \phi^\sigma(s)p(s)\Delta s$$

$$\geq \frac{\phi(t)\psi(t)}{c\phi(\sigma^2(b))} \int_a^{t_0} \phi^\sigma(s)p(s)\Delta s$$

since $\phi$ is monotone increasing. Define $c_2$ by

$$\frac{1}{c_2} := \min \left\{ \frac{1}{c\phi(\sigma^2(b))} \int_a^{t_0} \phi^\sigma(s)p(s)\Delta s, \frac{1}{c\psi(a)} \int_{t_0}^{\sigma(b)} \psi^\sigma(s)p(s)\Delta s \right\}.$$

Then $y(t) \geq \frac{\psi(t)\phi(t)}{c_2}$. Also,

$$y(t) = \frac{1}{c} \int_a^t \phi^\sigma(s)\psi(t)p(s)\Delta s + \frac{1}{c} \int_t^{\sigma(b)} \phi(t)\psi^\sigma(s)p(s)\Delta s$$

$$\leq \frac{1}{c} \psi(t) \int_a^t \phi^\sigma(s)p(s)\Delta s + \frac{1}{c} \phi(t)\psi^\sigma(t) \int_t^{\sigma(b)} p(s)\Delta s$$

$$\leq \frac{1}{c} \phi(t)\psi(t) \int_a^t p(s)\Delta s + \frac{1}{c} \phi(t)\psi(t) \int_t^{\sigma(b)} p(s)\Delta s$$

$$= \frac{\phi(t)\psi(t)}{c_1}.$$
where $c_1 := \frac{c}{\int_a^{\sigma(b)} p(s) \Delta s}$. Hence the lemma holds. ■

**Lemma 2.4.** $0 \leq G(t, s) \leq G(\sigma(s), s)$ for all $a \leq t \leq \sigma^2(b), a \leq s \leq b$.

**Proof.** From equation (1.3), we have

$$G(t, s) = \frac{1}{c} \begin{cases} \phi(t) \psi^\sigma(s) & t \leq s \\ \phi^\sigma(s) \psi(t) & \sigma(s) \leq t. \end{cases}$$

We have shown $\phi, \psi \geq 0$ in Lemma 1.1, therefore we have $G(t, s) \geq 0$ for all $a \leq t \leq \sigma^2(b), a \leq s \leq b$. Now,

$$G(\sigma(s), s) = \frac{1}{c} \phi(\sigma(s)) \psi(\sigma(s)).$$

Thus, if $t \leq s$, we have

$$G(t, s) = \frac{1}{c} \phi(t) \psi(\sigma(s)) \leq \frac{1}{c} \phi(\sigma(s)) \psi(\sigma(s)) = G(\sigma(s), s).$$

If $t \geq \sigma(s)$, we have

$$G(t, s) = \frac{1}{c} \phi(\sigma(s)) \psi(t) \leq \frac{1}{c} \phi(\sigma(s)) \psi(\sigma(s)) = G(\sigma(s), s).$$

Hence, the statement of the lemma holds. ■

Fix $0 < \xi < 1$ and choose $t_1, t_2 \in \mathbb{T}$ with $a < t_1 < t_2 \leq \sigma(t_2) < \sigma^2(b)$ such that

$$G(t, s) \geq \xi G(\sigma(s), s) \text{ for } t \in [t_1, \sigma(t_2)]. \quad (2.5)$$

We define the interval $I := [t_1, t_2]$ and a cone $K \subseteq C[a, \sigma^2(b)]$ by

$$K := \left\{ u \in C[a, \sigma^2(b)] : u(t) \geq 0, \text{ and } \min_{t \in I} u^\sigma(t) \geq \xi \|u\| \right\},$$

where $\| \cdot \|$ is the sup norm on $C[a, \sigma^2(b)]$.

**Lemma 2.5.** If (H2) holds, then $T_\lambda(K) \subseteq K$.

**Proof.** Note that as $G(t, s) \geq 0$ for all $t \in [a, \sigma^2(b)]$ and $s \in [a, b]$, and $f(t, u) \geq 0$ for $t \in (a, b)$ and $u \geq 0$, we get $(T_\lambda u)(t) \geq 0$ for all $t \in [a, \sigma^2(b)]$. Let $t \in I$ and $u \in K$, then...
then
\[
(T_\lambda u)^\sigma(t) = \lambda \int_a^{\sigma(b)} G(\sigma(t), s)f(s, u^\sigma(s)) \Delta s \\
\geq \xi \lambda \int_a^{\sigma(b)} G(\sigma(s), s)f(s, u^\sigma(s)) \Delta s \\
= \xi \max_{t \in [a, \sigma^2(b)]} \lambda \int_a^{\sigma(b)} G(t, s)f(s, u^\sigma(s)) \Delta s \\
= \xi \|T_\lambda u\|.
\]

The proof is complete.

We now state the following fixed-point theorem of cone compression/expansion type due to Krasnolselskii.

**Lemma 2.6.** Let \( E \) be a Banach space, \( K \subseteq E \) a cone and assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \Omega_1 \subset \Omega_2 \) and let \( T : K \cap (\Omega_2 \setminus \Omega_1) \rightarrow K \) be a completely continuous operator such that either

(i) \( \|Tu\| \geq \|u\|, u \in K \cap \partial \Omega_1 \) and \( \|Tu\| \leq \|u\|, u \in K \cap \partial \Omega_2 \) or

(ii) \( \|Tu\| \leq \|u\|, u \in K \cap \partial \Omega_1 \) and \( \|Tu\| \geq \|u\|, u \in K \cap \partial \Omega_2 \).

Then \( T \) has a fixed point in \( K \cap (\overline{\Omega_2} \setminus \Omega_1) \).

**Lemma 2.7.** Assume \((H_1), (H_2)\) and \((H_4)\) hold. Then there exists \( \lambda_0 > 0 \) such that \((1.1)-(1.2)\) has a positive solution \( z \) for \( \lambda = \lambda_0 \).

**Proof.** Let \( R > 0 \) be fixed. Choose \( \lambda_0 \) such that \( \lambda_0 \int_a^{\sigma(b)} G(\sigma(s), s)g_R(s) \Delta s \leq R \). Then for \( z \in \partial K_R \),
\[
\|T_{\lambda_0}z\| = \lambda_0 \int_a^{\sigma(b)} G(\sigma(s), s)f(s, z^\sigma(s)) \Delta s \\
\leq \lambda_0 \int_a^{\sigma(b)} G(\sigma(s), s)g_R(s) \Delta s \\
\leq R = \|z\|.
\]

Now, fix \( \tau_0 \in I \), and pick \( M > 0 \) sufficiently large so that
\[
\xi \lambda_0 M \int_I G(\tau_0, s)p_1(s) \Delta s \geq 1,
\]
where $p_1$ is the function in (H4). Now let $r = R_M$ in (H4). So for $0 \leq u \leq r$, we have $f(t, u) \geq Mp_1(t)u$. Then for $z \in \partial K_r$,

\[
(T_{\lambda_0} z)(\tau_0) = \lambda_0 \int_a^{\sigma(b)} G(\tau_0, s) f(s, z^\sigma(s)) \Delta s \\
\geq \lambda_0 M \int_I G(\tau_0, s) p_1(s)z^\sigma(s) \Delta s \\
\geq \lambda_0 M \xi \|z\| \int_I G(\tau_0, s) p_1(s) \Delta s \\
\geq \|z\|.
\]

The proof is complete.

3. Main Results

**Theorem 3.1.** Assume (H1)–(H4) hold. Then there exists $\lambda_0 > 0$ such that (1.1)–(1.2) has at least one positive solution for $0 < \lambda < \lambda_0$.

**Proof.** By the previous lemma, there exists $\lambda_0 > 0$ such that (1.1)–(1.2) has a positive solution $z$ for $\lambda = \lambda_0$. Now suppose $0 < \lambda < \lambda_0$. Then $Lz(t) = \lambda_0 f(t, z^\sigma(t)) \geq \lambda f(t, z^\sigma(t))$. Also, $w(t) \equiv 0$ solves $Lw = 0 \leq \lambda f(t, 0)$. Also, both $z$ and $w = 0$ satisfy the boundary conditions. So by Lemma 2.2 there is a nonnegative solution $z^*$ to (1.1)–(1.2). Now, if $\alpha > 0$ or $\beta > 0$, then $z^* > 0$ on $(a, t_0)$.

**Claim:** $t_0 = \sigma^2(b)$.

If not, $z^*(t_0) = 0 = \lambda \int_a^{\sigma(b)} G(t_0, s) f(s, z^\sigma(s)) \Delta s$. But by (H4), there is $\eta > 0$ such that $f(t, w) \geq p_1(t)w$ for $0 < w \leq \eta, t \in (a, b)$. Define $f_0(t, u) := \min\{f(t, u), f(t, \eta)\}$. Then, $f_0(t, u) \geq p_1(s) \min\{u, \eta\}$. Let $I = [t_1, t_2]$ so that (2.5) holds. If $\min_{s \in I} z^\sigma(s) < \eta$, we have

\[
0 = z^*(t_0) = \lambda \int_a^{\sigma(b)} G(t_0, s) f(s, z^\sigma(s)) \Delta s \\
= \lambda \int_I G(t_0, s) f_0(s, z^\sigma(s)) \Delta s \\
\geq \lambda \int_I G(t_0, s) f_0(s, z^\sigma(s)) \Delta s \\
\geq \lambda \int_I G(t_0, s) p_1(s) \min_{t \in I} z^\sigma(t) \Delta s
\]
\[ \geq \lambda \xi \|z^*\| \int_I G(t_0, s)p_1(s)\Delta s \]
\[ > 0 \]

which is a contradiction. If \( \min_{s \in I} z^{*\sigma}(s) \geq \eta \), we have

\[ 0 = z^*(t_0) = \lambda \int_a^b G(t_0, s)f(s, z^{*\sigma}(s))\Delta s \]
\[ \geq \lambda \int_I G(t_0, s)f(s, z^{*\sigma}(s))\Delta s \]
\[ \geq \lambda \int_I G(t_0, s)f_\eta(s, z^{*\sigma}(s))\Delta s \]
\[ \geq \lambda \int_I G(t_0, s)p_1(s)\eta\Delta s \]
\[ \geq \lambda \eta \int_I G(t_0, s)p_1(s)\Delta s \]
\[ > 0 \]

which is also a contradiction. Hence \( z^*(t) > 0 \) on \((a, \sigma^2(b))\). \( \blacksquare \)

**Theorem 3.2.** Assume (H1)–(H5) hold. Then there exists \( \lambda^* > 0 \) such that the BVP (1.1)–(1.2) has at least one positive solution for \( 0 < \lambda < \lambda^* \) and no positive solution for \( \lambda > \lambda^* \).

**Proof.** Let \( B \) be the set of all \( \lambda > 0 \) for which the BVP (1.1)–(1.2) has a positive solution in \( D \). Then by Lemma 2.7, we have that \( B \neq \emptyset \) and Theorem 3.1 shows that if \( \lambda_0 \in B \), then \( (0, \lambda_0) \subseteq B \). To show \( B \) is bounded, let \( \lambda \in B \) and let \( u_\lambda \) be the corresponding positive solution. By (H5), there exists \( e > 0 \) and a continuous function \( p_2 \) such that \( f(t, z) \geq ep_2(t)z \) for all \((t, z) \in (a, \sigma^2(b)) \times [0, \infty)\) and \( \int_a^{\sigma(b)} G(\sigma(s), s)p_2(s)\Delta s < \infty \). Now let \( y \) be the solution to the BVP (2.3)–(2.4) with \( p(t) = \phi^\sigma(t)\psi^\sigma(t)p_2(t) \). Then by Lemma 2.3, we have \( \phi(t)\psi(t) \leq c_2 y(t) \) and

\[ \int_a^{\sigma(b)} y^\sigma(t)Lu_\lambda(t)\Delta t = \lambda \int_a^{\sigma(b)} y^\sigma(t)f(t, u_\lambda^\sigma(t))\Delta t \]
\[ \geq e\lambda \int_a^{\sigma(b)} y^\sigma(t)u_\lambda^\sigma(t)p_2(t)\Delta t. \]
On the other hand, by Green’s formula ([3, Theorem 4.94]), we have
\[
\int_{a}^{\sigma(b)} y^{\sigma}(t) L u_\lambda(t) \Delta t = \int_{a}^{\sigma(b)} u_\lambda^{\sigma}(t) L y(t) \Delta t = \int_{a}^{\sigma(b)} u_\lambda^{\sigma}(t) \phi^{\sigma}(t) \psi^{\sigma}(t) p_2(t) \Delta t \leq c_2 \int_{a}^{\sigma(b)} y(t) u_\lambda^{\sigma}(t) p_2(t) \Delta t.
\]
So \( e\lambda \leq c_2 \). Hence \( \lambda \leq \frac{c_2}{e} \). Thus, \( B \) is bounded. Setting \( \lambda^* = \sup B \), we get that for \( 0 < \lambda < \lambda^* \) the BVP (1.1)–(1.2) has a positive solution, and for \( \lambda > \lambda^* \) there is no positive solution.

4. Example

Suppose that \( q > 1 \) and \( \mathbb{T} = \left\{ \frac{1}{q^n} : n \in \mathbb{N}_0 \right\} \cup \{0\} \). The singular BVP
\[
-u^{\Delta\Delta} = \lambda f(t, u^{\sigma}), \quad 0 < t < 1/q^2 \tag{4.1}
\]
\[
u(0) = 0 = u(1) \tag{4.2}
\]
satisfies Theorems 3.1 and 3.2, for
\[
f(t, z) = (t(1-t))^{-2+1/(z+1)} h(z),
\]
(note that \( f \) is singular at \( t = 0 \) and \( t = 1 \)), where
\[
h(z) = \begin{cases} \sqrt{z} & 0 \leq z \leq 1 \\ z^2 & z \geq 1. \end{cases}
\]
First, note that \( a = 0, \sigma^2(b) = 1, \alpha = \gamma = 1 \) and \( \beta = \delta = 0 \). Then \( \rho = 1 > 0 \), and Green’s function for (4.1)–(4.2) is
\[
G(t, s) = \begin{cases} t(1 - \sigma(s)) & t \leq s \\ \sigma(s)(1 - t) & \sigma(s) \leq t. \end{cases}
\]
Note that \( f(t, z) \geq 0 \) on \( (0, 1) \times [0, \infty) \), and is nondecreasing in \( z \). Thus \( (H_1) \) and \( (H_3) \) are satisfied. Also,
\[
f(t, z) \leq g_M(t) := M^2(t(1-t))^{-2+1/(M+1)}
\]
for \((t, z) \in (0, 1) \times [0, M]\). Also,

\[
\int_a^{\sigma(b)} G(\sigma(s), s)g_M(s)\Delta s
\]

\[= M^2 \int_0^{1/q} G(\sigma(s), s)[s(1-s)]^{-2+1/(M+1)} \Delta s \]

\[= M^2 \int_0^{1/q} \sigma(s)(1-\sigma(s))[s(1-s)]^{-2+1/(M+1)} \Delta s \]

\[= M^2 \sum_{k=2}^{\infty} \frac{1}{q^{k-1}} \left( 1 - \frac{1}{q^{k-1}} \right) \left( \frac{1}{q^k} \left( 1 - \frac{1}{q^k} \right) \right)^{-2+1/(M+1)} \left( \frac{q - 1}{q^k} \right) \]

\[= M^2 \sum_{k=2}^{\infty} \left( \frac{q^{k-1} - 1}{q^{2k-2}} \right) \left( \frac{q^k - 1}{q^k} \right)^{-2+1/(M+1)} \left( \frac{q - 1}{q^k} \right) \]

\[= M^2 q^2 \sum_{k=2}^{\infty} \left( \frac{q^{k-1} - 1}{q^{2k}} \right) \left( \frac{q - 1}{q^k} \right) \left( \frac{q^k - 1}{q^k} \right)^{2-1/(M+1)} \]

\[= M^2 q^2 \sum_{k=2}^{\infty} \left( \frac{q^{k-1} - 1}{q^{2k}} \right) \left( \frac{q - 1}{q^k} \right) \left( \frac{q^k - 1}{q^k} \right)^{2} \left( \frac{q^k - 1}{q^{2k}} \right)^{1/(M+1)} \]

\[\leq M^2 q^2 \sum_{k=2}^{\infty} \left( \frac{q^{k-1} - 1}{q^{2k}} \right) \left( \frac{q - 1}{q^k} \right) \left( \frac{q^k - 1}{q^k} \right)^{2} q^{2k} \left( \frac{q^k - 1}{q^{2k}} \right)^{1/(M+1)} \]

\[= M^2 q^2 \sum_{k=2}^{\infty} \left( \frac{q^k}{q^k - 1} \right)^2 \left( \frac{1}{q^k} \right)^{1/(M+1)} \]

Now, as the term \(\left( \frac{q^k}{q^k - 1} \right)^2\) approaches 1 as \(k \to \infty\), there is \(K\) such that for \(k \geq K\), we have \(\left( \frac{q^k}{q^k - 1} \right) < 2\). Therefore, we have:

\[
\int_0^{1/q} G(\sigma(s), s)g_M(s)\Delta s \leq M^2 q^2 \sum_{k=2}^{K} \left( \frac{q^k}{q^k - 1} \right)^2 \left( \frac{1}{q^k} \right)^{1/(M+1)}
\]

\[+ 2M^2 q^2 \sum_{k=K+1}^{\infty} \left( q^{-1/(M+1)} \right)^k \]

\[< \infty \]

since the last term is a convergent geometric series. Thus, \((H_2)\) is satisfied.
Next, we show (H4) holds. To see this, let \( d > 0 \) be given and let \( 0 < z \leq R_d \), where \( R_d := \frac{1}{d^2} \). It follows that \( z = \frac{1}{c^2} \) for some \( c \geq d \). Then, for \( t \in (0, 1) \cap \mathbb{T} \),

\[
 f(t, z) = (t(1-t))^{2+1/(z+1)}h(z) \\
= (t(1-t))^{-1}(t(1-t))^{-z/(z+1)}h(z) \\
\geq \frac{1}{t(1-t)}h(z) \\
\geq \frac{1}{t(1-t)\sqrt{z}} \\
= \frac{1}{t(1-t)c} \\
= \frac{1}{t(1-t)} \frac{1}{z^c} \\
\geq d \frac{1}{t(1-t)} \frac{1}{z^c} \\
\geq dp_1(t)z,
\]

where \( p_1(t) := \frac{1}{t(1-t)} \). So (H4) holds.

Finally, we show (H5) holds. First note that

\[
f(t, z) \geq (t(1-t))^{-1}h(z) \geq (t(1-t))^{-1}z = p_2(t)z,
\]

where \( p_2(t) = \frac{1}{t(1-t)} \). Also

\[
\int_a^{\sigma(b)} G(\sigma(s), s)p_2(s)\Delta s = \int_0^{1/q} \sigma(s)(1 - \sigma(s))(s(1-s))^{-1} \Delta s \\
= \sum_{k=2}^{\infty} \frac{1}{q^{k-1}} \left( 1 - \frac{1}{q^{k-1}} \right) \left( \frac{1}{q^k} \left( 1 - \frac{1}{q^k} \right) \right)^{-1} q - 1 \\
= \sum_{k=2}^{\infty} \frac{q^k(1 - 1/q^{k-1})}{q^{k-1}(1 - 1/q^k)} \left( \frac{q - 1}{q^k} \right)
\]
\[ = q(q - 1) \sum_{k=2}^{\infty} \frac{q^{k-1} - 1}{q^k - 1} \frac{q^k - 1}{q^{k-1}} \frac{1}{q^k} \]

\[ = q(q - 1) \sum_{k=2}^{\infty} \frac{q^k - q}{q^k - 1} \frac{1}{q^k} \]

\[ \leq q(q - 1) \sum_{k=2}^{\infty} q^{-k} \]

\[ = 1 < \infty. \]

Thus, (H5) holds.

References


