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The Fell Topology on the Space of Time Scales for Dynamic Equations[1]

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Abstract

We wish to examine the dynamics of solutions of dynamic equation on time scales as the time scales change. Toward this end, we examine the standard topologies on the space of time scales and show that the Fell topology is desirable.

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1. Introduction

In his 1988 dissertation [7], S. Hilger developed the calculus on times scales. A thorough introduction is contained in [1]. A time scale is a nonempty closed subset of $\mathbb{R}$. A derivative, called the Δ-derivative, is defined for a function f whose domain is a time scale $\mathbb{T}$ and is denoted $f^{\Delta}(t)$ at any $t \in \mathbb{T}$ $(t < \sup T)$. $f^{\Delta}(t)$ is designed to mimic the standard right-hand derivative $f'(t)$ when there exists a strictly decreasing sequence convergent to t in $\mathbb{T}$ and a scaled difference operator otherwise. In particular, $f^{\Delta}(t) = f'(t)$ on $\mathbb{R}$ and $f^{\Delta}(t) = \Delta f(t)$ on $\mathbb{Z}$. While the Δ-derivative is a "forward" operator, an analogous "backwards" operator exists called the ∇-derivative.

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As for differential and difference equations, we are interested in equations with Δ-derivatives, called dynamic equations. Given a dynamic equation, say the initial value problem

$$x^{\Delta} = f(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

the solution inherently depends on the time scale. Broadly, we would like to examine how the solution of (1.1) depends on the time scale that is its domain. A few things are known.

B. Garay, S. Hilger, and P. Kloeden showed in [4] that uniqueness of solutions of the initial value problem near a given compact time scale—guaranteed by a Lipschitz condition—implies the continuous dependence of the solutions over compact time scales. Nonuniqueness is not considered there; this is arguably the more interesting situation.

In [2] and [9], we considered a particular (logistic) initial value problem:

$$x^{\Delta} = 4x\left(\tfrac{3}{4} - x\right), \quad x(0) = x_0$$

on $\mathbb{R}_+$ as well as on the Eulerian time scales $\mu\mathbb{Z}_+$ (see [5]) for $0 < \mu \le 1$.

By definition of $x^{\Delta}(t)$, for every $t \in \mu\mathbb{Z}_+$,

$$\tfrac{x(t+\mu)-x(t)}{\mu} = 4x(t)\left(\tfrac{3}{4} - x(t)\right).$$

Hence,

$$\begin{aligned} x(t+\mu) &= 4\mu x(t)\left(\tfrac{3}{4} - x(t)\right) + x(t) \\ &= 4\mu x(t)\left(\tfrac{3\mu+1}{4\mu} - x(t)\right) \end{aligned}$$

and the solution is found by iterating

$$L_\mu(x) = 4\mu x\left(\tfrac{3\mu+1}{4\mu} - x\right).$$

starting from $x(0) = x_0$.

Note that when $\mu = 1$, $L_\mu(x) = 4x(1-x)$ and $\mu\mathbb{Z} = \mathbb{Z}$. On the other hand, as $\mu \to 0$, the solutions tend towards the solution of the logistic differential equation on $\mathbb{R}_+$.

The dynamics of the quadratic polynomial L_μ is easily understood. How the dynamics changes as the time scale changes (i.e., as μ varies between 0 and 1) is interesting.

Theorem 1.1. *L_μ is topologically conjugate to some $Q_c(x) = x^2 + c$, where every value of $\mu \in (0, 1]$ corresponds exactly to one value of $c \in \left[-2, 1/4\right)$, $\mu = 1$ corresponds to $c = -2$, and $c \to 1/4$ as $\mu \to 0$.*

The proof is a standard computation. The real interval $\left[-2, 1/4\right]$ is the real part of the Mandelbrot set for the family Q_c. Hence, passing through the time scales $\mu\mathbb{Z}$ from difference equation when $\mu = 1$ toward differential equation as $\mu \to 0$, we see all of the interesting dynamics of real quadratic polynomials and all of their bifurcations!

In the real case, the bifurcation sequence from $c = 1/4$ to $c = -2$ for Q_c is unique up to cancellation of inverses, *i.e.*, without going “back and forth" along the interval $\left[-2, 1/4\right]$. That is, there is a unique reduced bifurcation sequence.

In the complex case, the bifurcation sequence from $c = 1/4$ to $c = -2$ for Q_c is far from unique. For example, one could follow a path looping half-around the Mandelbrot set staying in the region of Cantor sets and one-sided shifts (horseshoes). If we take a path inside the Mandelbrot set, which is simply connected, then the matter is again uniquely determined.

In this example, we have realized the domain of the solutions on eulerian time scales as a parameter of a family of dynamical systems. We do not know what happens when non-Eulerian time scales (i.e., not $\mu\mathbb{Z}$) are allowed. We have also not had to worry about nonunique solutions.

This suggests the following approach. For any given initial value problem, treat the time scales as a parameter. Let A denote the set of all time scales and let B denote the set of all solutions of the initial value problem on all possible time scales. Consider the canonical projection:

$$\begin{array}{c} B \\ \big\downarrow{\scriptstyle \pi} \\ A \end{array}$$

That is, an element of B, a solution $f : \mathbb{T} \to \mathbb{R}$, projects to its domain, $\mathbb{T}$. What can we say about this projection, especially when there are nonunique solutions? Under what conditions is there unique lifting? Can we follow two different paths from the same starting point (a solution on the initial time scale) to different solutions following the same path of time scales? Can a loop in $\mathrm{CL}(\mathbb{R})$ lift to a path that is not a loop? How can this approach help us to understand the changes in dynamics of solutions caused by changes in their time scales? In order to make sense of these questions, we must first discuss the topologies on these sets.

2. Topologies on Sets of Closed Subsets

Given a topological space X, researchers in hyperspace theory use the following notation:

$$\mathrm{CL}(X) = \{ A \subset X \mid A \neq \emptyset \text{ and } A \text{ is closed in } X \}$$

and

$$2^X = \{ A \in \mathrm{CL}(X) \mid A \text{ is compact} \}.$$

Other authors use $\exp X$ and $\exp_c X$, respectively. We are especially interested in the set of time scales, $\mathrm{CL}(\mathbb{R})$, and the sets of functions on time scales.

There are several well-known topologies in use in hyperspace theory. Among these are the Hausdorff metric topology (for a metric space) introduced in [6] and the Vietoris topology introduced in [12]. See [8] for a good introduction to these hyperspace topologies.

Following a suggestion of N. Esty, we propose an alternative topology, the Fell topology, introduced in [3]. The Fell topology seems to be more appropriate for time scales.

Let X be a metrizable space and let d be a bounded metric on X. For example, for $X = \mathbb{R}$, d will denote the truncated metric $d(x, y) = \min\{1, |y - x|\}$. Let us consider these topologies on $\mathrm{CL}(X)$.

Definition 2.1. The *Hausdorff metric* on $\mathrm{CL}(X)$ with respect to d is defined by, for all $\mathbb{S}, \mathbb{T} \in \mathrm{CL}(X)$,

$$\begin{aligned} H_d(\mathbb{S}, \mathbb{T}) &= \sup \left\{ \sup_{s \in \mathbb{S}} d(s, \mathbb{T}), \sup_{t \in \mathbb{T}} d(\mathbb{S}, t) \right\} \\ &= \sup \left\{ \sup_{s \in \mathbb{S}} \inf_{t \in \mathbb{T}} d(s, t), \sup_{t \in \mathbb{T}} \inf_{s \in \mathbb{S}} d(s, t) \right\}. \end{aligned}$$

We denote the resulting topology by $\tau(H_d)$.

We use the following notation to define the Vietoris and Fell topologies.

Definition 2.2. For any $E \subset X$, let

$$E^- = \{A \in \mathrm{CL}(X) | A \cap E \neq \emptyset\}$$

and

$$E^+ = \{A \in \mathrm{CL}(X) | A \subset E\} = \{A \in \mathrm{CL}(X) | A \cap (X - E) = \emptyset\}.$$

In the literature, it is said that every $A \in E^-$ "hits" E and every $A \in E^+$ "misses" $X - E$; E^- is a "hit" set and E^+ is a "miss" set. Note that $E^+ \subset E_-$ for every E.

Call a subset of X *cocompact* if its complement is compact. We next define the Vietoris and Fell topologies.

Definition 2.3.

(a) A subbasis for the *lower Vietoris topology*, $\tau(V^-)$, consists of sets of the form U^- for all open subsets U of X.

(b) A subbasis for the *upper Vietoris topology*, $\tau(V^+)$, consists of sets of the form U^+ for all open subsets U of X.

(c) A subbasis for the *upper Fell topology*, $\tau(F^+)$, consists of sets of the form U^+ for all cocompact subsets U of X.

(d) The *Vietoris topology* is the join of the lower and upper Vietoris toplogies: $\tau(V) = \tau(V^-) \vee \tau(V^+)$.

(e) The *Fell topology* is the join of the lower Vietoris and upper Fell toplogies: $\tau(F) = \tau(V^-) \vee \tau(F^+)$.

The Vietoris and Fell topologies are known as hit-and-miss topologies. The following examples show that the Hausdorff and Vietoris topologies are not completely satisfactory on the space of time scales, $\mathrm{CL}(\mathbb{R})$. Recall that, on $\mathbb{R}$, we let d be the truncated metric.

Example 2.4. For $n \geq 0$, $H_d([0,n], \mathbb{R}_+) = 1$. Hence, $\{[0,n]\}$ does not converge to $\mathbb{R}_+$ in the Hausdorff topology.

Example 2.5. $\mathbb{Z} + 1/n$ does not converge to $\mathbb{Z}$ as $n \to \infty$ in the Vietoris topology since, for each $n > 1$, $\mathbb{Z} + 1/n \notin U^+$ where

$$U = \bigcup_{k=1}^{\infty} \left(k - \frac{1}{k}, k + \frac{1}{k} \right).$$

Notice that U^+ in Example 2.5 is Vietoris-open but not Fell-open since U is not cocompact. Let us further compare these three topologies on $\mathrm{CL}(X)$.

Theorem 2.6. Let X be a topological space.

(a) In general, the Vietoris topology is not metrizable.

(b) If X is compact metrizable, then the Hausdorff, Vietoris, and Fell topologies agree on $\mathrm{CL}(X)$.

(c) $\mathrm{CL}(X)$ is compact in the Fell topology.

Proof. Regarding (a), for a T_1 space X, $\tau(H_d) = \tau(V)$ if and only if X is compact (see [8]). See [11] for (b); also see Theorem 2.7 below. Fell proved (c) in [3]. ■

Theorem 2.7. Let X be a topological space.

(a) If X is metrizable, then, in general, the Hausdorff and Vietoris topologies are not comparable on $\mathrm{CL}(X)$.

(b) The Vietoris topology is always finer and, generally, strictly finer than the Fell topology.

(c) If X is metrizable, then the Hausdorff topology is always finer and, generally, strictly finer than the Fell topology.

Proof.

(a) We show that they are not comparable on $\mathrm{CL}(\mathbb{R})$.

Let $B \in \tau(H_d)$ be the open ball $B_{1/3}(\mathbb{N})$. From [8], we know that a basis for the Vietoris topology consists of sets of the form

$\langle U_1, \ldots, U_n \rangle = \{A \in \mathrm{CL}(X) | A \subset U_1 \cup \ldots U_n \text{ and } A \cap U_i \neq \emptyset \text{ for } i = 1, \ldots, n\}$, where $U_1, \ldots, U_n$ is any finite collection of open subsets of $\mathbb{R}$. Assume that $V = \langle U_1, U_2, \ldots, U_n \rangle$ is a basic open set in $\tau(V)$ containing $\mathbb{N}$. There exists k such that $i, j \in U_k \cap \mathbb{N}$ with $i \neq j$. Let $\mathbb{N}_1 = \mathbb{N} - \{i\} \in V$. Since $H_d(\mathbb{N}, \mathbb{N}_1) = 2/3$, $\mathbb{N}_1 \notin B$. Hence $V \not\subset B$. Therefore $\tau(H_d) \not\subset \tau(V)$. Let

$$U = \bigcup_{k=1}^{\infty} \left(k - \frac{1}{k}, k + \frac{1}{k} \right).$$

Then $\mathbb{N} \in U^+ = \langle U \rangle \in \tau(V)$. No open ball $B_\varepsilon(\mathbb{N})$ is a subset of $\langle U \rangle$ since the diameters of the intervals about k in U tend to 0 as $k \to \infty$. Therefore, $\tau(V) \not\subset \tau(H_d)$.

(b) Since cocompact sets are open, the Fell topology is coarser (i.e., weaker) than the Vietoris topology: $\tau(F) \subset \tau(V)$. In $\mathrm{CL}(\mathbb{R})$, U^+ for $U = (0, 1)$, is Vietoris-open, but not Fell-open, since U is not cocompact.

(c) $\tau(H_d) \not\subset \tau(F)$ on $\mathrm{CL}(\mathbb{R})$ since, otherwise, $\tau(H_d) \subset \tau(F) \subset \tau(V)$.

S. Naimpally has shown in [10] that $\tau(H_d)$ is also a hit-and-miss topology. In fact, the Hausdorff topology is the discrete-hit-and-far-miss topology (see [11]): $\tau(H_d) = \tau(L_d) \vee \tau(\delta^+)$, the join of the lower discrete and the upper far topologies. Since $\tau(V^-) \subset \tau(L_d)$ and $\tau(F^+) \subset \tau(\delta^+)$, the result follows. (This result is surprisingly difficult to prove directly from a metric definition.) ■

In particular, in the lattice of topologies on $\mathrm{CL}(\mathbb{R})$,

- $\tau(H_d) \not\subset \tau(V)$ and $\tau(V) \not\subset \tau(H_d)$;
- $\tau(F) \subset \tau(V)$ and $\tau(V) \not\subset \tau(F)$; and
- $\tau(F) \subset \tau(H_d)$ and $\tau(H_d) \not\subset \tau(F)$.

3. Convergence in $\mathrm{CL}(\mathbb{R})$ with the Fell Topology

Under the Fell topology on $\mathrm{CL}(\mathbb{R}^+)$, we wish to show that

$$[0, n] \to \mathbb{R}_+ \text{ as } n \to \infty \text{ and } \mathbb{Z} + \frac{1}{n} \to \mathbb{Z} \text{ as } n \to \infty.$$

In fact, we wish to prove that convergence under the Fell topology works as we would have it.

Definition 3.1. Let $\{\mathbb{T}_n\}$ be a sequence in $\mathrm{CL}(\mathbb{R})$ and let $t \in \mathbb{R}$.

(a) t is a *sequential limit point of* $\{\mathbb{T}_n\}$ if there exists a sequence $\{t_n\}$ such that, for all $n \in \mathbb{N}$, $t_n \in \mathbb{T}_n$ and $t_n \to t$ in $\mathbb{R}$ as $n \to \infty$.

(b) t is a *subsequential limit point* of $\{\mathbb{T}_n\}$ if t is a sequential limit point of a subsequence $\{\mathbb{T}_{n_i}\}$.

(c) Set $\mathbb{T}$ equal to the set set of all sequential limit points of $\{\mathbb{T}_n\}$ and $\mathbb{T}'$ equal to the set set of all subsequential limit points of $\{\mathbb{T}_n\}$.

We say that t_n converges to a sequential limit t "through the $\mathbb{T}_n$'s." Similarly, t_{n_i} converges to a subsequential limit t "through the $\mathbb{T}_{n_i}$'s." Clearly, $\mathbb{T} \subset \mathbb{T}'$.

Lemma 3.2. The sequential limit set $\mathbb{T}$ is closed, i.e., $\mathbb{T} \in \mathrm{CL}(\mathbb{R})$.

Proof. If a sequence $\{s_i\}$ in $\mathbb{T}$ converges to t, then, for every i, there exist sequences $\{t_{i,n}\}$ converging to s_i through the $\mathbb{T}_n$'s and the sequence $\{t_{n,n}\}$ converges to t through the $\mathbb{T}_n$'s. ■

The interesting situation appears to be when the sequential and subsequential limit sets of a given sequence in $\mathrm{CL}(\mathbb{R})$ are equal: $\mathbb{T} = \mathbb{T}'$.

Conjecture 3.3. $\mathbb{T}_n \to \mathbb{S}$ as $n \to \infty$ in $\mathrm{CL}(\mathbb{R})$ if and only if $\mathbb{S} = \mathbb{T} = \mathbb{T}'$.

Assuming the validity of this conjecture, is the Fell topology unique with respect to this property or in any sense optimal?

References

[1] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[2] E. Duke, K. Hall, and R. Oberste-Vorth, Changing time scales I: the continuous case as a limit, In N. Mastoraki, editor, *Proceedings of the Sixth WSEAS International Conference on Applied Mathematics*, 2004.

[3] J.M.G. Fell, A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space, *Proc. Amer. Math. Soc.*, 13:472–476, 1962.

[4] Barnabas M. Garay, Stefan Hilger, and Peter E. Kloeden, Continuous dependence in time scale dynamics, In *Proceedings of the Sixth International Conference on Difference Equations*, pages 279–287, Boca Raton, FL, 2004. CRC.

[5] K. Hall and R. Oberste-Vorth, Totally discrete and Eulerian time scales, In S. Elaydi, J. Cushing, R. Lasser, A. Ruffing, V. Papageorgious, and W. Van Assche, editors, *Difference Equations, Special Functions and Orthogonal Polynomials*, pages 462–470. World Scientific, 2007.

[6] Felix Hausdorff, *Grundzüge der Mengenlehre*, Chelsea Publishing Company, New York, N. Y., 1949.

[7] Stefan Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.

[8] Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces*, volume 216 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, 1999. Fundamentals and recent advances.

[9] B. Lawrence and R. Oberste-Vorth, Solutions of dynamic equations with varying time scales, In S. Elaydi, J. Cushing, R. Lasser, A. Ruffing, V. Papageorgious, and W. Van Assche, editors, *Difference Equations, Special Functions and Orthogonal Polynomials*, pages 452–461. World Scientific, 2007.

[10] Somshekhar Naimpally, All hypertopologies are hit-and-miss, *Appl. Gen. Topol.*, 3(1):45–53, 2002.

[11] Somshekhar Naimpally, What is a hit-and-miss topology? *Topol. Comment*, 8, 2003.

[12] Leopold Vietoris, Bereiche zweiter Ordnung, *Monatsh. Math. Phys.*, 32(1):258–280, 1922.