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Oscillation of Second Order Neutral Delay Differential Equations

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Abstract

We present new oscillation criteria for the second order nonlinear neutral delay differential equation

$$\left(a(t)\,(y(t)+p(t)\,y(t-\tau))'\right)' + q(t)\,|y(\sigma(t))|^{\alpha-1}\,y(\sigma(t)) = 0,$$

where $t \geq t_0$, τ, and α are positive constants and the functions $p, q, a, \sigma \in C([t_0, \infty), \mathbb{R})$. Our results generalize and improve some known results for oscillation of second order neutral delay differential equations. Our results are illustrated with an example.

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1. Introduction

Consider the second order nonlinear neutral delay differential equation

$$\left(a(t)\left(y(t)+p(t)\,y(t-\tau)\right)'\right)'+q(t)\left|y(\sigma(t))\right|^{\alpha-1}y(\sigma(t))=0, \tag{1.1}$$

where $t \geq t_0$, τ, and α are positive constants, $p, q, a, \sigma \in C([t_0,\infty),\mathbb{R})$.

Throughout this paper, we assume that (a) $0 \leq p(t) \leq 1,\ q(t) \geq 0,\ a(t) > 0,\ \alpha > 0$; (b) $\sigma(t) \leq t,\ \sigma'(t) > 0,\ \lim_{t\to\infty}\sigma(t)=\infty$; (c) $\displaystyle\int_{t_0}^{\infty}\frac{dt}{a(t)}=\infty$.

Second order neutral delay differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems (see Hale [10]).

Our attention is restricted to those solutions of equation (1.1) that satisfy $\sup\{|y(t)| : t \geq T\} > 0$. We make a standing hypothesis that (1.1) does possess such solutions. By a solution of equation (1.1) we mean a function $y \in C([\theta,\infty),\mathbb{R})$, $\theta = \min\{t_0-\tau, \sigma(t_0)\}$ in the sense that both $y(t)+p(t)\,y(t-\tau)$ and $a(t)\left(y(t)+p(t)\,y(t-\tau)\right)'$ are continuously differentiable for $t \geq t_0$ and $y(t)$ satisfies equation (1.1) on $[t_0,\infty)$. For further questions concerning existence and uniqueness of solutions of neutral delay differential equations see Hale [10].

A solution of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

In the last few decades, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different classes of second order neutral delay differential equations, see for example [2,6,7,9] and the references quoted therein. For oscillation of various functional differential equations we refer the reader to the monographs [1,7,9,20].

In particular, much work has been done on the following particular cases of (1.1):

$$y''(t)+q(t)\,y(t)=0, \tag{1.2}$$

$$\left(r(t)\,y'(t)\right)'+q(t)\,y(t)=0, \tag{1.3}$$

$$y''(t)+q(t)\,y(t-\sigma)=0, \tag{1.4}$$

$$\left(y(t)+p(t)\,y(t-\tau)\right)''+q(t)\,y(t-\sigma)=0. \tag{1.5}$$

An important tool in the study of the oscillatory behavior of solutions of these equations is the averaging technique which goes back as far as the classical result of Wintner [25] where it was proved that (1.2) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)\,dv\,ds=\infty. \tag{1.6}$$

Hartman [11] proved that the limit in (1.6) cannot be replaced by the limit supremum and proved that the condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v)\, dv ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v)\, dv ds \le \infty, \tag{1.7}$$

implies that every solution of (1.2) oscillates.

Kamenev [12] improved Wintner's result by proving that the condition

$$\lim_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds = \infty, \tag{1.8}$$

for some integer $n > 1$ is sufficient for the oscillation of (1.2).

Yan [26] proved that if

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds < \infty,$$

for some integer $n > 1$ and there exists a function ϕ on $[t_0, \infty)$ satisfying $\int_{t_0}^{\infty} \phi_+^2(t)\, dt = \infty$ where $\phi_+(t) = \max\{\phi(t), 0\}$ and

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds > \sup_{u\ge t_0} \phi(u), \tag{1.9}$$

then every solution of equation (1.2) oscillates.

Philos [18] further improved Kamenev's result by proving the following: Suppose there exist continuous functions $H,\ h : D \equiv \{(t,s) : t \ge s \ge t_0\} \to \mathbb{R}$ such that

$$\begin{aligned} H(t,t) &= 0, \quad t \ge t_0, \\ H(t,s) &> 0, \quad t > s \ge t_0, \end{aligned}$$

and H has a continuous and nonpositive partial derivative on D with respect to the second variable and satisfies

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \ge 0. \tag{1.10}$$

Further, suppose that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\, q(s) - \frac{1}{4} h^2(t,s) \right] ds = \infty. \tag{1.11}$$

Then every solution of equation (1.2) oscillates.

We note, however, that when $q(t) = \dfrac{\gamma}{t^2}$, (1.2) reduces to the well-known Euler–Cauchy equation

$$u''(t) + \frac{\gamma}{t^2} u(t) = 0, \quad t \ge 1, \tag{1.12}$$

to which none of the above mentioned oscillation criteria is applicable. In fact, the Euler–Cauchy equation (1.12) is oscillatory if $\gamma > \frac{1}{4}$, and nonoscillatory if $\gamma \leq \frac{1}{4}$, see [13]. For further results on the oscillation of superlinear and sublinear equations, we refer the reader to the papers [3–5, 23].

For oscillation of equation (1.3), Leighton [15] proved that if

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)\, dt = \infty, \tag{1.13}$$

then every solution of equation (1.3) oscillates.

Willett [24] used the transformation

$$\tau = \left(\int_t^{\infty} \frac{ds}{r(s)} \right)^{-1}, \quad u(t) = \tau^{-1}(y(t)),$$

to establish a new version of Leighton's criterion and obtained the following oscillation result: If

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t) \left(\int_t^{\infty} \frac{ds}{r(s)} \right)^2 dt = \infty, \tag{1.14}$$

then every solution of (1.3) oscillates.

We note, however, that the oscillation criteria of Leighton and Willett are not applicable to the equation

$$\left(t^2 u'(t)\right)' + \gamma u(t) = 0, \quad t > 0, \tag{1.15}$$

where γ is a positive constant. Kong [13], Li [16], Li and Yeh [17], Rogovchenko [19], and Yu [27] used the generalized Riccati technique and have given several sufficient conditions for oscillation of (1.3) which can be applied to (1.15); in fact every solution of (1.15) oscillates if $\gamma > \frac{1}{4}$, see [17, 18].

In [22], Waltman extended Leighton's criterion to equation (1.4) and showed that (1.4) is oscillatory if $q(t) \geq 0$ and

$$\int_{t_0}^{\infty} q(s)\, ds = \infty.$$

But, Travis [21] showed that Leighton's criterion is not enough to ensure the oscillation of equation (1.4). Hence, the oscillation analysis of the delay differential equations is more complicated than that of ordinary differential equations.

There has recently been an increased interest in the studying of the oscillation of second order neutral delay differential equations. The results of Waltman and Travis have been extended to neutral delay differential equations by Grammatikopoulos, Ladas and Meimaridou [8]. They proved that if

$$0 \leq p(t) \leq 1, \quad q(t) \geq 0,$$

and

$$\int_{t_0}^{\infty} q(s)\left[1-p(s-\sigma)\right] ds=\infty,$$

then equation (1.5) is oscillatory.

In this paper, we use the generalized Riccati transformation technique to establish some new sufficient conditions for the oscillation of equation (1.1). To the best of our knowledge nothing is known regarding the qualitative behavior of equation (1.1). The relevance of our results becomes clear in an example that we give in Section 2. In the sequel, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

2. Main Results

In this section, we will establish some new oscillation criteria for the oscillation of equation (1.1), which extend and improve some known results. Throughout this section, for any function $\phi\in C^1([t_0,\infty))$, we define

$$Q(t):=q(t)\left(1-p(\sigma(t))\right)^{\alpha},\ k(t)=\frac{N_1^2 a(\sigma(t))}{\alpha^2\sigma'(t)}\int_{t_1}^{\sigma(t)}\frac{a(\sigma(s))}{\sigma'(s)\Phi(s)a^2(s)}ds,$$

for some constant N_1 and $t_1\geq t_0$, *and*

$$\begin{aligned}\Phi(t)&=\exp\left(-2\int^{t}\phi(\zeta)\,d\zeta\right),\ \beta(t)=\phi(t)\left(1-\frac{a(t)}{k(t)}\right),\\ \psi(t)&=\Phi(t)\left(Q(t)+\frac{1}{k(t)}\left(a(t)\phi(t)\right)^2-\left(a(t)\phi(t)\right)'\right).\end{aligned}$$

Theorem 2.1. Suppose there exists a negative function $\phi\in C^1([t_0,\infty))$, such that

$$\left(\frac{\Phi'(t)\,a(\sigma(t))}{\sigma'(t)}\right)'\leq 0,\quad \text{for } t\geq t_0, \tag{2.1}$$

and

$$\liminf_{t\to\infty}\int_{t_0}^{t}\Phi(s)\,Q(s)\,ds>0, \tag{2.2}$$

and there exists continuous functions $H,\ h:D\to\mathbb{R}$ such that

(i) $H(t,t)=0,\quad$ *for* $t\geq t_0$,

(ii) $H(t,s)>0,\quad$ *for* $t>s\geq t_0$,

(iii) H has a continuous and nonpositive partial derivative on D with respect to the second variable.

Assume there exists a function $\upsilon \in C^1([t_0,\infty),(0,\infty))$ and $T \geq t_0$ such that

$$-\frac{\partial}{\partial s}[H(t,s)\upsilon(s)] + 2H(t,s)\,\upsilon(s)\,\phi(s)\left(1-\frac{a(s)}{k(s)}\right) = h(t,s)\sqrt{H(t,s)\,\upsilon(s)}.$$

Further assume for all sufficiently large T,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\,\upsilon(s)\,\psi(s)-\frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right]ds=\infty. \tag{2.3}$$

Then equation (1.1) is oscillatory for all $\alpha > 1$.

Proof. Suppose to the contrary that equation (1.1) possesses a nonoscillatory solution y on an interval $[t_0,\infty)$. Without loss of generality, we shall assume that $y(t) > 0$ for all $t \geq t_0$. Let

$$z(t) = y(t) + p(t)\,y(t-\tau). \tag{2.4}$$

Then in view of condition (a), we see that $z(t) > 0$, $\left(a(t)\,z'(t)\right)' \leq 0$ for $t \geq t_1 \geq t_0$. Therefore $a(t)\,z'(t)$ is a decreasing function of t, in view of Ruan [20, Theorem 1], we have immediately that $z'(t) \geq 0$ for $t \geq t_1$. Consequently,

$$z(t) > 0,\ z'(t) \geq 0,\ \left(a(t)\,z'(t)\right)' \leq 0, \quad \text{for } t \geq t_1. \tag{2.5}$$

Now, observe that from (1.1), we have

$$\left(a(t)\,z'(t)\right)' + q(t)\,y^{\alpha}(\sigma(t)) = 0, \tag{2.6}$$

Now using (2.4) and (2.5), we get

$$\begin{aligned} y(t) &= z(t) - p(t)\,y(t-\tau) \\ &= z(t) - p(t)\left(z(t-\tau) - p(t-\tau)\,y(t-2\tau)\right) \\ &\geq z(t) - p(t)\,z(t-\tau) \geq z(t)\left(1-p(t)\right). \end{aligned} \tag{2.7}$$

Using (2.6), (2.7) and using the definition of the function $Q(t)$, we have

$$\left(a(t)\,z'(t)\right)' + Q(t)\,z^{\alpha}(\sigma(t)) \leq 0, \quad t \geq t_1. \tag{2.8}$$

We now define the function

$$w(t) = \Phi(t)\,a(t)\left(\frac{z'(t)}{z^{\alpha}(\sigma(t))} + \phi(t)\right). \tag{2.9}$$

This and (2.8) imply for $t \geq t_1$ that

$$\begin{aligned} w'(t) &\leq -2\phi(t)\,w(t) \\ &\quad + \Phi(t)\left\{-Q(t) + \left(a(t)\,\phi(t)\right)' - \frac{\alpha a(t)\,\sigma'(t)\,z'(t)\,z'(\sigma(t))}{z^{\alpha+1}(\sigma(t))}\right\}. \end{aligned} \tag{2.10}$$

From (2.5), and assumption (b), we get

$$a(\sigma(t))\,z'(\sigma(t)) \geq a(t)\,z'(t),$$

From this and (2.10), we get for $t \geq t_1$

$$\begin{aligned} w'(t) &\leq -2\phi(t)\,w(t) \\ &\quad + \Phi(t)\left\{-Q(t) + (a(t)\,\phi(t))' - \frac{\alpha\sigma'(t)}{a(\sigma(t))}\left(\frac{a(t)\,z'(t)}{z^{\gamma}(\sigma(t))}\right)^2\right\}, \end{aligned} \tag{2.11}$$

where $\gamma = \dfrac{\alpha+1}{2}$. Also, from (2.5), and assumption (b), we can write (2.11) in the form

$$\begin{aligned} w'(t) \leq \Phi(t)&\Bigg\{-Q(t) - 2a(t)\,\phi^2(t) \\ &+ (a(t)\,\phi(t))' - \frac{2\phi(t)\,a(t)\,z'(t)}{z^{\alpha}(\sigma(t))} - \frac{\alpha\sigma'(t)}{a(\sigma(t))}\left(\frac{a(t)\,z'(t)}{z^{\gamma}(t)}\right)^2\Bigg\}. \end{aligned} \tag{2.12}$$

Integrating (2.12) from t_1 to t $(t > t_1)$, we get

$$\begin{aligned} \Phi(t)\,\frac{a(t)\,z'(t)}{z^{\alpha}(\sigma(t))} &\leq C - \int_{t_1}^{t} \Phi(s)Q(s)\,ds + \int_{t_1}^{t} \frac{\Phi'(s)\,a(s)\,z'(s)}{z^{\alpha}(\sigma(s))}ds \\ &\quad - \int_{t_1}^{t} \frac{\alpha\sigma'(s)\,\Phi(s)}{a(\sigma(s))}\left(\frac{a(s)\,z'(s)}{z^{\gamma}(s)}\right)^2 ds, \end{aligned} \tag{2.13}$$

where $C = w(t_1) - a(t_1)\,\Phi(t_1)\,\phi(t_1)$. Since $\lim_{t\to\infty} \sigma(t) = \infty$, for t sufficiently large $\sigma(t) > t_1$, $a(s)\,z'(s) \leq a(\sigma(s))\,z'(\sigma(s))$. Then by Bonnet's theorem, since $\dfrac{\Phi'(t)\,a(\sigma(t))}{\sigma'(t)}$ is nonincreasing, for a fixed $t \geq t_1$, there exists $\xi \in [t_1, t]$ such that

$$\begin{aligned} \int_{t_1}^{t} \frac{\Phi'(s)\,a(s)\,z'(s)}{z^{\alpha}(\sigma(s))}ds &\leq \int_{t_1}^{t} \frac{\Phi'(s)\,a(\sigma(s))}{\sigma'(s)}\frac{z'(\sigma(s))\,\sigma'(s)}{z^{\alpha}(\sigma(s))}ds \\ &= \frac{\Phi'(t_1)\,a(\sigma(t_1))}{\sigma'(t_1)}\int_{t_1}^{\xi} \frac{z'(\sigma(s))\,\sigma'(s)}{z^{\alpha}(\sigma(s))}ds \\ &= \frac{\Phi'(t_1)\,a(\sigma(t_1))}{\sigma'(t_1)}\int_{z(\sigma(t_1))}^{z(\sigma(\xi))} u^{-\alpha}du \\ &= \frac{\Phi'(t_1)\,a(\sigma(t_1))}{(1-\alpha)\,\sigma'(t_1)}\left(z^{1-\alpha}(\sigma(\xi)) - z^{1-\alpha}(\sigma(t_1))\right) \\ &< \frac{a(\sigma(t_1))\,\Phi'(t_1)\,z^{1-\alpha}(\sigma(t_1))}{(\alpha-1)\,\sigma'(t_1)} = M. \end{aligned} \tag{2.14}$$

Thus, for $t \geq t_1$, we find from (2.13), that

$$\Phi (t) \frac{a (t) z' (t)}{z^{\alpha} (\sigma (t))} \leq L - \int_{t_1}^{t} \Phi (s) Q (s) ds - \int_{t_1}^{t} \frac{\alpha \sigma' (s) \Phi (s)}{a (\sigma (s))} \left(\frac{a (s) z' (s)}{z^{\gamma} (s)} \right)^2 ds,$$

where $L = C + M > 0$ and hence, since $\Phi (t) \dfrac{a (t) z' (t)}{z^{\alpha} (\sigma (t))} > 0$, we have

$$\begin{aligned} &\int_{t_1}^{t} \frac{\alpha \sigma' (s) \Phi (s)}{a (\sigma (s))} \left(\frac{a (s) z' (s)}{z^{\gamma} (s)} \right)^2 ds \\ &\quad \leq L - \Phi (t) \frac{a (t) z' (t)}{z^{\alpha} (\sigma (t))} - \int_{t_1}^{t} \Phi (s) Q (s) ds \\ &\quad < L - \int_{t_1}^{t} \Phi (s) Q (s) ds. \end{aligned} \tag{2.15}$$

From (2.2) and (2.15), we have that the integral

$$\int_{t_1}^{t} \frac{\alpha \sigma' (s) \Phi (s)}{a (\sigma (s))} \left(\frac{a (s) z' (s)}{z^{\gamma} (s)} \right)^2 ds,$$

converges as $t \to \infty$. Thus, there exists a positive constant N such that

$$\int_{t_1}^{t} \frac{\alpha \sigma' (s) \Phi (s)}{a (\sigma (s))} \left(\frac{a (s) z' (s)}{z^{\gamma} (s)} \right)^2 ds \leq N, \quad \text{for all } t \geq t_1. \tag{2.16}$$

By Schwarz's inequality, we get

$$\begin{aligned} &\left| \int_{t_1}^{t} \frac{z'(s)}{z^{\gamma}(s)} ds \right|^2 = \left| \int_{t_1}^{t} \sqrt{\frac{a(\sigma(s))}{\alpha\sigma'(s)\Phi(s)a^2(s)}} \sqrt{\frac{\alpha\sigma'(s)\Phi(s)}{a(\sigma(s))}} \frac{a(s)z'(s)}{z^{\gamma}(s)} ds \right|^2 \\ &\leq \int_{t_1}^{t} \frac{a (\sigma (s))}{\alpha \sigma' (s) \Phi (s) a^2 (s)} ds \left(\int_{t_1}^{t} \frac{\alpha \sigma' (s) \Phi (s)}{a (\sigma (s))} \left(\frac{a (s) z' (s)}{z^{\gamma} (s)} \right)^2 ds \right) \\ &\leq N \int_{t_1}^{t} \frac{a (\sigma (s))}{\alpha \sigma' (s) \Phi (s) a^2 (s)} ds. \end{aligned}$$

Hence, for $t \geq t_1$

$$\left| z^{1-\gamma} (t) - z^{1-\gamma} (t_1) \right| \leq (\gamma - 1) N^{\frac{1}{2}} \left(\int_{t_1}^{t} \frac{a (\sigma (s))}{\alpha \sigma' (s) \Phi (s) a^2 (s)} ds \right)^{\frac{1}{2}}.$$

Therefore there exists a constant N_1 and $t_2 > t_1$ such that

$$\left|z^{1-\gamma}(t)\right| \le N_1 \left(\int_{t_1}^{t} \frac{a(\sigma(s))}{\alpha\sigma'(s)\,\Phi(s)\,a^2(s)} ds\right)^{\frac{1}{2}}, \quad \text{for } t \ge t_2.$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can assume that there exists a $T \ge t_2$ such that $\sigma(t) \ge t_1$ for all $t \ge T$. Hence

$$\left|z^{1-\gamma}(\sigma(t))\right| \le N_1 \left(\int_{t_1}^{\sigma(t)} \frac{a(\sigma(s))}{\alpha\sigma'(s)\,\Phi(s)\,a^2(s)} ds\right)^{\frac{1}{2}}, \quad \text{for } t \ge T,$$

or

$$\left|z^{\gamma}(\sigma(t))\right| \le \left|z^{\alpha}(\sigma(t))\right| N_1 \left(\int_{t_1}^{\sigma(t)} \frac{a(\sigma(s))}{\alpha\sigma'(s)\,\Phi(s)\,a^2(s)} ds\right)^{\frac{1}{2}}, \text{ for } t \ge T. \tag{2.17}$$

From (2.17), (2.11) and the definition of $k(t)$, we get, for $t \ge T$

$$w'(t) \le -2\phi(t)\,w(t) + \Phi(t)\left\{-Q(t) + (a(t)\,\phi(t))' - \frac{1}{k(t)}\left(\frac{a(t)\,z'(t)}{z^{\alpha}(\sigma(t))}\right)^2\right\}.$$

Equation (2.9) yields

$$\begin{aligned} w'(t) &\le -2\phi(t)\,w(t) \\ &\qquad + \Phi(t)\left\{-Q(t) + (a(t)\,\phi(t))' - \frac{1}{k(t)}\left(\frac{w(t)}{\Phi(t)} - a(t)\,\phi(t)\right)^2\right\} \\ &= -\psi(t) - 2\phi(t)\left(1 - \frac{a(t)}{k(t)}\right) w(t) - \frac{1}{k(t)\,\Phi(t)} w^2(t). \end{aligned} \tag{2.18}$$

Multiplying both sides of (2.18) by $H(t,s)\,\upsilon(s)$ and integrating from T to t, we have, for all $t \ge T \ge t_1$,

$$\begin{aligned} &\int_T^t H(t,s)\,\upsilon(s)\,\psi(s)\,ds \\ &\quad \le -\int_T^t H(t,s)\,\upsilon(s)\,w'(s)\,ds - 2\int_T^t H(t,s)\,\upsilon(s)\,\phi(s)\left(1 - \frac{a(s)}{k(s)}\right) w(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&-\int_T^t \frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)} w^2(s)\,ds \\
&= H(t,T)\,\upsilon(T)\,w(T) \\
&\quad -\int_T^t \left[-\frac{\partial}{\partial s}\left(H(t,s)\,\upsilon(s)\right) + 2H(t,s)\,\upsilon(s)\,\phi(s)\left(1-\frac{a(s)}{k(s)}\right)\right] w(s)\,ds \\
&\quad -\int_T^t \frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)} w^2(s)\,ds \\
&= H(t,T)\,\upsilon(T)\,w(T) \\
&\quad -\int_T^t \left[\sqrt{\frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)}} w(s) + \frac{1}{2}\sqrt{\Phi(s)\,k(s)}h(t,s)\right]^2 ds \\
&\quad +\frac{1}{4}\int_T^t \Phi(s)\,k(s)\,h^2(t,s)\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_T^t \left[H(t,s)\,\upsilon(s)\,\psi(s) - \frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right] ds \\
&\quad \leq H(t,T)\,\upsilon(T)\,w(T) - \int_T^t \left[\sqrt{\frac{H(t,s)\upsilon(s)}{k(s)\Phi(s)}} w(s) + \frac{1}{2}\sqrt{\Phi(s)k(s)}h(t,s)\right]^2 ds.
\end{aligned}
\tag{2.19}
$$

By this equation, we have, for $t \geq T$

$$
\begin{aligned}
&\int_T^t \left[H(t,s)\,\upsilon(s)\,\psi(s) - \frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right] ds \\
&\quad \leq H(t,T)\,\upsilon(T)\,|w(T)|.
\end{aligned}
\tag{2.20}
$$

It follows from (2.20) that

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\,\upsilon(s)\,\psi(s) - \frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right] ds \\
&\quad \leq \upsilon(T)\,|\psi(T)|,
\end{aligned}
$$

which contradicts assumption (2.3). Therefore, equation (1.1) is oscillatory. ■

Remark 2.2. The conclusion of Theorem 2.1 remains valid if assumption (2.3) is replaced by the two conditions

$$
\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\,\upsilon(s)\,\psi(s)\,ds = \infty,
$$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Phi(s)\, k(s)\, h^2(t,s)\, ds < \infty.$$

Remark 2.3. With the appropriate choice of the functions H, υ and h, we can deduce from Theorem 2.1 a number of oscillation criteria for equation (1.1). Consider, for example,

$$H(t,s) = (t-s)^n, \quad (t,s) \in D, \ \upsilon(s) = s,$$

where n is an integer greater than one. Then H is continuous on D and satisfies

$$H(t,t) = 0, \quad \text{for } t \ge t_0,$$

$$H(t,s) > 0, \quad \text{for } t > s \ge t_0.$$

Moreover, H has a continuous and nonpositive partial derivative on D with respect to the second variable. Clearly, the function

$$h(t,s) = \frac{(t-s)^{(n-2)/2}}{\sqrt{s}} \left[(n+1)s - t + 2s(t-s)\beta(s)\right], \quad t \ge s \ge t_0,$$

is continuous and satisfies for $t \ge s \ge t_0$

$$-\frac{\partial}{\partial s}\left(H(t,s)\,\upsilon(s)\right) + 2H(t,s)\,\upsilon(s)\,\beta(s) = h(t,s)\sqrt{H(t,s)\,\upsilon(s)}.$$

Therefore, by Theorem 2.1, we get the following oscillation criterion.

Corollary 2.4. Let assumptions (2.1) and (2.2) hold. If for all sufficiently large T,

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_T^t \left[A(t,s) - B(t,s)\right] ds = \infty,$$

where

$$A(t,s) = (t-s)^n\, s\psi(s),$$
$$B(t,s) = \frac{1}{4}\Phi(s)\, k(s)\, \frac{(t-s)^{n-2}}{s} \left[(n+1)s - t + 2s(t-s)\beta(s)\right]^2,$$

for some integer $n > 1$, then the equation (1.1) is oscillatory for all $\alpha > 1$.

Theorem 2.5. Let assumptions (2.1) and (2.2) hold, the functions H, h, υ be defined as in Theorem 2.1, and suppose that

$$0 < \inf_{s\ge t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \le \infty. \tag{2.21}$$

If there exists a function $\varphi \in C^1([t_0,\infty), \mathbb{R})$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Phi(s)\, k(s)\, h^2(t,s)\, ds < \infty, \tag{2.22}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\,\upsilon(s)\,\psi(s)-\frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right]ds\geq\varphi(T),\tag{2.23}$$

$$\limsup_{t\to\infty}\int_T^t\frac{\varphi_+^2(s)}{\upsilon(s)\,\Phi(s)\,k(s)}ds=\infty,\tag{2.24}$$

where $\varphi_+(s)=\max\{\varphi(t),0\}$, then equation (1.1) is oscillatory for all $\alpha>1$.

Proof. As in Theorem 2.1, without loss of generality we may assume that there exists a solution x of equation (1.1) such that $x(t)>0$ on $[t_1,\infty)$ for some $t_1\geq t_0$. Again defining the function w as in Theorem 1, we arrive at (2.19) which yields for $t>T\geq t_1$,

$$\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\,\upsilon(s)\,\psi(s)-\frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right]ds$$
$$\leq\upsilon(T)\,w(T)-\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)\,k(s)}h(t,s)\right]^2ds.$$

Thus,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\,\upsilon(s)\,\psi(s)-\frac{1}{4}\Phi(s)\,k(s)\,h^2(t,s)\right]ds$$
$$\leq\upsilon(T)\,w(T)$$
$$-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)\,k(s)}h(t,s)\right]^2ds.$$

From (2.23), we have

$$\upsilon(T)\,w(T)\geq\varphi(T)$$
$$+\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)\,k(s)}h(t,s)\right]^2ds.$$

Then, for $T\geq t_1$,

$$\upsilon(T)\,w(T)\geq\varphi(T)\tag{2.25}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{H(t,s)\,\upsilon(s)}{k(s)\,\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)\,k(s)}h(t,s)\right]^2ds<\infty.$$

Thus

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\frac{H(t,s)\upsilon(s)}{k(s)\Phi(s)}w^2(s)+\sqrt{H(t,s)\upsilon(s)}w(s)h(t,s)\right]ds\\
&\quad\le\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{H(t,s)\upsilon(s)}{k(s)\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)k(s)}h(t,s)\right]^2ds\\
&\quad<\infty.
\end{aligned}\tag{2.26}$$

Define

$$u(t)=\frac{1}{H(t,T)}\int_T^t\frac{H(t,s)\upsilon(s)}{k(s)\Phi(s)}w^2(s)\,ds,$$

and

$$v(t)=\frac{1}{H(t,T)}\int_T^t\sqrt{H(t,s)\upsilon(s)}w(s)\,h(t,s)\,ds,$$

for $t\ge t_1$. Then (2.26) implies that

$$\liminf_{t\to\infty}[u(t)+v(t)]<\infty.\tag{2.27}$$

Now, we claim that

$$\int_T^\infty\frac{\upsilon(s)w^2(s)}{k(s)\Phi(s)}ds<\infty.\tag{2.28}$$

Suppose to the contrary that

$$\int_T^\infty\frac{\upsilon(s)w^2(s)}{k(s)\Phi(s)}ds=\infty.\tag{2.29}$$

By (2.21), there is a positive constant M_1 such that

$$\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]>M_1>0.\tag{2.30}$$

Let M_2 be any arbitrary positive number. It follows from (2.29) that there exists a $t_2>T$ such that

$$\int_T^t\frac{\upsilon(s)w^2(s)}{k(s)\Phi(s)}ds\ge\frac{M_2}{M_1},\quad\text{for all } t\ge t_2.$$

Consequently, for all $t\ge t_2$,

$$\begin{aligned}
u(t)&=\frac{1}{H(t,T)}\int_T^t H(t,s)\,d\left[\int_T^s\frac{\upsilon(\zeta)w^2(\zeta)}{k(\zeta)\Phi(\zeta)}d\zeta\right],\\
&=\frac{1}{H(t,T)}\int_T^t\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_T^s\frac{\upsilon(\zeta)w^2(\zeta)}{k(\zeta)\Phi(\zeta)}d\zeta\right]ds,\\
&\ge\frac{1}{H(t,T)}\int_{t_2}^t\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_T^s\frac{\upsilon(\zeta)w^2(\zeta)}{k(\zeta)\Phi(\zeta)}d\zeta\right]ds,\\
&\ge\frac{M_2}{M_1}\frac{1}{H(t,T)}\int_{t_2}^t\left[-\frac{\partial H(t,s)}{\partial s}\right]ds=\frac{M_2}{M_1}\frac{H(t,t_2)}{H(t,T)}.
\end{aligned}$$

By (2.30), we have

$$\frac{H(t,s)}{H(t,t_0)} \geq M_1, \quad \text{for all } t \geq T,$$

so that

$$u(t) \geq M_2, \quad \text{for all } t \geq T.$$

Since M_2 is an arbitrary constant, we conclude that

$$\lim_{t\to\infty} u(t) = \infty. \tag{2.31}$$

Consider a sequence $\{t_n\}_{n=1}^{\infty} \in (t_1,\infty)$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} [u(t_n) + v(t_n)] = \liminf_{t\to\infty} [u(t) + v(t)].$$

By (2.27), there exists a number M such that

$$u(t_n) + v(t_n) \leq M, \quad \text{for } n = 1, 2, \ldots. \tag{2.32}$$

It follows from (2.31) that

$$\lim_{n\to\infty} u(t_n) = \infty. \tag{2.33}$$

Thus, (2.32) yields

$$\lim_{n\to\infty} v(t_n) = -\infty. \tag{2.34}$$

It follows from (2.33) and (2.34) that for large values of n,

$$\frac{v(t_n)}{u(t_n)} < \epsilon - 1 < 0, \tag{2.35}$$

where $\epsilon \in (0,1)$. Thus, by (2.34) and (2.35), we conclude that

$$\lim_{n\to\infty} \frac{v(t_n)}{u(t_n)} v(t_n) = \infty. \tag{2.36}$$

On the other hand, by the Schwarz inequality, we get

$$\begin{aligned}
v^2(t_n) &= \left\{\frac{1}{H(t_n,T)} \int_T^{t_n} \sqrt{H(t_n,s)\,\upsilon(s)}\, w(s)\, h(t_n,s)\, ds\right\}^2 \\
&\leq \left\{\frac{1}{H(t_n,T)} \int_T^{t_n} \frac{H(t_n,s)\,\upsilon(s)}{k(s)\,\Phi(s)} w^2(s)\, ds\right\} \\
&\quad \left\{\frac{1}{H(t_n,T)} \int_T^{t_n} k(s)\,\Phi(s)\, h^2(t_n,s)\, ds\right\} \\
&\leq u(t_n) \left\{\frac{1}{H(t_n,T)} \int_T^{t_n} k(s)\,\Phi(s)\, h^2(t_n,s)\, ds\right\},
\end{aligned}$$

for any positive integer n. Consequently for n large enough,

$$\frac{v^2(t_n)}{u(t_n)} \leq \frac{1}{H(t_n, T)} \int_T^{t_n} k(s)\, \Phi(s)\, h^2(t_n, s)\, ds.$$

By (2.36), we have

$$\lim_{n\to\infty} \frac{1}{H(t_n, T)} \int_T^{t_n} k(s)\, \Phi(s)\, h^2(t_n, s)\, ds = \infty.$$

Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_T^{t} k(s)\, \Phi(s)\, h^2(t, s)\, ds = \infty,$$

which contradicts assumption (2.22). Therefore, (2.29) fails to hold and we have proved that (2.28) holds. Hence, by (2.25),

$$\int_T^{\infty} \frac{\varphi_+^2(s)}{\upsilon(s)\, \Phi(s)\, k(s)} ds \leq \int_T^{\infty} \frac{\upsilon(s)\, w^2(s)}{\Phi(s)\, k(s)} ds < \infty,$$

which contradicts (2.24). This completes our proof. ■

Corollary 2.6. Let assumptions (2.1) and (2.2) hold, the functions H, h, υ be defined as in Theorem 2.1, and let (2.21) hold. If there exists a function $\varphi \in C^1([t_0, \infty), \mathbb{R})$ such that (2.24) and

$$\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_T^{t} H(t, s)\, \psi(s)\, ds < \infty, \tag{2.37}$$

$$\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_T^{t} \left[H(t, s)\, \upsilon(s)\, \psi(s) - \frac{1}{4} \Phi(s)\, k(s)\, h^2(t, s) \right] ds \geq \varphi(T), \tag{2.38}$$

where $\varphi_+(s) = \max\{\varphi(t), 0\}$, then equation (1.1) is oscillatory for all $\alpha > 1$.

Remark 2.7. If we take

$$H(t, s) = (t - s)^n,$$

where n is an integer with $n > 1$ as in Remark 2.3,

$$\upsilon(s) = 1, \; h(t, s) = (t - s)^{\frac{n-2}{2}} (n + 2(t - s)\, \beta(s)),$$

then the following oscillation criterion can be obtained from Theorem 2.5.

Corollary 2.8. Let assumptions (2.1) and (2.2) hold. If

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_T^{t} L(t, s) ds < \infty,$$

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_T^t\left[(t-s)^n\,\psi(s)-\frac{1}{4}L(t,s)\right]ds\geq\varphi(T),$$

where

$$L(t,s):=\Phi(s)\,k(s)\,(t-s)^{n-2}\,(n+2\,(t-s)\,\beta(s))^2$$

for some integer $n>1$, and (2.24) holds, then equation (1.1) is oscillatory for all $\alpha>1$.

Remark 2.9. When we choose $\phi(t)=0$ in the above results, we get another result for the oscillation of equation (1.1) for all $\alpha>0$.

Corollary 2.10. Suppose that there exist continuous functions $H,\ h:D\to\mathbb{R}$ such that

(i) $H(t,t)=0,\quad$ for $t\geq t_0$,

(ii) $H(t,s)>0,\quad$ for $t>s\geq t_0$,

(iii) H has a continuous and nonpositive partial derivative on D with respect to the second variable.

Suppose there exists a function $\upsilon\in C^1([t_0,\infty),(0,\infty))$, $T\geq t_0$ such that, for some $t_1\geq t_0,\ \sigma(t)\geq t_1$ for $t\geq T$ and

$$-\frac{\partial}{\partial s}\left(H(t,s)\,\upsilon(s)\right)=h(t,s)\sqrt{H(t,s)\,\upsilon(s)}.$$

If

$$\liminf_{t\to\infty}\int_{t_0}^t Q(s)\,ds>0,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\,\upsilon(s)\,Q(s)-\frac{1}{4}k(s)\,h^2(t,s)\right]ds=\infty,$$

then equation (1.1) is oscillatory for all $\alpha>0$.

Example 2.11. Consider the following second order neutral delay differential equation, for $t\geq 3,\ \alpha>0$,

$$\left(\frac{1}{\sqrt{t}}\left[y(t)+\frac{1}{\sqrt{t-1}}y(t-1)\right]'\right)'+\frac{t^{\alpha+1}\,(2+\cos t)}{(t-2)^{\alpha}}\left|y\left(\frac{t}{3}\right)\right|^{\alpha}\operatorname{sgn}y\left(\frac{t}{3}\right)=0. \tag{2.39}$$

Let us take $H(t,s)=(t-s)^2$ and $v(t)=\dfrac{1}{t}$. Then $t_1=t_0=3$, $T=9$,

$$h(t,s)=\frac{t}{s\sqrt{s}}+\frac{1}{\sqrt{s}},\ \ Q(t)=\frac{t^{\alpha+1}\,(2+\cos t))}{(t-2)^{\alpha}}\left(1-\frac{\sqrt{3}}{\sqrt{t-3}}\right)^{\alpha},$$

and

$$k(t)=\frac{\sqrt{3}N_1^2}{\alpha^2}\left(2t-\frac{81}{\sqrt{t}}\right).$$

Now, we can prove that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{4H(t,T)}\int_T^t k(s)\,h^2(t,s)\,ds\\
&= \lim_{t\to\infty}\frac{\sqrt{3}N_1^2}{4\alpha^2(t-9)^2}\int_9^t\left(2s-\frac{81}{\sqrt{s}}\right)\left(\frac{t}{s\sqrt{s}}+\frac{1}{\sqrt{s}}\right)^2 ds\\
&= \frac{N_1^2}{15\sqrt{3}\alpha^2} < \infty,
\end{aligned}$$

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\,\upsilon(s)\,Q(s)\,ds\\
&= \lim_{t\to\infty}\frac{1}{(t-9)^2}\int_9^t\frac{(t-s)^2}{s}\frac{s^{\alpha+1}(2+\cos(s))}{(s-2)^\alpha}\left(1-\frac{\sqrt{3}}{\sqrt{s-3}}\right)^\alpha ds\\
&= \infty.
\end{aligned}$$

Then all the hypotheses of Corollary 2.10 are satisfied. Hence equation (2.39) is oscillatory for $\alpha > 0$. Note that none of the above mentioned oscillation criteria can be applied to (2.39).

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