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Existence of Nontrivial Solutions of a Two Point Boundary Value Problem for a $2n$th Order Nonlinear Difference Equation

Heidi Berger
Department of Mathematics, Simpson College,
701 N.C. Street, Indianola, IA 50125, USA
E-mail: Heidi.berger@simpson.edu

Abstract

We will show how to use the mountain pass theorem to obtain nontrivial solutions of a certain two point BVP for the $2n$th order, $n \in \mathbb{N}$ (formally) self-adjoint nonlinear difference equation

$$\sum_{i=0}^{n} \Delta^i [r_i(t) \Delta^i u(t-i)] = f(t, u(t)), \quad t \in [a, b]_{\mathbb{Z}}.$$

No periodicity assumptions will be placed on r_i, $i = 0, 1, \ldots, n$ or f and it will be assumed that f grows superlinearly both at the origin and at infinity. An example of our results will be given.

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1. Introduction and Preliminary Results

A great deal of work has been done concerning the existence to discrete boundary value problems. Recently, techniques from critical point theory have been employed to show the existence of nontrivial solutions to discrete boundary value problems [1–5]. These techniques are complementary to the fixed point theory that has also been utilized to study this area.

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We will be concerned with finding nontrivial solutions to the $2n$th order nonlinear BVP:

$$\sum_{i=0}^{n} \Delta^i [r_i(t)\Delta^i u(t-i)] - f(t, u(t)) = 0, \quad t \in [a, b]_{\mathbb{Z}} \tag{1.1}$$

$$u(a-i) = 0 = u(b+i), \qquad 1 \le i \le n. \tag{1.2}$$

We assume a, b are integers and we define $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. Moreover, $f : [a, b]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous in x and $\Delta u(t) := u(t+1) - u(t)$ defines the forward difference operator Δ. Throughout this paper, we let

$$F(t, x) = \int_0^x f(t, s)\, ds.$$

We also assume throughout:

$$(-1)^i r_i(t) \ge 0 \quad \text{for} \quad 1 \le i \le n \quad \text{for} \quad t \in [a-n, b+n]_{\mathbb{Z}} \tag{1.3}$$

$$r_0(t) > 0 \quad \forall t \in [a-n, b+n]_{\mathbb{Z}}. \tag{1.4}$$

This boundary value problem generalizes the important Sturm–Liouville problem when $n = 1$ and a beam bending problem when $n = 2$ [8].

In this section, we establish the variational framework for (1.1), (1.2). Before proceeding, we need a few useful definitions.

Definition 1.1. Let E be a real Banach space and let $\varphi : E \to \mathbb{R}$ be a mapping. We say φ is *Fréchet differentiable* at $u \in E$ if there exists a continuous linear map $L = L(u) : E \to \mathbb{R}$ satisfying

$$\lim_{x \to u} \frac{\varphi(x) - \varphi(u) - L[x-u]}{\|x-u\|_E} = 0.$$

The mapping L will be denoted by $\varphi'(u)$. A *critical point* u of φ is a point at which $\varphi'(u) = 0$, i.e., $\varphi'(u)v = 0$ for all $v \in E$. We write $\varphi \in C^1(E, \mathbb{R})$ provided $\varphi'(u)$ is continuous for all $u \in E$.

The following remark will be useful.

Remark 1.2. Assume $\varphi : \mathbb{R}^n \to \mathbb{R}$ has continuous first-order partial derivatives. Then for any $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$,

$$\varphi'(u) = \nabla_u \varphi(u),$$

where

$$\nabla_u \varphi := \left(\frac{\partial \varphi}{\partial u_1}, \cdots, \frac{\partial \varphi}{\partial u_n} \right)$$

is the gradient of φ with respect to u.

Definition 1.3. Let E be a real Banach space. The function $\varphi \in C^1(E, \mathbb{R})$ satisfies the *Palais–Smale condition* if every sequence $\{u_j\}$ in E such that $\{\varphi(u_j)\}$ is bounded and $\varphi'(u_j) \to 0$ as $j \to \infty$ contains a convergent subsequence.

We state the mountain pass theorem, which is instrumental to proving the main results of the paper. Let $B(0, \rho)$ denote the open ball in a Banach space E of radius ρ centered at 0.

Theorem 1.4. (Mountain Pass Theorem [5]) Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$. Suppose φ satisfies the Palais–Smale condition and $\varphi(0) = 0$. If
(A_1) there exist constants $\rho, \alpha > 0$ such that

$$\varphi\Big|_{\partial B(0,\rho)} \geq \alpha$$

and
(A_2) there exists $e \in E \backslash \overline{B(0, \rho)}$ such that $\varphi(e) \leq 0$,
then φ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \varphi(g(s)), \tag{1.5}$$

where

$$\Gamma = \{g \in C\left([0, 1], E\right) \mid g(0) = 0,\ g(1) = e\}. \tag{1.6}$$

Theorem 1.5. (Ljusternik–Schnirelman [2]) Let $\varphi \in C^1(\mathbb{R}^N, \mathbb{R})$ be an even function. Then the restriction of φ to the unit sphere S^{N-1} of $\mathbb{R}^N$ possesses at least N distinct pairs of critical points.

Let $E = \{u : [a-n, b+n]_{\mathbb{Z}} \to \mathbb{R} : u(a-i) = 0 = u(b+i), 1 \leq i \leq n\}$. Then E is a Hilbert space with inner product

$$\langle u, v\rangle_E := \sum_{t=a}^{b+n} \sum_{i=0}^{n} |r_i(t)| \Delta^i u(t-i) \Delta^i v(t-i),\ u, v \in E$$

and corresponding norm

$$\|u\|_E^2 = \langle u, u\rangle_E = \sum_{t=a}^{b+n} \sum_{i=0}^{n} |r_i(t)| \left(\Delta^i u(t-i)\right)^2.$$

Let $N = b - a + 1$. Then we can associate E with $\mathbb{R}^N$ by the relation

$$(0, \ldots, 0, u(a), u(a+1), \ldots, u(b), 0, \ldots, 0) \leftrightarrow (x_1, \ldots, x_N),$$

where $x_i = u(a + i - 1)$, $1 \leq i \leq N$.

Define a map $\varphi : E \to \mathbb{R}$ by:

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\sum_{t=a}^{b+n}\sum_{i=0}^{n} |r_i(t)| \left(\Delta^i u(t-i)\right)^2 - \sum_{t=a}^{b} F(t, u(t)) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} F(t, u(t)).
\end{aligned}$$

For our application, we will be interested in computing the Fréchet derivative of φ. Here is a remark to aid in this calculation.

Remark 1.6. Let $\mathcal{H}$ be a real Hilbert space, let $f : \mathcal{H} \to \mathbb{R}$ be the function defined by $f(x) = \|x\|^2$ and let $u \in \mathcal{H}$. Then the Fréchet derivative of f at u is the linear functional on $\mathcal{H}$ given by $f'(u)x := 2\langle x, u\rangle$.

One could use Remark 1.2 to prove Remark 1.6. It is also an easy exercise to prove Remark 1.6 using the definition of the Fréchet derivative.

With the aid of Remark 1.6, we calculate the Fréchet derivative of our functional φ.

Theorem 1.7. For $u, v \in E$, the Fréchet derivative of φ at u is given by

$$\begin{aligned}
\varphi'(u)v &= \langle u, v\rangle_E - \sum_{t=a}^{b+n} f(t, u(t))v(t) \\
&= \sum_{t=a}^{b+n}\left[\sum_{i=0}^{n} \Delta^i[r_i(t)\Delta^i u(t-i)] - f(t, u(t))\right] v(t).
\end{aligned}$$

Proof. Let $u, v \in E$. Define $\varphi_1(u) = \frac{1}{2}\|u\|_E^2$ and $\varphi_2(u) = \sum_{t=a}^{b} F(t, u(t))$. Then $\varphi = \varphi_1 - \varphi_2$. By Remark 1.6, $\varphi_1'(u)v = \langle u, v\rangle_E$. Also, as the Fréchet derivative of functions in $C^1(\mathbb{R}^N, \mathbb{R})$ is the gradient of that function,

$$\begin{aligned}
\varphi_2'(u)v &= < \nabla_u \varphi_2(u), v >_E \\
&= < \nabla_u \left(\sum_{\tau=a}^{b}\int_0^{u(\tau)} f(\tau, s)\, ds\right), v >_E \\
&= < f(u), v >_{\mathbb{R}^N} \\
&= \sum_{t=a}^{b} f(t, u(t))v(t) \\
&= \sum_{t=a}^{b+n} f(t, u(t))v(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi'(u)v &= \langle u, v\rangle_E - \sum_{t=a}^{b+n} f(t, u(t))v(t)\\ &= \sum_{t=a}^{b+n}\left[\sum_{i=0}^{n} |r_i(t)|\Delta^i u(t-i)\Delta^i v(t-i) - f(t, u(t))v(t)\right].\end{aligned}$$

It remains to show that

$$\sum_{t=a}^{b+n}\left[\sum_{i=0}^{n} |r_i(t)|\Delta^i u(t-i)\Delta^i v(t-i)\right] = \sum_{t=a}^{b+n}\left[\sum_{i=0}^{n}\Delta^i[r_i(t)\Delta^i u(t-i)]\right] v(t). \tag{1.7}$$

We verify that (1.7) holds by finite induction on n. If $n = 0$, then (1.7) clearly holds.

Suppose (1.7) holds for a fixed n. We will adopt the convention that for any $u \in E$, we extend the domain of u to $\mathbb{Z}$ by defining $u(t) \equiv 0$ for all $t > b + n$ and for all $t < a - n$. Then

$$\begin{aligned}&\sum_{t=a}^{b+n+1}\sum_{i=0}^{n+1} |r_i(t)|\Delta^i u(t-i)\Delta^i v(t-i)\\ &= \left(\sum_{t=a}^{b+n} + \sum_{t=b+n+1}^{b+n+1}\right)\left(\sum_{i=0}^{n} |r_i(t)|\Delta^i u(t-i)\Delta^i v(t-i)\right.\\ &\qquad \left. + |r_{n+1}(t)|\Delta^{n+1} u(t-n-1)\Delta^{n+1} v(t-n-1)\right)\\ &= \sum_{t=a}^{b+n}\sum_{i=0}^{n} |r_i(t)|\Delta^i u(t-i)\Delta^i v(t-i)\\ &\qquad + \sum_{t=a}^{b+n} |r_{n+1}(t)|\Delta^{n+1} u(t-n-1)\Delta^{n+1} v(t-n-1)\\ &\qquad + \sum_{i=0}^{n} |r_i(b+n+1)|\Delta^i u(b+n-i+1)\Delta^i v(b+n-i+1)\\ &\qquad + |r_{n+1}(b+n+1)|\Delta^{n+1} u(b)\Delta^{n+1} v(b)\\ &= \sum_{t=a}^{b+n}\sum_{i=0}^{n}\Delta^i[r_i(t)\Delta^i u(t-i)]v(t) \sum_{t=a}^{b+n} |r_{n+1}(t)|\Delta^{n+1}\\ &\qquad \times u(t-n-1)\Delta^{n+1} v(t-n-1) + |r_{n+1}(b+n+1)|u(b)v(b)\end{aligned}$$

by the boundary assumptions on u and v and the induction hypothesis. Note that by summation by parts [3, Theorem 2.8] and the boundary conditions on u and v:

$$\begin{aligned}&\sum_{t=a}^{b+n} |r_{n+1}(t)|\Delta^{n+1}u(t-n-1)\Delta^{n+1}v(t-n-1)\\&\quad = -|r_{n+1}(b+n+1)|u(b)v(b) + \sum_{t=a}^{b+n} \Delta^{n+1}[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t).\end{aligned}$$

Indeed, using a summation by parts formula we get:

$$\begin{aligned}&\sum_{t=a}^{b+n} |r_{n+1}(t)|\Delta^{n+1}u(t-n-1)\Delta^{n+1}v(t-n-1)\\
&\quad = (-1)^{n+1}\sum_{t=a}^{b+n} r_{n+1}(t)\Delta^{n+1}u(t-n-1)\Delta^{n+1}v(t-n-1)\\
&\quad = (-1)^{n+1}\Bigg[r_{n+1}(t)\Delta^{n+1}u(t-n-1)\Delta^{n}v(t-n-1)\Big|_a^{b+n+1}\\
&\qquad -\sum_{t=a}^{b+n} \Delta[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]\Delta^{n}v(t-n)\Bigg]\\
&\quad = -|r_{n+1}(b+n+1)|u(b)v(b)\\
&\qquad +(-1)^{n+1}\Bigg[-\sum_{t=a}^{b+n} \Delta[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]\Delta^{n}v(t-n)\Bigg],\end{aligned}$$

using $u(t)=0, t\le a-1, u(t)=0, t\ge b+1$. Again by summation by parts we get:

$$\begin{aligned}&\sum_{t=a}^{b+n} |r_{n+1}(t)|\Delta^{n+1}u(t-n-1)\Delta^{n+1}v(t-n-1)\\
&\quad = -|r_{n+1}(b+n+1)|u(b)v(b)\\
&\qquad +(-1)^{n+1}\Bigg[-\Delta[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]\Delta^{n-1}v(t-n)\Big)\Big|_a^{b+n+1}\\
&\qquad +\sum_{t=a}^{b+n} \Delta^2[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]\Delta^{n-1}v(t-n)\Bigg]\\
&\quad = -|r_{n+1}(b+n+1)|u(b)v(b)\\
&\qquad +(-1)^{n+1}\sum_{t=a}^{b+n} \Delta^2[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]\Delta^{n-1}v(t-n),\end{aligned}$$

where we have used the boundary conditions on u, v to simplify. Continuing in this fashion we see:

$$\begin{aligned}
&\sum_{t=a}^{b+n} |r_{n+1}(t)|\Delta^{n+1}u(t-n-1)\Delta^{n+1}v(t-n-1) \\
&\quad = (-1)^{n+1}\Bigg[(-1)^n\Delta^n[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t-1)\Big|_a^{b+1+n} \\
&\qquad +(-1)^{n+1}\sum_{t=a}^{b+n}\Delta^{n+1}[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t)\Bigg] \\
&\quad = -|r_{n+1}(b+n+1)|u(b)v(b) + \sum_{t=a}^{b+n}\Delta^{n+1}[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{t=a}^{b+n+1}\sum_{i=0}^{n+1}|r_i(t)|\Delta^i u(t-i)\Delta^i v(t-i) \\
&\quad = \sum_{t=a}^{b+n}\sum_{i=0}^{n}\Delta^i[r_i(t)\Delta^i u(t-i)]v(t) + \sum_{t=a}^{b+n}\Delta^{n+1}[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t) \\
&\quad = \sum_{t=a}^{b+n}\sum_{i=0}^{n}\Delta^i[r_i(t)\Delta^i u(t-i)]v(t) + \sum_{t=a}^{b+n}\Delta^{n+1}[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t) \\
&\qquad + \sum_{i=0}^{n}\Delta^i[r_i(t)\Delta^i u(t-i)]\Big|_{t=b+n+1} v(b+n+1) \\
&\qquad + \Delta^{n+1}[r_{n+1}(t+n+1)\Delta^{n+1}u(t)]\Big|_{t=b} v(b+n+1) \\
&\quad = \left(\sum_{t=a}^{b+n} + \sum_{t=b+n+1}^{b+n+1}\right)\Bigg(\sum_{i=0}^{n}\Delta^i[r_i(t)\Delta^i u(t-i)]v(t) \\
&\qquad + \Delta^{n+1}[r_{n+1}(t)\Delta^{n+1}u(t-n-1)]v(t)\Bigg) \\
&\quad = \sum_{t=a}^{b+n+1}\sum_{i=0}^{n+1}\Delta^i[r_i(t)\Delta^i u(t-i)]v(t).
\end{aligned}$$

Hence, by induction, (1.7) holds and the theorem is proved. ∎

Corollary 1.8. Let $u \in E$. The following are equivalent:

1. u is a critical point of φ.

2. u is a solution of (1.1), (1.2).

Furthermore, $\varphi \in C^1(E, \mathbb{R})$.

Proof. By definition and Theorem 1.7,

$$u \quad \text{is a critical point of} \quad \varphi$$

iff

$$\varphi'(u)v = 0 \quad \forall\, v \in E$$

iff

$$\sum_{t=a}^{b+n}\left[\sum_{i=0}^{n} \Delta^i[r_i(t)\Delta^i u(t-i)] - f(t, u(t))\right] v(t) = 0 \quad \forall\, v \in E \tag{1.8}$$

iff

$$u \quad \text{is a solution of (1.1), (1.2).}$$

To see that the last statement holds, for any $m \in [a, b]_{\mathbb{Z}}$, let

$$v_m(t) = \begin{cases} 1 & \text{if } t = m \\ 0 & \text{if } t \neq m. \end{cases}$$

Then $v_m \in E$ and $\varphi'(u)v_m = 0$ for all $m \in [a, b]_{\mathbb{Z}}$. But this implies:

$$\sum_{i=0}^{n} \Delta^i[r_i(t)\Delta^i u(t-i)] - f(t, u(t)) = 0, \quad \forall\, t \in [a, b]_{\mathbb{Z}},$$

so that these critical points correspond to solutions of the BVP (1.1), (1.2). Conversely, if u is a solution to BVP (1.1), (1.2), then (1.8) clearly holds. As E and $\mathbb{R}$ are Euclidean spaces, the continuity of f guarantees that $\varphi \in C^1(E, \mathbb{R})$. ■

2. Main Results

First, we introduce some notation. As E is a finite dimensional vector space, we know that all norms defined on E are equivalent. In particular, there exist constants d_1, d_2, and $B > 0$ such that

$$d_1\|u\|_2^2 \leq \|u\|_E^2 \leq d_2\|u\|_2^2 \tag{2.1}$$

and

$$\|u\|_\infty \leq B\|u\|_E \quad \forall\, u \in E. \tag{2.2}$$

Here, $\|u\|_2 = \left(\sum_{t=a}^{b} u^2(t)\right)^{\frac{1}{2}}$ and $\|u\|_\infty = \max_{t\in[a,b]_{\mathbb{Z}}} |u(t)|$.

Now we state and prove the main theorems.

Theorem 2.1. Suppose

$$\lim_{x\to 0} \frac{F(t,x)}{x^2} = 0; \quad t \in [a,b]_{\mathbb{Z}}, \tag{2.3}$$

and there exists $\beta > 2$ such that

$$0 < \beta \int_0^x f(t,s)\,ds \le xf(t,x) \;\; \forall\,(t,x) \in [a-n,b+n]_{\mathbb{Z}} \times \mathbb{R}\backslash\{0\}. \tag{2.4}$$

Then there exists a nontrivial solution to the BVP (1.1), (1.2).

Remark 2.2. Assumption (2.4) implies that for each $t \in [a,b]_{\mathbb{Z}}$, there exists a real number $\alpha(t) > 0$ such that

$$F(t,x) \ge \alpha(t)|x|^{\beta} \quad \text{for } |x| \ge 1, \;\; t \in [a,b]_{\mathbb{Z}}. \tag{2.5}$$

Proof. To verify this remark, assume $x \ge 1$. Then from (2.4),

$$\begin{aligned}
&\beta \int_0^x f(t,s)\,ds \le xf(t,x)\\
&\quad\Rightarrow \frac{\beta}{x} \le \frac{f(t,x)}{\int_0^x f(t,s)\,ds}\\
&\quad\Rightarrow \int_1^x \frac{\beta}{\tau}\,d\tau \le \int_1^x \frac{f(t,\tau)}{\int_0^\tau f(t,s)\,ds}\,d\tau\\
&\quad\Rightarrow \int_1^x \frac{\beta}{\tau}\,d\tau \le \ln\left|\int_0^\tau f(t,s)\,ds\right|\Bigg|_1^x\\
&\quad\Rightarrow \beta \ln|x| \le \ln\left|\int_0^x f(t,s)\,ds\right| - \ln\left|\int_0^1 f(t,s)\,ds\right|\\
&\quad\Rightarrow \beta \ln|x| + C(t) \le \ln\left(\int_0^x f(t,s)\,ds\right), \quad \text{where } C(t) := \ln\left|\int_0^1 f(t,s)\,ds\right|\\
&\quad\Rightarrow e^{\beta \ln|x| + C(t)} \le \int_0^x f(t,s)\,ds\\
&\quad\Rightarrow |x|^{\beta}\alpha(t) \le \int_0^x f(t,s)\,ds = F(t,x),
\end{aligned}$$

where $\alpha(t) = e^{C(t)}$. The case for $x \le -1$ is similar. ■

Remark 2.3. The inequality (2.5) implies

$$\lim_{x\to\infty} \frac{F(t,x)}{x^2} = \infty; \quad t \in [a,b]_{\mathbb{Z}}. \tag{2.6}$$

Proof. By (2.5):

$$\frac{\alpha(t)|x|^\beta}{x^2} \le \frac{F(t,x)}{x^2}.$$

Since $\lim_{x\to\infty} \alpha(t)|x|^{\beta-2} = \infty$, it follows that $\lim_{x\to\infty} \dfrac{F(t,x)}{x^2} = \infty,\ t \in [a,b]_{\mathbb{Z}}$. ∎

Proof of Theorem 2.1. We will prove the existence of a nontrivial critical point of φ using Theorem 1.4. We know that $\varphi \in C^1(E,\mathbb{R})$ and $\varphi(0) = 0$.

We verify that the Palais–Smale condition holds. Let $\{u_k\}$ be a sequence in E such that

$$\{\varphi(u_k)\} \quad \text{is bounded and} \quad \varphi'(u_k) \to 0 \ \text{ as } \ k \to \infty. \tag{2.7}$$

We will show that $\{u_k\}$ possesses a convergent subsequence.

By (2.7), $\varphi'(u_k) \to 0$ as $k \to \infty$, so that

$$\sup_{v\in E, v\neq 0} \left| \frac{\varphi'(u_k)v}{\|v\|_E} \right| \to 0.$$

Without loss of generality, we can assume that $u_k \neq 0$ for an k. Hence, $\{\varphi(u_k)\}$ and $\left\{ \dfrac{1}{\|u_k\|_E} \varphi'(u_k)u_k \right\}$ are bounded sequences of real numbers. So there exist constant $M, N \ge 0$ such that

$$M + \frac{N}{\beta}\|u_k\|_E \ge \varphi(u_k) - \frac{1}{\beta}\varphi'(u_k)u_k.$$

By (2.4),

$$F(t,u_k(t)) - \frac{1}{\beta}u_k(t)f(t,u_k(t)) \le 0.$$

Thus,

$$\begin{aligned} M + \frac{N}{\beta}\|u_k\|_E &\ge \varphi(u_k) - \frac{1}{\beta}\varphi'(u_k)u_k \\ &= \left(\frac{1}{2} - \frac{1}{\beta}\right)\|u_k\|_E^2 - \sum_{t=a}^{b+n}\left[F(t,u_k(t)) - \frac{1}{\beta}f(t,u_k(t))u_k(t)\right] \\ &\ge \left(\frac{1}{2} - \frac{1}{\beta}\right)\|u_k\|_E^2. \end{aligned}$$

Thus,

$$\alpha\|u_k\|_E^2 - \frac{N}{\beta}\|u_k\|_E - M \le 0\ , \text{ where } \alpha := \frac{1}{2} - \frac{1}{\beta} > 0.$$

As $\left(\frac{N}{\beta}\right)^2 + 4\alpha M \geq 0$, we see that $\{u_k\}$ is bounded in E. As E is finite dimensional, $\{u_k\}$ has a convergent subsequence. Hence, φ satisfies the Palais–Smale condition.

Now we prove that (A_1) in the mountain pass theorem (Theorem 1.4) holds. Let B, d_1 be as in (2.2), (2.1), respectively. By (2.3), there is a $\delta > 0$ such that $F(t,x) \leq \frac{1}{4}d_1x^2$ whenever $|x| < \delta t \in [a,b]_{\mathbb{Z}}$. Let $\rho := \frac{\delta}{B}$ and suppose $\|u\|_E \leq \rho$. Then we have $\|u\|_\infty \leq \frac{\delta}{B}B = \delta$, so that $F(t,u(t)) \leq \frac{1}{4}d_1u^2(t)$ for all $t \in [a,b]_{\mathbb{Z}}$. Thus, if $\|u\|_E = \rho$:

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} F(t,u(t)) \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{4}d_1\|u\|_2^2 \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{4}\|u\|_E^2, \text{ by (2.1)} \\
&= \frac{1}{4}\rho^2 := \alpha > 0.
\end{aligned}$$

Thus, (A_1) holds. Finally, we verify that (A_2) holds. Condition (2.6) implies that there exists $\delta_1 > B\rho$ such that $F(t,x) \geq d_2x^2$ for all $|x| \geq \delta_1,\ t \in [a,b]_{\mathbb{Z}}$. (Here, B and d_2 are defined in (2.2) and (2.1).) Let $e \in E$ be such that $|e(t)| \geq \delta_1$ on $[a,b]_{\mathbb{Z}}$. Then $\|e\|_\infty \geq \delta_1$ so that $\|e\|_E > \rho$. In this case,

$$\begin{aligned}
\varphi(e) &= \frac{1}{2}\|e\|_E^2 - \sum_{t=a}^{b} F(t,e(t)) \\
&\leq \frac{1}{2}\|e\|_E^2 - d_2\|e\|_2^2 \\
&\leq \frac{1}{2}\|e\|_E^2 - \|e\|_E^2, \text{ by (2.1)} \\
&= -\frac{1}{2}\|e\|_E^2 < 0.
\end{aligned}$$

Thus, (A_2) holds.

By the mountain pass theorem, there exists $c > 0$, a critical value of φ. The corresponding critical point $u \in E$ is a solution of the BVP (1.1), (1.2) by construction of φ. As $\varphi(0) = 0$ and $\varphi(u) = c > 0$, $u \neq 0$. So u is a nontrivial solution to our BVP and the result holds. ■

Theorem 2.4. Suppose

$$\lim_{x\to 0} \frac{F(t,x)}{x^2} = \infty;\ t \in [a,b]_{\mathbb{Z}}, \tag{2.8}$$

and there exists $0 < \gamma < 2$ such that

$$0 < xf(t,x) \leq \gamma \int_0^x f(t,s)\,ds \;\; \forall\,(t,x) \in [a-n, b+n]_{\mathbb{Z}} \times \mathbb{R}\backslash\{0\}. \tag{2.9}$$

Then there exists a nontrivial solution to the BVP (1.1), (1.2).

Remark 2.5. As in Theorem 2.1, for each $t \in [a-n, b+n]_{\mathbb{Z}}$, there exists a real number $\alpha_1(t) > 0$ such that

$$F(t,x) \geq \alpha_1(t)|x|^\gamma \quad \text{for } \; |x| \geq 1 \tag{2.10}$$

so that

$$\lim_{x\to\infty} \frac{F(t,x)}{x^2} = 0; \;\; t \in [a,b]_{\mathbb{Z}}. \tag{2.11}$$

The proofs of (2.10) and (2.11) are similar to the proofs of (2.5) and (2.6), respectively.

Proof of Theorem 2.4. We will show that the functional $-\varphi$ satisfies all the conditions of the mountain pass theorem. We know that $-\varphi \in C^1(E,\mathbb{R})$ and $-\varphi(0)=0$.

We verify that the Palais–Smale condition holds. Let $\{u_k\}$ be a sequence in E such that

$$\{-\varphi(u_k)\} \quad \text{is bounded and} \quad -\varphi'(u_k) \to 0 \;\text{ as }\; k \to \infty. \tag{2.12}$$

We will show that $\{u_k\}$ possesses a convergent subsequence.

By (2.12), $-\varphi'(u_k) \to 0$ as $k \to \infty$, so that

$$\sup_{v\in E, v\neq 0} \left| \frac{-\varphi'(u_k)v}{\|v\|_E} \right| \to 0.$$

Without loss of generality, we can assume that $u_k \neq 0$ for a k. Hence, $\{-\varphi(u_k)\}$ and $\left\{\frac{1}{\|u_k\|_E}\varphi'(u_k)u_k\right\}$ are bounded sequences of real numbers. So there exist constant $M_1, N \geq 0$ such that

$$M_1 + \frac{N}{\gamma}\|u_k\|_E \geq \varphi(u_k) + \frac{1}{\gamma}\varphi'(u_k)u_k.$$

By (2.9), $F(t,u_k(t)) - \frac{1}{\gamma}u_k(t)f(t,u_k(t)) \geq 0$. Thus,

$$\begin{aligned} M_1 + \frac{N}{\gamma}\|u_k\|_E &\geq -\varphi(u_k) + \frac{1}{\gamma}\varphi'(u_k)u_k \\ &= \left(-\frac{1}{2} + \frac{1}{\gamma}\right)\|u_k\|_E^2 + \sum_{t=a}^{b+n}\left[F(t,u_k(t)) - \frac{1}{\gamma}f(t,u_k(t))u_k(t)\right] \\ &\geq \left(-\frac{1}{2} + \frac{1}{\gamma}\right)\|u_k\|_E^2. \end{aligned}$$

Thus,

$$\alpha_1 \|u_k\|_E^2 - \frac{N}{\gamma}\|u_k\|_E - M_1 \leq 0, \text{ where } \alpha_1 := -\frac{1}{2} + \frac{1}{\gamma} > 0.$$

As $\dfrac{N^2}{\gamma^2} + 4M_1\alpha_1 \geq 0$, $\{u_k\}$ is bounded in E. Thus, as E is finite dimensional, $\{u_k\}$ has a convergent subsequence. Hence, $-\varphi$ satisfies the Palais–Smale condition.

Now we prove that (A_1) in the mountain pass theorem holds. By (2.11), there is a $\beta > 0$ such that $F(t,x) \geq d_2x^2$ whenever $|x| < \beta$, $t \in [a,b]_{\mathbb{Z}}$. Let $\rho_1 := \dfrac{\beta}{B}$, where B is as in (2.2), d_2 is defined in (2.1), and suppose $\|u\|_E \leq \rho_1$. Then we have $\|u\|_\infty \leq \dfrac{\beta}{B}B = \beta$, so that $F(t,u(t)) \geq d_2u^2(t)$ for all $t \in [a,b]_{\mathbb{Z}}$. Thus, if $\|u\|_E = \rho_1$:

$$\begin{aligned} -\varphi(u) &= -\frac{1}{2}\|u\|_E^2 + \sum_{t=a}^{b} F(t,u(t)) \\ &\geq -\frac{1}{2}\|u\|_E^2 + d_2\|u\|_2^2 \\ &\geq -\frac{1}{2}\|u\|_E^2 + \|u\|_E^2, \text{ by (2.1)} \\ &= \frac{1}{2}\rho_1^2 := \alpha_1 > 0. \end{aligned}$$

Thus, (A_1) holds.

Finally, we verify that condition (A_2) holds. Condition (2.8) implies that there exists $\beta_1 > B\rho_1$, where B is as in (2.2), such that $F(t,x) \leq d_1x^2 \ \forall\ |x| \geq \beta_1$, where d_1 is as in (2.1). Let $e_1 \in E$ be such that $|e_1(t)| \geq \beta_1$ on $[a,b]_{\mathbb{Z}}$. Then $\|e_1\|_\infty \geq \beta_1$ so that $\|e_1\|_E > \rho_1$. In this case,

$$\begin{aligned} -\varphi(e_1) &= -\frac{1}{2}\|e_1\|_E^2 + \sum_{t=a}^{b} F(t,e_1(t)) \\ &\leq -\frac{1}{2}\|e_1\|_E^2 + \frac{1}{4}d_1\|e_1\|_2^2 \\ &\leq -\frac{1}{2}\|e_1\|_E^2 + \frac{1}{4}\|e_1\|_E^2, \text{ by (2.1)} \\ &= -\frac{1}{4}\|e_1\|_E^2 < 0. \end{aligned}$$

Thus, (A_2) holds.

By the mountain pass theorem, there exists $c > 0$, a critical value of $-\varphi$. The corresponding critical point $u \in E$ is a solution of the BVP (1.1), (1.2) by construction of $-\varphi$. As $-\varphi(0) = 0$ and $-\varphi(u) = c > 0$, $u \neq 0$. So u is a nontrivial solution to the BVP (1.1), (1.2) and the result holds. ■

Next, we obtain a result concerning the existence of multiple solutions to the BVP (1.1), (1.2) by applying Theorem 1.5:

Theorem 2.6. Assume

$$f(t, -x) = -f(t, x). \tag{2.13}$$

Then the BVP (1.1), (1.2) has at least $N := b-a+1$ distinct pairs of nontrivial solutions.

Remark 2.7. Note that if u is a solution to (1.1), (1.2), then under the assumptions of Theorem 2.6, $-u$ is also a solution to the BVP. In this case, call $(u, -u)$ a pair of solutions to the BVP (1.1), (1.2).

Proof of Theorem 2.6. By condition (2.13), we see that φ is even:

$$\begin{aligned}
\varphi(-u) &= \frac{1}{2}\|-u\|_E^2 - \sum_{t=a}^{b} F(t, -u(t)) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} \int_0^{-u(t)} f(t, s)\, ds \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} \left(-\int_{-u(t)}^{0} f(t, s)\, ds\right) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} \int_{-u(t)}^{0} f(t, -s)\, ds \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} \int_{u(t)}^{0} -f(t, \tau)\, d\tau \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} \int_0^{u(t)} f(t, \tau)\, d\tau \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t=a}^{b} F(t, u(t)) \\
&= \varphi(u).
\end{aligned}$$

By Theorem 1.5, the restriction of φ to S^{N-1} has at least N distinct pairs of critical points. So there exist $u_j \in E$ such that $(u_j(a), \cdots, u_j(b)) \in S^{N-1}$, $1 \leq j \leq 2N$ such that

$$\varphi'(u_j)v = \sum_{t=a}^{b+n} \left[\sum_{i=0}^{n} \Delta^i [r_i(t)\Delta^i u_j(t-i)] - f(t, u_j(t)) \right] v(t) = 0 \quad \forall\, v \in E$$

such that $(v(a), \cdots, v(b)) \in S^{N-1}$, $1 \le j \le 2N$. For $m \in [a, b]_{\mathbb{Z}}$, let

$$v_m(t) = \begin{cases} 1 & \text{if } t = m \\ 0 & \text{if } t \ne m. \end{cases}$$

Then $v_m \in E$ and $(v_m(a), \ldots, v_m(b)) \in S^{N-1}$ and $\varphi'(u_j)v_m = 0$, $1 \le j \le 2N$, $m \in [a, b]_{\mathbb{Z}}$. But this implies

$$\sum_{i=0}^{n} \Delta^i [r_i(t)\Delta^i u(t-i)] - f(t, u(t)) = 0, \quad \forall\, t \in [a, b]_{\mathbb{Z}},$$

so that these critical points correspond to solutions of the BVP (1.1), (1.2). ■

Example 2.8. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$\sum_{i=0}^{n} \Delta^i [(-1)^i \Delta^i u(t-i)] = x^{2n+1} e^t, \quad t \in [0, 1]_{\mathbb{Z}} \tag{2.14}$$

$$u(-i) = 0 = u(1+i), \qquad 1 \le i \le n. \tag{2.15}$$

Here $f(t, x) = x^{2n+1} e^t$. Then f satisfies the conditions of Theorem 2.1 with $\beta = 2n$, so that BVP (2.14), (2.15) has a nontrivial solution. Moreover, f satisfies (2.13) in Theorem 2.6, so that BVP (2.14), (2.15) has two distinct pairs of nontrivial solutions.

Example 2.9. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$\sum_{i=0}^{n} \Delta^i [(-1)^i \Delta^i u(t-i)] = x^{\frac{1}{2n+1}} sin^2(t), \quad t \in [0, 1]_{\mathbb{Z}} \tag{2.16}$$

$$u(-i) = 0 = u(1+i), \qquad 1 \le i \le n. \tag{2.17}$$

Here, $f(t, x) = x^{\frac{1}{2n+1}} \sin^2(t)$ satisfies the conditions of Theorem 2.4 with $\gamma = \dfrac{2n+2}{2n+1}$. Also, f satisfies (2.13) in Theorem 2.6. In the first case, BVP (2.16), (2.17) would have a nontrivial solution and in the second case, this BVP would have two distinct pairs of nontrivial solutions.

References

[1] Dingyong Bai and Yuantong Xu, Nontrivial solutions of boundary value problems of second-order difference equations, *J. Math. Anal. Appl.*, 326(1):297–302, 2007.

[2] Youssef Jabri, *The mountain pass theorem*, volume 95 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2003. Variants, generalizations and some applications.

[3] Walter G. Kelley and Allan C. Peterson, *Difference equations*, Harcourt/Academic Press, San Diego, CA, second edition, 2001. An introduction with applications.

[4] Manjun Ma and Zhiming Guo, Homoclinic orbits for second order self-adjoint difference equations, *J. Math. Anal. Appl.*, 323(1):513–521, 2006.

[5] Jean Mawhin and Michel Willem, *Critical point theory and Hamiltonian systems*, volume 74 of *Applied Mathematical Sciences*, Springer-Verlag, New York, 1989.

[6] Gen-qiang Wang and Sui Sun Cheng, Elementary variational approach to zero — free solutions of a nonlinear eigenvalue problem, 2007. Preprint.

[7] Gen-qiang Wang and Sui Sun Cheng, Multiple steady state solutions of a circular neural network. 2007. Preprint.

[8] C. Ray Wylie, *Advanced engineering mathematics*, McGraw-Hill Book Co., New York, fourth edition, 1975.

[9] Qin Qin Zhang, Existence of solutions for a nonlinear system with applications to difference equations, *Appl. Math. E-Notes*, 6:153–158 (electronic), 2006.