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Intermediate Solutions for Emden–Fowler Type Equations: Continuous Versus Discrete[1]

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Abstract

The existence of a class of nonoscillatory solutions for a generalized Emden–Fowler differential equation is studied. Some analogies and discrepancies between the continuous and discrete case are also discussed.

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1. Introduction

Consider the second order differential equation

$$\big(a(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)\big)' + b(t)|x(t)|^{\beta} \operatorname{sgn} x(t) = 0, \tag{1.1}$$

where $\alpha,\ \beta$ are two positive constants, $\alpha \neq \beta$, a, b are positive continuous functions for $t \geq 0$ such that

$$I_a = \int_0^{\infty} \left(\frac{1}{a(t)}\right)^{1/\alpha} dt = \infty, \quad I_b = \int_0^{\infty} b(t)\, dt < \infty.$$

The prototype of (1.1) is the *Emden–Fowler equation*

$$x''(t) + b(t)|x(t)|^{\beta} \operatorname{sgn} x(t) = 0, \quad \beta > 0,\ \beta \neq 1. \tag{1.2}$$

Equation (1.1) is usually called the generalized Emden–Fowler equation and both equations have been widely investigated in the literature. Recent developments in the asymptotic behavior are in [4, 8, 9, 12, 14, 18, 20, 22, 25]. Other contributions can be found in the monographs [1, 2, 10, 17, 21] and references therein.

Some of the quoted papers deal with a classification of nonoscillatory solutions, based on suitable integral criteria. In particular, in [12, 20, 22], nonoscillatory solutions are classified as subdominant, intermediate or dominant solutions, according to their asymptotic behavior (see below for the definition). Nevertheless, the existence of the intermediate solutions is a difficult problem, see, e.g., [2, page 241], as well as their possible coexistence with different types of nonoscillatory solutions, see, e.g., [14, page 213]. Such a problem has been completely resolved in [4] for the *half-linear equation*, i.e. (1.1) with $\alpha = \beta$, which reads as

$$\big(a(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)\big)' + b(t)|x(t)|^{\alpha} \operatorname{sgn} x(t) = 0. \tag{1.3}$$

Our aim is to extend these recent results to the case $\alpha \neq \beta$, especially as it concerns the existence or nonexistence of the intermediate solutions. Our results generalize the well-known results, stated for (1.2), by Moore and Nehari for the superlinear case and Belohorec for the sublinear one (see, e.g., [24, Theorems 6.3, 6.4]). Our approach is based on a comparison with the half-linear case and on some integral inequalities and it is completely different from those used in papers by Moore–Nehari and Belohorec, in which the nonexistence of intermediate solutions of (1.2) is obtained using some asymptotic estimates for nonoscillatory solutions of (1.2).

We also discuss some analogies and discrepancies between the continuous and the discrete case. In particular, we will show that some recent nonoscillation criteria, stated for the differential equation (1.1), cannot be carried over *verbatim* to the difference equation

$$\Delta\big(a_n|\Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n\big) + b_n|x_{n+1}|^{\beta} \operatorname{sgn} x_{n+1} = 0, \tag{1.4}$$

where $\alpha,\ \beta$ are two positive constants, $\alpha \neq \beta$, $\{a_n\}$,$\{b_n\}$ are positive real sequences for $n \geq 0$ such that

$$Y_a = \sum_{n=0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\alpha} = \infty, \quad Y_b = \sum_{n=0}^{\infty} b_n < \infty.$$

Some notations are in order. Let λ, μ be two positive numbers and define

$$A(t) = \left(\frac{1}{a(t)}\right)^{1/\alpha}, \tag{1.5}$$

$$J_\lambda = \int_0^\infty A(t) \left(\int_t^\infty b(s)ds\right)^{1/\lambda}, \quad K_\mu = \int_0^\infty b(t) \left(\int_0^t A(\sigma)\, d\sigma\right)^{\mu} dt.$$

2. Preliminaries

Throughout this paper we shall consider only the solutions of (1.1) which exist on some ray $[\tau, \infty)$, where $\tau \geq 0$ may depend on the particular solution. As usual, a solution x of (1.1) is said to be *nonoscillatory* if $x(t) \neq 0$ for large t and *oscillatory* otherwise. The equation (1.1) is called *nonoscillatory* if all its solutions are nonoscillatory. This terminology holds also for (1.3), but, in such a case, some stronger results hold. In particular, any solution of (1.3) is defined in the whole half-line and the existence of a nonoscillatory solution implies the nonoscillation of (1.3) (see, e.g., [10]).

For any solution x of (1.1), denote by $x^{[1]}$ the *quasiderivative of* x, i.e., the function

$$x^{[1]}(t) = a(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t). \tag{2.1}$$

Since $I_a = \infty$, it is easy to verify that any nonoscillatory solution x of (1.1) is eventually monotone and verifies

$$x(t)x^{[1]}(t) > 0 \text{ for large } t;$$

we denote this property by saying that x is of class $\mathbb{M}^+$. Thus any nonoscillatory solution x is either eventually positive increasing such that $x^{[1]}$ is positive decreasing, or x is eventually negative decreasing such that $x^{[1]}$ is negative decreasing, and we can divide the class $\mathbb{M}^+$ of all nonoscillatory solutions into the three subclasses:

$$\mathbb{M}^+_{\infty,\ell} = \left\{x \in \mathbb{M}^+ : |x(\infty)| = \infty, x^{[1]}(\infty) = \ell_x, 0 < |\ell_x| < \infty\right\},$$

$$\mathbb{M}^+_{\infty,0} = \left\{x \in \mathbb{M}^+ : |x(\infty)| = \infty, x^{[1]}(\infty) = 0\right\},$$

$$\mathbb{M}^+_{\ell,0} = \left\{x \in \mathbb{M}^+ : x(\infty) = \ell_x, x^{[1]}(\infty) = 0,\ 0 < |\ell_x| < \infty\right\}.$$

Following [22], solutions in $\mathbb{M}^+_{\infty,\ell}$, $\mathbb{M}^+_{\infty,0}$, $\mathbb{M}^+_{\ell,0}$ are called *dominant solutions, intermediate solutions* and *subdominant solutions*, respectively. Indeed, if $x \in \mathbb{M}^+_{\infty,\ell}$, $y \in \mathbb{M}^+_{\infty,0}$, $z \in \mathbb{M}^+_{\ell,0}$, then we have

$$|x(t)| > |y(t)| > |z(t)| \text{ for large } t.$$

Concerning the existence of solutions in the classes $\mathbb{M}^+_{\infty,\ell}$ and $\mathbb{M}^+_{\ell,0}$, the following holds.

Theorem 2.1. i_1) Equation (1.1) has solutions in the class $\mathbb{M}^+_{\ell,0}$ if and only if

$$J_\alpha = \int_0^\infty A(t)\left(\int_t^\infty b(s)ds\right)^{1/\alpha} < \infty.$$

i_2) Equation (1.1) has solutions in the class $\mathbb{M}^+_{\infty,\ell}$ if and only if

$$K_\beta = \int_0^\infty b(t)\left(\int_0^t A(\sigma)\,d\sigma\right)^\beta dt < \infty.$$

Proof. The assertion follows from [11] (see also, e.g., [2, Theorems 3.13.11, 3.13.12] or [23, Theorems 4.3, 4.4]). ■

Remark 2.2. The above classification of nonoscillatory solutions is valid also when $\alpha = \beta$, i.e. in the half-linear case. In particular, Theorem 2.1 continues to hold for (1.3), but, clearly, in such a case, in claim i_2) the existence in $\mathbb{M}^+_{\infty,\ell}$ depends on the convergence of K_α, instead of K_β (see, e.g., [4]).

The existence of intermediate solutions for (1.1) is a more difficult problem, since for these solutions we do not have sharp upper or lower bounds. Nevertheless, such a problem has been completely solved in the half-linear case ([4]). In particular, the following result has been proved.

Theorem 2.3. ([4, Theorems 6 and 7]) Assume (1.3) is nonoscillatory. Then for (1.3) we have $M^+_{\infty,0} = \emptyset$ if and only if $J_\alpha < \infty$ and $K_\alpha < \infty$.

We close this section by recalling some useful relations on the change of integration for double integrals, which play an important tool in the asymptotic theory of differential equations.

In view of the Fubini theorem, we have $J_1 = K_1$. In general, the following holds.

Lemma 2.4. i_1) If $\lambda \geq 1$ and $J_\lambda < \infty$, then $K_\lambda < \infty$.
i_2) If $\lambda \leq 1$ and $K_\lambda < \infty$, then $J_\lambda < \infty$.
i_3) If $\lambda < \mu$ and $K_\mu < \infty$, then $J_\lambda < \infty$.
i_4) If $\lambda > \mu$ and $J_\lambda < \infty$, then $K_\mu < \infty$.

Proof. Claims i_1), i_2) are proved in [9, Corollary 1] and claims i_3), i_4) in [8, Lemmas 1, 2]. ■

In virtue of claims i_3), i_4) in Lemma 2.4, the possible cases of the mutual behavior of the integrals J_α, K_β ($\alpha \neq \beta$) are the following four:

C_1) $J_\alpha = \infty,\ K_\beta = \infty$;

C_2) $J_\alpha = \infty,\ K_\beta < \infty$ and $\alpha > \beta$;

C_3) $J_\alpha < \infty$, $K_\beta = \infty$ and $\alpha < \beta$;

C_4) $J_\alpha < \infty$, $K_\beta < \infty$.

3. An Existence Result

If the case C_1) occurs, then (1.1) is oscillatory (see, e.g., [21, Theorems 11.3, 11.4]). Clearly, this fact is not true in the half-linear case, as the Euler equation illustrates, see, e.g., [10, Theorem 1.4.4].

In virtue of Theorem 2.1, if any of the case C_i), $i = 2, 3, 4$, holds, then the class $\mathbb{M}^+$ is nonempty. It is well known that, when the case C_2) holds, then (1.1) has intermediate solutions (see, e.g., [22, Theorem 1.3] if $a \equiv 1$, or [20, Theorem 2.4]). In order to study the existence of these solutions in case C_3), the following uniqueness result for subdominant solutions is useful and makes Theorem 2.1-i_1) more complete.

Theorem 3.1. Assume $J_\alpha < \infty$. Then for any $L \neq 0$ there exists a unique solution x of (1.1) satisfying the boundary conditions

$$x(\infty) = L, \quad x^{[1]}(\infty) = 0. \tag{3.1}$$

Proof. Without loss of generality, assume $L > 0$. Put

$$h_1 = \begin{cases} (1/2)^{\beta-1} L^{\beta-1} & \text{if} \quad \beta \leq 1, \\ L^{\beta-1} & \text{if} \quad \beta > 1 \end{cases} \tag{3.2}$$

and

$$h_2 = \begin{cases} (1/2)^{(1-\alpha)\beta/\alpha}\, L^{(1-\alpha)\beta/\alpha} & \text{if} \quad \alpha \leq 1, \\ L^{(1-\alpha)\beta/\alpha} & \text{if} \quad \alpha > 1. \end{cases} \tag{3.3}$$

Choose $t_0 \geq 0$ large so that

$$\int_{t_0}^{\infty} A(t) \left(\int_t^{\infty} b(\sigma) d\sigma \right)^{1/\alpha} dt < \frac{1}{2} L^{1-(\beta/\alpha)} \tag{3.4}$$

and

$$h = \frac{\beta}{\alpha} h_1 h_2 \int_{t_0}^{\infty} A(s) \left(\int_s^{\infty} b(\sigma) d\sigma \right)^{1/\alpha} ds \ < 1, \tag{3.5}$$

where $A(t)$ is defined by (1.5). Set

$$\Omega = \left\{ u \in C[t_0, \infty),\ L/2 \leq u(t) \leq L \right\}$$

and consider the operator T given by

$$T(u)(t) = L - \int_t^{\infty} A(s) \left(\int_s^{\infty} b(\sigma) u^{\beta}(\sigma) d\sigma \right)^{1/\alpha} ds.$$

In view of (3.4), for any $u \in \Omega$ we have

$$\int_{t_0}^{\infty} A(s) \left(\int_s^{\infty} b(\sigma) u^{\beta}(\sigma) d\sigma \right)^{1/\alpha} ds \le L^{\beta/\alpha} \int_{t_0}^{\infty} A(s) \left(\int_s^{\infty} b(\sigma) d\sigma \right)^{1/\alpha} ds \le \frac{L}{2}$$

and so $T(u)(t) \ge L/2$. Since $T(u)(t) \le L$, we obtain $T(\Omega) \subset \Omega$.

Let BC be the metric space of bounded continuous functions on $[t_0, \infty)$ with distance given by $d(f, g) = \sup_{t \in [t_0, \infty)} |f(t) - g(t)|$. Now we prove that T is a contraction in Ω with respect to d. From the mean value theorem we have for $M, N > 0$ and $q > 0$

$$|M^q - N^q| \le q|M - N| \max \left\{ N^{q-1}, M^{q-1} \right\}. \tag{3.6}$$

So for $u, v \in \Omega$, in view of (3.2), we obtain for $\sigma \ge t_0$

$$|u^{\beta}(\sigma) - v^{\beta}(\sigma)| \le \beta h_1 |u(\sigma) - v(\sigma)|. \tag{3.7}$$

For any $w \in \Omega$ we have

$$\frac{1}{2^{\beta}} L^{\beta} \int_s^{\infty} b(\sigma) d\sigma \le \int_s^{\infty} b(\sigma) w^{\beta}(\sigma)\, d\sigma \le L^{\beta} \int_s^{\infty} b(\sigma) d\sigma$$

and so, from (3.3) we obtain for $s \ge t_0$

$$\left(\int_s^{\infty} b(\sigma) w^{\beta}(\sigma)\, d\sigma \right)^{(1-\alpha)/\alpha} \le h_2 \left(\int_s^{\infty} b(\sigma) d\sigma \right)^{(1-\alpha)/\alpha}. \tag{3.8}$$

Applying again the inequality (3.6) with

$$M = \int_s^{\infty} b(\sigma) u^{\beta}(\sigma)\, d\sigma, \quad N = \int_s^{\infty} b(\sigma) v^{\beta}(\sigma)\, d\sigma, \quad q = 1/\alpha$$

and, taking into account (3.8), we get for $s \ge t_0$

$$\begin{aligned} &\left| \left(\int_s^{\infty} b(\sigma) u^{\beta}(\sigma) d\sigma \right)^{1/\alpha} - \left(\int_s^{\infty} b(\sigma) v^{\beta}(\sigma) d\sigma \right)^{1/\alpha} \right| \\ &\le \frac{1}{\alpha} h_2 \left(\int_s^{\infty} b(\sigma) d\sigma \right)^{(1-\alpha)/\alpha} \int_s^{\infty} b(\sigma) |u^{\beta}(\sigma) - v^{\beta}(\sigma)|\, d\sigma. \end{aligned} \tag{3.9}$$

Thus, in view of (3.5), (3.7) and (3.9), we have

$$
\begin{aligned}
&|T(u)(t) - T(v)(t)| \\
&\le \int_t^\infty A(s) \left| \left(\int_s^\infty b(\sigma) u^\beta(\sigma) d\sigma \right)^{1/\alpha} - \left(\int_s^\infty b(\sigma) v^\beta(\sigma) d\sigma \right)^{1/\alpha} \right| ds \\
&\le \frac{1}{\alpha} h_2 \int_t^\infty A(s) \left(\int_s^\infty b(\sigma) d\sigma \right)^{(1-\alpha)/\alpha} \left(\int_s^\infty b(\sigma) |u^\beta(\sigma) - v^\beta(\sigma)| \, d\sigma \right) ds \\
&\le \frac{\beta}{\alpha} h_1 h_2 \int_t^\infty A(s) \left(\int_s^\infty b(\sigma) d\sigma \right)^{(1-\alpha)/\alpha} \left(\int_s^\infty b(\sigma) |u(\sigma) - v(\sigma)| \, d\sigma \right) ds \\
&\le h \, d(u, v).
\end{aligned}
$$

Hence the operator T is a contraction in Ω. Since $T(\Omega) \subset \Omega$, by applying the contraction theorem, we obtain the existence of a unique fixed point of T in Ω. It is easy to verify that every eventually positive solution x of (1.1) satisfying (3.1) belongs to Ω (by choosing a suitable large t_0) and satisfies the equation $x = T(x)$. So the proof is complete. ∎

In order to prove the existence of intermediate solutions, the following nonoscillation criterion, which extends to (1.1) a result stated for (1.2) by Kiguradze ([16]), plays a crucial role.

Theorem 3.2. **([21, Theorem 14.3])** Assume $\beta > \alpha$ and let the functions a, b be absolutely continuous on every finite interval. If $I_a = \infty$ and there exists $\varepsilon > 0$ such that the function

$$
b(t) A^{-1/\alpha}(t) \left(\int_0^t A(\sigma) d\sigma \right)^{\delta+\varepsilon} \text{ is nonincreasing for large } t, \tag{3.10}
$$

where

$$
\delta = \frac{1 + \alpha\beta + 2\alpha}{\alpha + 1},
$$

then (1.1) is nonoscillatory.

From Theorems 2.1, 3.1 and 3.2 the following result follows.

Theorem 3.3. Assume $J_\alpha < \infty$ and $K_\beta = \infty$, i.e. the case C_3) occurs, and let the functions a, b be absolutely continuous on every finite interval. If there exists $\varepsilon > 0$ such that (3.10) is satisfied, then (1.1) has solutions in the class $\mathbb{M}^+_{\infty,0}$.

Proof. In virtue of Theorem 3.1, the class $\mathbb{M}^+_{\ell,0}$ is one parametric family of solutions with the parameter different from zero. By Theorem 2.1 we have $\mathbb{M}^+_{\infty,\ell} = \emptyset$. Applying Theorem 3.2, all solutions of (1.1) are nonoscillatory and, consequently, $\mathbb{M}^+_{\infty,0} \neq \emptyset$. ∎

4. Some Nonexistence Results

As we showed in Section 3, the intermediate solutions exist in cases C$_2$) and C$_3$). In this section we investigate these solutions in case C$_4$). By Theorem 2.3, in case C$_4$) intermediate solutions do not exists if $\alpha = \beta$. Using a comparison method, we can extend this result to (1.1).

Theorem 4.1. Equation (1.1) does not have solutions in the class $\mathbb{M}^{+}_{\infty,0}$ if any of the following conditions is satisfied:

i$_1$) $K_\beta < \infty, \quad \beta \ge \alpha, \;\; 0 < \alpha \le 1;$

i$_2$) $J_\alpha < \infty, \quad \alpha \ge \beta, \;\; \alpha \ge 1.$

Proof. Let x be a solution of (1.1) in the class $\mathbb{M}^{+}_{\infty,0}$.

Claim i$_1$). Since $K_\beta < \infty$, from Theorem 2.1-i$_2$) we have $\mathbb{M}^{+}_{\infty,\ell} \ne \emptyset$. Let z be a solution of (1.1) in the class $\mathbb{M}^{+}_{\infty,\ell}$ and, without loss of generality, suppose $z(t) > x(t) > 0$, for $t \ge t_0 \ge 0$. Consider the half-linear equations ($t \ge t_0$)

$$(a(t)|v'|^\alpha \operatorname{sgn} v')' + b_x(t)|v|^\alpha \operatorname{sgn} v = 0, \tag{4.1}$$

$$(a(t)|w'|^\alpha \operatorname{sgn} w')' + b_z(t)(t)|w|^\alpha \operatorname{sgn} w = 0, \tag{4.2}$$

where

$$b_x(t) = b(t)x^{\beta-\alpha}(t), \;\; b_z(t) = b(t)z^{\beta-\alpha}(t).$$

Obviously, x is a solution of (4.1) and z solution of (4.2). Integrating (1.1) we obtain

$$\int_{t_0}^{\infty} b(s)z^\beta(s)ds < \infty,$$

which yields

$$\int_{t_0}^{\infty} b_z(s)ds < \infty.$$

Hence for the half-linear equation (4.2) we have $I_{b_z} < \infty$, $I_a = \infty$. Since (4.2) has a solution in the class $\mathbb{M}^{+}_{\infty,\ell}$ (indeed z is such a solution), from Theorem 2.1-i$_2$) and Remark 2.2 we have

$$\int_{t_0}^{\infty} b_z(t)\Big(\int_{t_0}^{t} A(s)\,ds\Big)^\alpha dt < \infty$$

and so

$$\int_{t_0}^{\infty} b_x(t)\Big(\int_{t_0}^{t} A(s)\,ds\Big)^\alpha dt < \infty.$$

Since $\alpha \le 1$, using Lemma 2.4-i$_2$) we obtain

$$\int_{t_0}^{\infty} A(t)\Big(\int_{t}^{\infty} b_x(s)\,ds\Big)^{1/\alpha} dt < \infty.$$

Thus, by applying Theorem 2.3 to the half-linear equation (4.1), we obtain that (4.1) does not have intermediate solutions. This is a contradiction, since x is a solution of (4.1).

Claim i$_2$). Without loss of generality, suppose $x(t) > 1$ for $t \geq t_0 \geq 0$. Consider again the half-linear equation (4.1). Since $\alpha \geq \beta$, we have for $t \geq t_0$

$$b_x(t) < b(t)$$

and so, because $J_\alpha < \infty$, we obtain

$$\int_{t_0}^{\infty} A(t)\Big(\int_t^{\infty} b_x(s)\,ds\Big)^{1/\alpha} dt < \infty.$$

Since $\alpha \geq 1$, using Lemma 2.4-i_1) we have

$$\int_{t_0}^{\infty} b_x(t)\,\Big(\int_{t_0}^{t} A(s)\,ds\Big)^{\alpha} dt < \infty.$$

Thus, again by applying Theorem 2.3 to the half-linear equation (4.1), we obtain that (4.1) does not have intermediate solutions. This is a contradiction, since x is a solution of (4.1). ■

Remark 4.2. Under the assumptions of Theorem 4.1, we have by Lemma 2.4-i$_3$), i$_4$) that the case C$_4$) holds.

From Theorem 4.1 we get for (1.2) the following result stated by Belohorec (if $\beta < 1$) and Moore–Nehari (if $\beta > 1$), see, e.g., [24, Theorems 6.3 and 6.4].

Corollary 4.3. Equation (1.2) does not have intermediate solutions if $J_1 < \infty$ and $K_\beta < \infty$.

When $\beta > \alpha > 1$ or $\beta < \alpha < 1$, Theorem 4.1 cannot be applied. Assuming stronger conditions, we can prove that also in these cases intermediate solutions do not exist.

Theorem 4.4. Equation (1.1) does not have solutions in the class $\mathbb{M}^{+}_{\infty,0}$ if any of the following conditions is satisfied:

i$_1$) $\beta > \alpha > 1$, $K_\beta < \infty$ and

$$J_\beta = \int_0^{\infty} A(t)\left(\int_t^{\infty} b(s)ds\right)^{1/\beta} dt < \infty.$$

i$_2$) $\beta < \alpha < 1$, $J_\alpha < \infty$ and

$$K_\alpha = \int_0^{\infty} b(t)\left(\int_0^{t} A\,(\sigma)\,d\sigma\right)^{\alpha} dt < \infty.$$

In order to prove Theorem 4.4, the following lemma will be needed.

Lemma 4.5. Consider the half-linear differential equation

$$\left(c(t)|x'(t)|^{\gamma}\operatorname{sgn} x'(t)\right)' + b(t)|x(t)|^{\gamma}\operatorname{sgn} x(t) = 0, \tag{4.3}$$

where c is a positive continuous function for $t \geq t_0 \geq 0$ and $\gamma > 0$. If (4.3) has a nonoscillatory unbounded solution, then

$$\int_{t_0}^{\infty}\left(\frac{1}{c(t)}\right)^{1/\gamma} dt = \infty.$$

Proof. Let x be a nonoscillatory unbounded solution of (4.3). Without loss of generality assume $x(t) > 0$ and $x^{[1]}(t) > 0$ for $t \geq t_0$. Then there exists $h > 0$ such that for $t \geq t_0$

$$\frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq h. \tag{4.4}$$

Integrating this inequality, we obtain

$$\ln\frac{x(t)}{x(t_0)} < h^{1/\gamma}\int_{t_0}^{t}\left(\frac{1}{c(s)}\right)^{1/\gamma} ds,$$

and so the assertion follows. ■

Proof of Theorem 4.4. Let x be a solution of (1.1) in the class $\mathbb{M}^{+}_{\infty,0}$ and, without loss of generality, suppose $x(t) > 0$, $x'(t) > 0$ for $t \geq t_0 \geq 0$.

Claim i$_1$). Consider for $t \geq t_0$ the half-linear equation

$$(a_x(t)|z'|^{\beta}\operatorname{sgn} z')' + b(t)|z|^{\beta}\operatorname{sgn} z = 0, \tag{4.5}$$

where

$$a_x(t) = a(t)(x'(t))^{\alpha-\beta}.$$

Clearly, x is an intermediate solution of (4.5). Since $x^{[1]}$ is positive decreasing for $t \geq t_0$, from (2.1) there exists $k > 0$ such that $x'(t) < kA(t)$. Then

$$\left(\frac{1}{a_x(t)}\right)^{1/\beta} = A^{\alpha/\beta}(t)(x'(t))^{(\beta-\alpha)/\beta} < k_1 A(t). \tag{4.6}$$

where $k_1 = k^{(\beta-\alpha)/\beta}$. Moreover, since $J_\beta < \infty$, we obtain

$$\int_{t_0}^{\infty}\left(\frac{1}{a_x(t)}\right)^{1/\beta}\left(\int_t^{\infty} b(s)ds\right)^{1/\beta} dt < k_1\int_{t_0}^{\infty} A(t)\left(\int_t^{\infty} b(s)ds\right)^{1/\beta} dt < \infty.$$

Because (4.5) has a (nonoscillatory) unbounded solution, from Lemma 4.5 we get

$$\int_{t_0}^{\infty}\left(\frac{1}{a_x(t)}\right)^{1/\beta} dt = \infty.$$

So, applying Theorem 2.3 to (4.5), we obtain

$$\int_{t_0}^{\infty} b(t)\Big(\int_{t_0}^{t}\Big(\frac{1}{a_x(s)}\Big)^{1/\beta} ds\Big)^{\beta} dt = \infty.$$

Since

$$\int_{t_0}^{\infty} b(t)\Big(\int_{t_0}^{t}\Big(\frac{1}{a_x(s)}\Big)^{1/\beta} ds\Big)^{\beta} dt \le k_1^{\beta}\int_{t_0}^{\infty} b(t)\Big(\int_{t_0}^{t} A(s)ds\Big)^{\beta} dt,$$

we have $K_\beta = \infty$, which is a contradiction.

Claim i$_2$). Consider the half-linear equation (4.1) on $[t_0, \infty)$. Reasoning as in the proof of Theorem 4.1-i$_2$) we obtain

$$\int_{t_0}^{\infty} A(t)\Big(\int_{t}^{\infty} b_x(s)\, ds\Big)^{1/\alpha} dt < \infty.$$

Since $\alpha > \beta$, we get $b_x(t) < b(t)$, and so, because $K_\alpha < \infty$, we get

$$\int_{t_0}^{\infty} b_x(t)\Big(\int_{t_0}^{t} A(s)\, ds\Big)^{\alpha} dt < \int_{t_0}^{\infty} b(t)\Big(\int_{t_0}^{t} A(s)\, ds\Big)^{\alpha} dt < \infty.$$

Applying Theorem 2.3 to the half-linear equation (4.1), we get that it does not have intermediate solutions, which is a contradiction. ∎

Remark 4.6. Obviously, under the assumptions of Theorem 4.4, the case C$_4$) holds. Indeed, if $\beta > \alpha$, we have by Lemma 2.4-i$_4$)

$$J_\beta < \infty \Rightarrow J_\alpha < \infty$$

and similarly, if $1 > \alpha > \beta$, we have by Lemma 2.4-i$_3$)

$$K_\alpha < \infty \Rightarrow K_\beta < \infty.$$

Remark 4.7. Theorem 4.1 is applicable when integrals J_β and K_β have the same asymptotic behavior, i.e., are both either convergent or divergent. This happens for instance if

$$A(t) = O(t^{\sigma}), \quad b(t) = O(t^{\rho}) \quad \text{for large } t, \tag{4.7}$$

where $\sigma \ge -1$, $\rho < -1$ (see, e.g., [9]). If (4.7) is satisfied, from Theorem 4.4 intermediate solutions do not exist. However, in general, it can happen $K_\beta < \infty$ and $J_\beta = \infty$ and in view of Theorems 4.1 and 4.4 the following problem remains open.

Open Problem. Do intermediate solutions of equation (1.1) in the case C$_4$) assuming

$$\beta > \alpha > 1, \quad J_\alpha < \infty, \quad K_\beta < \infty, \quad J_\beta = \infty$$

or

$$\beta < \alpha < 1, \quad J_\alpha < \infty, \quad K_\beta < \infty, \quad K_\alpha = \infty$$

exist? For instance, does the equation

$$\big(|x'(t)|^{2/3}\operatorname{sgn} x'(t)\big)' + \frac{1}{(t+2)^{5/3}(\log(t+2))^{5/6}}|x(t)|^{1/3}\operatorname{sgn} x(t) = 0$$

have intermediate solutions?

5. Discrete Versus Continuous Case

In this section we consider the difference equation (1.4) and we show some similarities and discrepancies in the asymptotic behavior of solutions between the continuous case and the discrete one.

As usual, a solution x of (1.4), is said to be *nonoscillatory* if $x_n x_{n+1} > 0$ for large n and *oscillatory* otherwise. The equation (1.4) is called *nonoscillatory* if all its solutions are nonoscillatory. Similarly to the continuous case, all the nonoscillatory solutions x of (1.4) are of class $\mathbb{M}^+$, that is they are eventually monotone and satisfy the condition

$$x_n x_n^{[1]} > 0 \text{ for large } n,$$

where the sequence $x^{[1]} = \{x_n^{[1]}\}$, $x_n^{[1]} = a_n|\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n$, is called the *quasidifference of* x.

The classification of these solutions into the subclasses $\mathbb{M}^+_{\infty,\ell}$, $\mathbb{M}^+_{\infty,0}$, $\mathbb{M}^+_{\ell,0}$, stated for (1.1), continues to hold also for (1.4) by replacing $x(t)$ and $x^{[1]}(t)$ with x_n and $x_n^{[1]}$, respectively. Theorem 2.1 on existence of dominant or subdominant solutions for (1.1) can be formulated for (1.4) in terms of the series

$$S_\alpha = \sum_{n=0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\alpha} \left(\sum_{k=n}^{\infty} b_k\right)^{1/\alpha}, \quad T_\beta = \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n} \left(\frac{1}{a_k}\right)^{1/\alpha}\right)^{\beta}, \tag{5.1}$$

instead of the integrals J_α, K_β, see, e.g., [19, Theorems 2 and 3] with minor changes.

Concerning the mutual behavior of the series (5.1), using the discrete version of Lemma 2.4 ([6, Theorem 1], [7, Lemma 2]) the possible cases are again the following:

C$_1$) $S_\alpha = \infty$, $T_\beta = \infty$;

C$_2$) $S_\alpha = \infty$, $T_\beta < \infty$ and $\alpha > \beta$;

C$_3$) $S_\alpha < \infty$, $T_\beta = \infty$ and $\alpha < \beta$;

C$_4$) $S_\alpha < \infty$, $T_\beta < \infty$.

When the case C$_1$) holds, (1.4) is oscillatory, as it follows, e.g., from [15, Theorems 1, 2], with minor changes. When C$_2$) holds, (1.4) has solutions in the class $\mathbb{M}^+_{\infty,0}$, see, e.g., [3, Theorem 2], in which a more general equation than (1.4) is considered. When the case C$_3$) occurs, the situation is different with respect to the continuous case. This is due to the fact that Theorem 3.2 does not have a discrete analogy, as the following example shows.

Example 5.1. Consider the difference equation

$$\Delta\big(|\Delta x_n|^2 \operatorname{sgn} \Delta x_n\big) + b_n |x_{n+1}|^{9/2} \operatorname{sgn} x_{n+1} = 0, \tag{5.2}$$

where
$$b_n = \frac{8n^4 + 48n^3 + 120n^2 + 144n + 68}{\sqrt{(n^2 + 3n + 2)^9}}.$$
For this equation the discrete counterpart of (3.10) becomes

$$b_n n^{\delta+\varepsilon} \text{ is nonincreasing for large } n, \tag{5.3}$$

where $\delta = 14/3$. Clearly, $Y_a = \infty$ and (5.3) is satisfied if $0 < \varepsilon < 1/3$, but (5.2) has the oscillatory (unbounded) solution $x = \{(-1)^n n(n+1)\}$. Nevertheless, Theorem 3.2 can be applied to the corresponding differential equation

$$\big(|x(t)|^2 \operatorname{sgn} x(t)\big)' + b(t)|x(t)|^{9/2} \operatorname{sgn} x(t) = 0, \tag{5.4}$$

where b is a continuous positive decreasing function such that $b(n) = b_n$. Thus (5.4) is nonoscillatory and, by Theorems 3.1 and 3.3, we have $\mathbb{M}^+ = \mathbb{M}^+_{\ell,0} \cup \mathbb{M}^+_{\infty,0}$, where $\mathbb{M}^+_{\ell,0}$ is one-parametric family of solutions satisfying (3.1). Finally, observe that the case C_3) holds.

Due to the lack of a discrete counterpart of Theorem 3.2, it is an open problem whenever intermediate solutions can exist in the discrete case, when C_3) holds. The following example shows that this fact can occur.

Example 5.2. Consider the difference equation

$$\Delta^2 x_n + b_n x^5_{n+1} = 0,$$

where
$$b_n = \frac{2(n+1)^{2/3} - (n+2)^{2/3} - n^{2/3}}{(n+1)^{10/3}}.$$
In virtue of the concavity of $u^{2/3}$ on $[0, \infty)$, the sequence b is positive and $S_1 < \infty$, $T_5 = \infty$. So the case C_3) holds and $x = \{n^{2/3}\}$ is an intermediate solution.

In general, several nonoscillation criteria, stated for the differential equation (1.1), cannot be carried over to the difference equation (1.4). This fact, already pointed out in [13] for the classical nonoscillation results by Atkinson and Heidel, remains to hold for more recent nonoscillation criteria. For instance, in the continuous case the following holds.

Theorem 5.3. [18, 25] Equation (1.2) is nonoscillatory if any of the following two conditions is satisfied.

i_1) the function b is absolutely continuous and there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} \psi(t) = k > 0, \quad \int^\infty |\psi'(t)|dt < \infty,$$

where $\psi(t) = t^{(\beta+3)/2+\varepsilon} b(t)$.

i$_2$) $\beta \in (0,1)$, the function $\varphi(t) = t^{(\beta+3)/2}b(t)$ is nonincreasing and $\lim\limits_{t\to\infty} t^{(\beta+3)/2}b(t) = 0$.

Observe that Theorem 5.3-i$_1$) holds both in the superlinear and sublinear case. The following example shows that this result does not have a discrete counterpart.

Example 5.4. Consider the differential equation

$$x'' + \frac{4t^2 + 8t + 6}{(t+1)^6}x^3 = 0. \tag{5.5}$$

It is easy to verify that for $\varepsilon = 1$ all assumptions of Theorem 5.3-i$_1$) are verified, and so all the solutions of (5.5) are nonoscillatory. Such a result is not true for the corresponding discrete equation

$$\Delta^2 x_n + \frac{4(n+1)^2 + 8(n+1) + 6}{(n+2)^6}x_{n+1}^3 = 0,$$

since $x = \{(-1)^n(n+1)^2\}$ is an its oscillatory solution. Observe that for $\varepsilon = 1$ the sequence $\psi = \{n^4 b_n\}$ satisfies $\lim\limits_n \psi_n = 4$ and $\sum\limits_n |\Delta\psi_n| < \infty$.

A similar example can be produced in the case of Theorem 5.3-i$_2$). For instance, for the difference equation ($n \ge N$)

$$\Delta^2 x_n + b_n |x_{n+1}|^{1/2} \operatorname{sgn} x_{n+1} = 0,$$

where

$$b_n = \frac{9\sqrt{2}}{4}2^{-n/2},$$

the sequence $\{n^{7/4}b_n\}$ is nonincreasing and tends to zero, but this equation has the oscillatory solution $x = \{(-1)^n 2^{-n}\}$.

6. Concluding Remark

When the case C$_4$) holds, it is an open problem whether intermediate solutions of (1.4) do not exist. In particular, the nonexistence of such solutions under the additional conditions as in the continuous case is an open problem. Observe that Theorems 4.1 and 4.4 are based on a comparison with the half-linear case, and so a crucial role is played by Theorem 2.3. The study of intermediate solutions of nonlinear difference equations is the subject of the forthcoming paper [5].

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