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Existence of Positive Solutions of Second Order Neutral Dynamic Equations on Time Scales

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Abstract

In this paper, a class of second order neutral dynamic equations on time scales is studied. Various types of eventually positive solutions are given and the existence of these eventually positive solutions are then derived by means of Kranoselskii's fixed point theorem.

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1. Introduction

Recall that an open problem in [7] is under what conditions there will exist positive solutions to the equation

$$[x(t) + p(t)x(g(t))]^{\Delta} + q(t)x(h(t)) = 0$$

on a time scale. To solve this problem, Zhu and Wang [8] considered the first order neutral dynamic equation of the form

$$[x(t) + p(t)x(g(t))]^{\Delta} + f(t, x(h(t))) = 0.$$

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A new problem now emerges that under what conditions there will exist positive solutions to higher order neutral dynamic equations. Following the trend, in this paper we consider second order neutral functional dynamic equations of the form

$$[x(t) + p(t)x(g(t))]^{\Delta\Delta} + f(t, x(h(t))) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale with $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$.

Referring to [1,3,4], we see that a time scale is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. There are many results on time scales, and we can find the details in the monographs [3,4]. In particular, we remark that the theory for multiple integration on time scales has been established in [2].

Let $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ denote the set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$. Throughout we assume that

(H1) $g, h \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $g(t) \le t$, $\lim_{t\to\infty} g(t)/t = 1$, $\lim_{t\to\infty} h(t) = \infty$, and there exists a sequence $\{c_n\}_{n\ge 0}$ such that $\lim_{n\to\infty} c_n = \infty$ and $g(c_{n+1}) = c_n$.

(H2) $p \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and there exists a constant p_0 with $|p_0| < 1$ and $\lim_{t\to\infty} p(t) = p_0$.

(H3) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $f(t, x)$ is nondecreasing in x and $xf(t, x) > 0$ for $t \in \mathbb{T}$ and $x \ne 0$.

As usual, by a solution of (1.1) we mean a continuous function x which is defined on $\mathbb{T}$ and satisfies (1.1) for $t \ge t_1 \ge t_0$. Among the solutions of (1.1), one is said to be nonoscillatory if it is eventually positive or eventually negative. We will restrict our attention to eventually positive solutions since dual statements for eventually negative solutions can be readily stated.

2. Preliminaries

For $T_0, T_1 \in \mathbb{T}$, let $[T_0, \infty)_{\mathbb{T}} := \{t \in \mathbb{T} : t \ge T_0\}$ and $[T_0, T_1]_{\mathbb{T}} := \{t \in \mathbb{T} : T_0 \le t \le T_1\}$. Further, let $r_\lambda(t) = t^{2\lambda}$ for some integer $\lambda \in \{0, 1\}$, $T_0 > 0$, and $C([T_0, \infty)_{\mathbb{T}}, \mathbb{R})$ denote all continuous functions mapping $[T_0, \infty)_{\mathbb{T}}$ into $\mathbb{R}$, and

$$BC[T_0, \infty)_{\mathbb{T}} := \left\{ x \in C[T_0, \infty)_{\mathbb{T}} : \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \left| \frac{x(t)}{r_\lambda(t)} \right| < \infty \right\}. \tag{2.1}$$

When endowed with the usual linear structure and norm $||x|| = \sup_{t\in[T_0,\infty)_{\mathbb{T}}} |x(t)/r_\lambda(t)|$, $(BC[T_0, \infty)_{\mathbb{T}}, ||\cdot||)$ is a Banach space. A set $X \subset BC[T_0, \infty)_{\mathbb{T}}$ is said to be uniformly Cauchy if for any given $\varepsilon > 0$, there exists a $T_1 \in [T_0, \infty)_{\mathbb{T}}$ such that for any $x \in X$,

$$\left| \frac{x(t_1)}{r_\lambda(t_1)} - \frac{x(t_2)}{r_\lambda(t_2)} \right| < \varepsilon \quad \text{for all} \quad t_1, t_2 \in [T_1, \infty)_{\mathbb{T}}.$$

X is said to be equicontinuous on $[a, b]_{\mathbb{T}}$ if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x(t_1) - x(t_2)| < \varepsilon, \quad |t_1 - t_2| < \delta \quad \text{with} \quad t_1, t_2 \in [a, b]_{\mathbb{T}}$$

for any $x \in X$.

Lemma 2.1. Suppose that x is continuous and $x(t)/t^{\lambda}$ is eventually positive on $\mathbb{T}$ for some integer $\lambda \in \{0, 1\}$. Suppose further that $z(t) = x(t) + p(t)x(g(t))$ and $\lim_{t\to\infty} z(t)/t^{\lambda} = b$. Then $\lim_{t\to\infty} x(t)/t^{\lambda} = b/(1 + p_0)$.

Proof. In view of the assumptions (H1) and (H2), there exist $T_0 \in \mathbb{T}$ and $|p_0| < p_1 < 1$ such that $x(t) > 0$, $x(g(t)) > 0$ and $|p(t)| \le p_1$ for $t \in [T_0, \infty)_{\mathbb{T}}$. Next there will be two cases to consider.

In case b is finite, we assert that $x(t)/t^{\lambda}$ is bounded on $\mathbb{T}$. Otherwise, there exists $\{t_n\} \subset \mathbb{T}$ with $t_n \ge T_0$ and $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{t\to\infty} \frac{x(t)}{t^{\lambda}} = \infty, \quad x(t) \le x(t_n) \quad \text{for all} \quad t \in [T_0, t_n]_{\mathbb{T}}.$$

Note that since $g(t) \le t$, it follows that

$$\frac{z(t_n)}{t_n^{\lambda}} = \frac{x(t_n)}{t_n^{\lambda}} + p(t_n)\frac{x(g(t_n))}{t_n^{\lambda}} \ge (1 - p_1)\frac{x(t_n)}{t_n^{\lambda}} \to \infty$$

as $t \to \infty$. This contradicts with the fact that b is finite. Now we may assume that

$$\limsup_{t\to\infty} \frac{x(t)}{t^{\lambda}} = \overline{x}, \quad \liminf_{t\to\infty} \frac{x(t)}{t^{\lambda}} = \underline{x}.$$

Then in $\mathbb{T}$ there exist sequences $\{\overline{s}_n\}$ and $\{\underline{s}_n\}$ satisfying respectively $\overline{s}_n \to \infty$ and $\underline{s}_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{x(\overline{s}_n)}{\overline{s}_n^{\lambda}} = \overline{x}, \quad \lim_{n\to\infty} \frac{x(\underline{s}_n)}{\underline{s}_n^{\lambda}} = \underline{x}.$$

Hence we have when $0 \le p_0 < 1$,

$$b = \liminf_{n\to\infty} \left(\frac{x(\overline{s}_n)}{\overline{s}_n^{\lambda}} + p(\overline{s}_n)\frac{x(g(\overline{s}_n))}{\overline{s}_n^{\lambda}} \right) \ge \overline{x} + p_0\underline{x} \tag{2.2}$$

as well as

$$b = \limsup_{n\to\infty} \left(\frac{x(\underline{s}_n)}{\underline{s}_n^{\lambda}} + p(\underline{s}_n)\frac{x(g(\underline{s}_n))}{\underline{s}_n^{\lambda}} \right) \ge \underline{x} + p_0\overline{x}. \tag{2.3}$$

From (2.2) and (2.3), we obtain $\overline{x} \le \underline{x}$, which implies that $\overline{x} = \underline{x}$ when $0 \le p_0 < 1$. When $-1 < p_0 < 0$, similar to the arguments as above we can find that $b \ge \overline{x} + p_0\overline{x}$

and $b \le \underline{x} + p_0\underline{x}$, which induces again $\overline{x} = \underline{x}$. Now we see that $\lim_{t\to\infty} x(t)/t^\lambda$ exists. It is easy to verify that $\lim_{t\to\infty} x(t)/t^\lambda = b/(1+p_0)$.

In case b is infinite, we assert that $b = -\infty$ cannot hold. Otherwise, there exists $T_1 \in \mathbb{T}$ with $T_1 \ge T_0$ such that $z(t) < 0$ on $[T_1, \infty)_{\mathbb{T}}$. As a consequence, it follows that

$$x(t) < -p(t)x(g(t)) \le p_1 x(g(t)) \quad \text{for} \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{2.4}$$

Using the assumption (H1), we can choose some positive integer n_0 such that $c_n \ge T_1$ for all $n \ge n_0$. Then for any $n \ge n_0 + 1$, we have from (2.4) that

$$\begin{aligned} x(c_n) &< p_1 x(g(c_n)) = p_1 x(c_{n-1}) < p_1^2 x(g(c_{n-1})) = p_2^2 x(c_{n-2}) \\ &< \ldots < p_1^{n-n_0} x(g(c_{n_0+1})) = p_1^{n-n_0} x(c_{n_0}), \end{aligned}$$

and hence we have $\lim_{n\to\infty} x(c_n)/c_n^\lambda = 0$. Therefore $\lim_{n\to\infty} z(c_n)/c_n^\lambda = 0$, which is a contradiction. Note that since

$$\frac{z(t)}{t^\lambda} \le \frac{x(t)}{t^\lambda} + p_1 \frac{x(g(t))}{t^\lambda}, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

we see that $\lim_{n\to\infty} x(t)/t^\lambda = \infty$. The proof is complete. ■

The following result [5, Lemma 3] will be needed.

Lemma 2.2. If $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $a, t \in \mathbb{T}$ with $a \le t$, then

$$\int_a^t \int_a^\tau f(s)\Delta s \Delta\tau = \int_a^t (t - \sigma(\tau)) f(\tau)\Delta\tau,$$

where σ is the forward jump operator on $\mathbb{T}$ defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}.$$

In the sense of the norm $||\cdot||$ defined as above, the following result holds and its proof is similar to [8, Lemma 4], so we omit it.

Lemma 2.3. Suppose that $X \subset BC[T_0, \infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Suppose further that X is equicontinuous on $[T_0, T_1]_{\mathbb{T}}$ for any $T_1 \in [T_0, \infty)_{\mathbb{T}}$. Then X is relatively compact.

In the next section, we will employ Kranoselskii's fixed point theorem (see [6]) to establish the existence of eventually positive solutions for (1.1). Here we cite it as follows.

Lemma 2.4. Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Suppose further that there exist two operators $U, S : X \to \Omega$ such that

(i) $Ux + Sy \in X$ for all $x, y \in X$;

(ii) U is a contraction mapping;

(iii) S is completely continuous.

Then $U + S$ has a fixed point in X.

Note that the conclusion of Lemma 2.4 holds when the operator $U = 0$. Hence we have the following presentation.

Corollary 2.5. Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Suppose further that there exists an operator $S : X \to \Omega$ such that

(i) $Sx \in X$ for all $x \in X$;

(ii) S is completely continuous.

Then S has a fixed point in X.

3. Classifications of Positive Solutions

For the sake of convenience, we set

$$z(t) = x(t) + p(t)x(g(t)) \tag{3.1}$$

whenever it is defined. Then (1.1) can be written as

$$z^{\Delta\Delta}(t) + f(t, x(h(t))) = 0, \quad t \in \mathbb{T}. \tag{3.2}$$

Let S^+ denote the set of all eventually positive solutions of (1.1). We now introduce the following notations for classifying these eventually positive solutions. Let

$$A_1(\alpha, \beta) = \left\{ x \in S^+ : \lim_{t\to\infty} x(t) = \alpha, \ \lim_{t\to\infty} \frac{x(t)}{t} = \beta \right\}$$

and

$$A_0(\alpha) = \left\{ x \in S^+ : \lim_{t\to\infty} x(t) = \alpha \right\}.$$

Theorem 3.1. If x is an eventually positive solution of (1.1), then either $x \in A_0(0)$ or there exists a real number $a > 0$ such that x belongs to $A_1(\infty, a)$, $A_1(\infty, 0)$ or $A_1(a, 0)$.

Proof. By the assumption for x and (3.2), we may take a sufficient large $T_0 \in \mathbb{T}$ such that $z^{\Delta\Delta}(t) < 0$ for all $t \in [T_0, \infty)_{\mathbb{T}}$. Consequently, both $z(t)$ and $z^{\Delta}(t)$ are eventually of fixed sign. For simplicity, we may assume that $z(t)$ and $z^{\Delta}(t)$ are all of fixed sign on $[T_0, \infty)_{\mathbb{T}}$.

Note that the proof of Lemma 2.1 has shown that $\lim_{t\to\infty} z(t) = -\infty$ cannot hold. In addition, if $z^{\Delta}(t) < 0$ on $[T_0, \infty)_{\mathbb{T}}$, then there exists a constant $M > 0$ such that $z^{\Delta}(t) \le -M$ for all $t \in [T_0, \infty)_{\mathbb{T}}$. This means that

$$z(t) = z(T_0) + \int_{T_0}^{t} z^{\Delta}(s)\Delta s \le z(T_0) - (t - T_0)M \to -\infty \quad \text{as} \quad t \to \infty,$$

which is contrary to the fact that $\lim_{t\to\infty} z(t) = -\infty$ cannot hold. Thus $z^{\Delta}(t) > 0$ on $[T_0, \infty)_{\mathbb{T}}$.

Assume $z(t) < 0$ on $[T_0, \infty)_{\mathbb{T}}$. Since $z^{\Delta}(t) > 0$ on $[T_0, \infty)_{\mathbb{T}}$, we see that $\lim_{t\to\infty} z(t)$ exists and $\lim_{t\to\infty} z(t) \le 0$. By the proof of Lemma 2.1, $\lim_{t\to\infty} z(t) = 0$, which means that $\lim_{t\to\infty} x(t) = 0$. That says x belongs to $A_0(0)$.

Now assume $z(t) > 0$ on $[T_0, \infty)_{\mathbb{T}}$. Hence we may assume that

$$\lim_{t\to\infty} z(t) = L_0, \quad \lim_{t\to\infty} z^{\Delta}(t) = L_1,$$

where $0 < L_0 \le \infty$ and $0 \le L_1 < \infty$.

- If $0 < L_1 < \infty$, then, due to L'Hôpital's rule (see [3, Theorem 1.120]), we have

  $$\lim_{t\to\infty} \frac{z(t)}{t} = L_1,$$

  which, together with Lemma 2.1, shows that

  $$\lim_{t\to\infty} \frac{x(t)}{t} = \frac{L_1}{1 + p_0}.$$

  On the other hand, it is obvious that $\lim_{t\to\infty} x(t) = \infty$. So x belongs to $A_1(\infty, a)$, where $a > 0$.

- If $L_1 = 0$ and $L_0 = \infty$, then L'Hôpital's rule implies that $\lim_{t\to\infty} z(t)/t = 0$. By Lemma 2.1, we obtain $\lim_{t\to\infty} x(t)/t = 0$ and $\lim_{t\to\infty} x(t) = \infty$. Hence x belongs to $A_1(\infty, 0)$.

- If $L_1 = 0$ and $0 < L_0 < \infty$, then by the same reason as above we see that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t)/t = L_0/(1 + p_0)$. Hence x belongs to $A_1(a, 0)$, where $a > 0$.

The proof is complete. ■

4. Existence Criteria

In this section we will give existence criteria for each type of solution which has been classified according to Theorem 3.1.

Theorem 4.1. If (1.1) has a solution in $A_1(\infty, a)$ for $a > 0$, then there exists some $K > 0$ such that

$$\int_{t_0}^{\infty} f(\tau, Kh(\tau))\Delta\tau < \infty. \tag{4.1}$$

The converse is also true.

Proof. Suppose that x is an eventually positive solution of (1.1) satisfying $\lim_{t\to\infty} x(t)/t = a$. Then there exists $T_0 \in \mathbb{T}$ such that

$$x(h(t)) \geq \frac{ah(t)}{2}, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{4.2}$$

Noticing $\lim_{t\to\infty} z(t)/t = (1 + p_0)a$, we have

$$\lim_{t\to\infty} z^{\Delta}(t) = (1 + p_0)a. \tag{4.3}$$

From (3.2) we obtain that

$$z^{\Delta}(t) - z^{\Delta}(T_0) = -\int_{T_0}^{t} f(\tau, x(h(\tau)))\Delta\tau. \tag{4.4}$$

Invoking (4.3), (4.4) yields that

$$\int_{T_0}^{\infty} f(\tau, x(h(\tau)))\Delta\tau < \infty. \tag{4.5}$$

In view of (4.2), (4.5), and the monotonicity of $f(t, x)$ in x, we see that (4.1) holds, where $K = a/2$.

Conversely, suppose that (4.1) holds for some constant $K > 0$. To accomplish our proof, we proceed in two steps.

In case $0 \leq p_0 < 1$, take p_1 satisfying $p_0 < p_1 < (1 + 4p_0)/5 < 1$. Then

$$p_0 > \frac{5p_1 - 1}{4}.$$

Note that since $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} p(t)g(t)/t = p_0$ and (4.1) holds, there exists $T_0 \in \mathbb{T}$ large enough such that

$$\frac{p(t)g(t)}{t} \geq \frac{5p_1 - 1}{4}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{4.6}$$

$$\frac{5p_1 - 1}{4} \le p(t) \le p_1, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{4.7}$$

and

$$\int_{T_0}^{\infty} f(\tau, Kh(\tau))\Delta\tau \le \frac{(1 - p_0)K}{8}. \tag{4.8}$$

Note that the assumption (H1) implies that there exists $T_1 \in \mathbb{T}$ with $T_1 > T_0$ such that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Let $\lambda = 1$. Define the Banach space $BC[T_0, \infty)_{\mathbb{T}}$ as in (2.1) and let

$$X = \left\{ x \in BC[T_0, \infty)_{\mathbb{T}} : \frac{Kt}{2} \le x \le Kt \right\}. \tag{4.9}$$

Then it is clear that X is a bounded, convex and closed subset of $BC[T_0, \infty)_{\mathbb{T}}$, and for any $x \in X$ we have

$$f(t, x(h(t))) \le f(t, Kh(t)), \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{4.10}$$

Define two operators U and $S : X \to BC[T_0, \infty)_{\mathbb{T}}$ by

$$(Ux)(t) = \begin{cases} \dfrac{3Kp_1 t}{4} - \dfrac{p(T_1)x(g(T_1))t}{T_1}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3Kp_1 t}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}} \end{cases} \tag{4.11}$$

and

$$(Sx)(t) = \begin{cases} \dfrac{3Kt}{4}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3Kt}{4} + \displaystyle\int_{T_1}^{t} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \tag{4.12}$$

Next our task is to show that U and S satisfy the conditions in Lemma 2.4.

(i) We first prove that $Ux + Sy \in X$ for all $x, y \in X$. Note that since $Kt/2 \le x \le Kt$ and $Kt/2 \le y \le Kt$ for any $x, y \in X$, for $t \in [T_1, \infty)_{\mathbb{T}}$, we have from (4.7) and

(4.8) that

$$\begin{aligned}
(Ux)(t)+(Sy)(t) &= \frac{3(1+p_1)Kt}{4} - p(t)x(g(t)) \\
&\quad + \int_{T_1}^{t}\int_{\tau}^{\infty} f(s,x(h(s)))\Delta s\Delta\tau \\
&\geq \frac{3(1+p_1)Kt}{4} - p_1Kt \\
&= \frac{(3-p_1)Kt}{4} \geq \frac{Kt}{2}, \\
(Ux)(t)+(Sy)(t) &\leq \frac{3(1+p_1)Kt}{4} - \frac{p(t)Kt}{2} + \frac{(1-p_1)Kt}{8} \\
&\leq \frac{3(1+p_1)Kt}{4} - \frac{5p_1-1}{4}\cdot\frac{Kt}{2} + \frac{(1-p_1)Kt}{8} \\
&= Kt.
\end{aligned}$$

When $t \in [T_0, T_1]_{\mathbb{T}}$, by $g(t) \leq t$, (4.6) and (4.7), we have

$$\begin{aligned}
(Ux)(t)+(Sy)(t) &= \frac{3(1+p_1)Kt}{4} - p(T_1)\frac{x(g(T_1))t}{T_1} \\
&\geq \frac{3(1+p_1)Kt}{4} - p_1\frac{KT_1t}{T_1} \\
&\geq \frac{Kt}{2}
\end{aligned}$$

and

$$\begin{aligned}
(Ux)(t)+(Sy)(t) &\leq \frac{3(1+p_1)Kt}{4} - p(T_1)\frac{Kg(T_1)t}{2T_1} \\
&= \frac{3(1+p_1)Kt}{4} - \frac{p(T_1)g(T_1)}{T_1}\cdot\frac{Kt}{2} \\
&\leq \frac{3(1+p_1)Kt}{4} - \frac{5p_1-1}{4}\cdot\frac{Kt}{2} \\
&\leq Kt.
\end{aligned}$$

That is, $Ux + Sy \in X$ for any $x, y \in X$.

(ii) For the property of U being a contraction mapping, the proof is similar to the proof of [8, Theorem 8] and hence we skip it.

(iii) We must prove that S is a completely continuous mapping. Indeed, by analogous arguments to the proof of [8, Theorem 8] we may prove that S maps X into X and is continuous on X. Next, to prove that SX is relatively compact, we need to verify that SX satisfies all the conditions in Lemma 2.3. Note that since S maps

X into X, we have $||Sx|| = \sup_{t\in[T_0,\infty)_{\mathbb{T}}} |(Sx)(t)/t^2| \le K/T_0$ for any $x \in X$, hence SX is bounded. For any $\varepsilon > 0$, take $T_2 \in [T_1, \infty)_{\mathbb{T}}$ so that

$$\frac{1}{t} \le \frac{\varepsilon}{3K} \quad \text{for all} \quad t \in [T_2, \infty)_{\mathbb{T}}.$$

Then, for any $x \in X$ and $t_1, t_2 \in [T_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} \left| \frac{(Sx)(t_1)}{t_1^2} - \frac{(Sx)(t_2)}{t_2^2} \right| &\le \frac{3K}{4}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) + \frac{(1-p_1)K}{8}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{12} < \varepsilon. \end{aligned}$$

Thus, SX is uniformly Cauchy. Let $T_3 \in [T_0, \infty)_{\mathbb{T}}$ be arbitrary. Without loss of generality, set $T_3 > T_1$. We now show that SX is equicontinuous on $[T_0, T_3]_{\mathbb{T}}$. For any $x \in X$, we have

$$|(Sx)(t_1) - (Sx)(t_2)| = \frac{3K|t_1 - t_2|}{4}$$

for $t_1, t_2 \in [T_0, T_1]_{\mathbb{T}}$ and

$$|(Sx)(t_1) - (Sx)(t_2)| \le \frac{(1-p_1)K|t_1 - t_2|}{8}$$

for $t_1, t_2 \in [T_1, T_3]_{\mathbb{T}}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that when for $t_1, t_2 \in [T_0, T_3]_{\mathbb{T}}$ with $|t_1 - t_2| < \delta$,

$$|(Sx)(t_1) - (Sx)(t_2)| < \varepsilon \quad \text{for all} \quad x \in X.$$

Hence SX is equicontinuous on $[T_0, T_3]_{\mathbb{T}}$ for any $T_3 \in [T_0, \infty)_{\mathbb{T}}$. By means of Lemma 2.3, SX is relatively compact and hence S is a completely continuous mapping.

To sum up, there exists $x \in X$ by Lemma 2.4 such that $(U + S)x = x$. Therefore, we have

$$x(t) = \frac{3(1+p_1)Kt}{4} - p(t)x(g(t)) + \int_{T_1}^{t}\int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{4.13}$$

This means that x is an eventually positive solution of (1.1). Note that (4.10) implies

$$\frac{1}{t}\int_{T_1}^{t}\int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau \le \frac{1}{t}\int_{T_1}^{t}\int_{\tau}^{\infty} f(s, Kh(s))\Delta s \Delta\tau,$$

which, together with (4.1), induces that

$$\lim_{t\to\infty} \frac{1}{t} \int_{T_1}^{t} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau = 0.$$

Hence, from (4.13), it follows that $\lim_{t\to\infty} z(t)/t = 3(1 + p_1)K/4$. Thus, Lemma 2.1 implies that $\lim_{t\to\infty} x(t)/t = 3(1 + p_1)K/(4 + 4p_0)$ and hence $\lim_{t\to\infty} x(t) = \infty$. This says that (1.1) has a solution in $A_1(\infty, a)$ when $0 \le p_0 < 1$.

In case $-1 < p_0 < 0$, take p_1 so that $-p_0 < p_1 < (1 - 4p_0)/5 < 1$. Then $p_0 < (1 - 5p_1)/5$. Similarly, we can choose $T_0 \in \mathbb{T}$ large enough such that (4.8) holds and

$$\frac{p(t)g(t)}{t} \le -\frac{5p_1 - 1}{4}, \quad \frac{5p_1 - 1}{4} \le -p(t) \le p_1, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

Take $T_1 \in [T_0, \infty)_{\mathbb{T}}$ so that $g(t) \ge T_0$ and $h(t) \ge T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Now we introduce the Banach space $BC[T_0, \infty)_{\mathbb{T}}$ and its subset X as above. Define the operator S as in (4.12) and the operator U by

$$(Ux)(t) = \begin{cases} -\dfrac{3Kp_1 t}{4} - \dfrac{p(T_1)x(g(T_1))t}{T_1}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ -\dfrac{3Kp_1 t}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

The rest of the proof is similar to the proof in the case $0 \le p_0 < 1$ and is omitted. By Lemma 2.4, there exists $x \in X$ such that

$$x(t) = \frac{3(1 - p_1)Kt}{4} - p(t)x(g(t)) + \int_{T_1}^{t} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau, \quad t \in [T_1, \infty)_{\mathbb{T}},$$

which means that x is an eventually positive solution of (1.1) and $\lim_{t\to\infty} z(t)/t = 3(1 - p_1)K/4$. Thus we have $\lim_{t\to\infty} x(t)/t = 3(1-p_1)K/(4+4p_0)$, which yields $\lim_{t\to\infty} x(t)/t = \infty$. In summary, we see that (1.1) has a solution in $A_1(\infty, a)$ when $-1 < p_0 < 0$. The proof is complete. ■

Theorem 4.2. If (1.1) has a solution in $A_1(a, 0)$ for $a > 0$, then there exists some $K > 0$ such that

$$\int_{t_0}^{\infty} (\sigma(\tau) - t_0) f(\tau, K)\Delta\tau < \infty. \tag{4.14}$$

The converse is also true.

Proof. First of all, we note that Lemma 2.2 implies that

$$\int_{t_0}^{\infty} (\sigma(\tau) - t_0) f(\tau, K)\Delta\tau = \int_{t_0}^{\infty} \int_{\tau}^{\infty} f(s, K)\Delta s \Delta\tau. \tag{4.15}$$

Suppose that x is a solution of (1.1) in $A_1(a, 0)$. Then there exists $T_0 \in \mathbb{T}$ such that

$$x(t) \geq \frac{a}{2}, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{4.16}$$

In addition, it follows that

$$\lim_{t\to\infty} \frac{z(t)}{t} = 0, \quad \lim_{t\to\infty} z(t) = (1 + p_0)a,$$

which means that

$$\lim_{t\to\infty} z^{\Delta}(t) = 0.$$

Hence, from (3.2) we have

$$z^{\Delta}(t) = \int_t^{\infty} f(s, x(h(s)))\Delta s,$$

and this yields

$$z(t) - z(T_0) = \int_t^{\infty} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s. \tag{4.17}$$

Since $\lim_{t\to\infty} z(t)$ exists, (4.17) shows

$$\int_t^{\infty} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s < \infty. \tag{4.18}$$

Combining (4.15), (4.16), and (4.18), we see that (4.14) holds, where $K = a/2$.

Conversely, when $0 \leq p_0 < 1$, we use the notations from the proof of Theorem 4.1 such that (4.6) and (4.7) hold and

$$\int_{T_0}^{\infty} \int_{\tau}^{\infty} f(s, K)\Delta s \Delta\tau \leq \frac{(1 - p_1)K}{8}.$$

Let $\lambda = 0$ and define the Banach space as in (2.1). Let

$$X = \left\{ x \in BC[T_0, \infty)_{\mathbb{T}} : \frac{K}{2} \leq x(t) \leq K \right\}.$$

Now define two operators U and S on X by

$$(Ux)(t) = \begin{cases} \dfrac{3Kp_1}{4} - p(T_1)x(g(T_1)), & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3Kp_1}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}} \end{cases}$$

and

$$(Sx)(t) = \begin{cases} \dfrac{3K}{4} - \displaystyle\int_{T_1}^{\infty} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3K}{4} - \displaystyle\int_{t}^{\infty} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta\tau, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, we can prove that all the conditions in Lemma 2.4 are satisfied. So there exists $x \in X$ such that

$$x(t) = \frac{3(1+p_1)K}{4} - p(t)x(g(t)) - \int_t^\infty \int_\tau^\infty f(s, x(h(s)))\Delta s \Delta\tau, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{4.19}$$

From (4.14) and (4.19) we have

$$\lim_{t\to\infty} z(t) = \frac{3(1+p_1)K}{4}, \quad \lim_{t\to\infty} \frac{z(t)}{t} = 0.$$

Further, by Lemma 2.1 we obtain that

$$\lim_{t\to\infty} x(t) = \frac{3(1+p_1)K}{4(1+p_0)}, \quad \lim_{t\to\infty} \frac{x(t)}{t} = 0.$$

That is, x is a solution of (1.1) in $A_1(a, 0)$. We may use similar steps to prove the case of $-1 < p_0 < 0$. The proof is complete. ■

Theorem 4.3. If (1.1) has a solution in $A_1(\infty, 0)$, then

$$\int_{t_0}^\infty f(\tau, 1)\Delta\tau < \infty, \tag{4.20}$$

$$\int_{t_0}^\infty (\sigma(\tau) - t_0) f(\tau, h(\tau))\Delta\tau = \infty. \tag{4.21}$$

Conversely, if $\lim_{t\to\infty} p(t)t = 0$ and

$$\int_{t_0}^\infty f(\tau, h(\tau))\Delta\tau < \infty, \tag{4.22}$$

$$\int_{t_0}^\infty (\sigma(\tau) - t_0) f(\tau, 1)\Delta\tau = \infty, \tag{4.23}$$

then (1.1) has a solution in $A_1(\infty, 0)$.

Proof. We first demonstrate the necessity. Let x be an eventually positive solution of (1.1) in $A_1(\infty, 0)$. Since $\lim_{t\to\infty} x(t)/t = 0$ and $\lim_{t\to\infty} x(t) = \infty$, there exists $T_0 \in \mathbb{T}$ such that

$$x(t) \le t, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{4.24}$$

$$x(t) \ge 1, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{4.25}$$

For simplicity, we deem that $h(t) \ge T_0$ on $[T_0, \infty)_{\mathbb{T}}$. Note that since $\lim_{t\to\infty} z(t)/t = 0$ and $\lim_{t\to\infty} z(t) = \infty$, we have

$$\lim_{t\to\infty} z^\Delta(t) = 0, \quad \lim_{t\to\infty} z(t) = \infty. \tag{4.26}$$

Now integrating (3.2) from T_0 to t, we obtain that

$$z^{\Delta}(t) - z^{\Delta}(T_0) = -\int_{T_0}^{t} f(\tau, x(h(\tau)))\Delta\tau. \tag{4.27}$$

Using (4.25) and (4.26), we find (4.20). Since $\lim_{t\to\infty} z^{\Delta}(t) = 0$, (4.27) implies that

$$z^{\Delta}(t) = \int_{t}^{\infty} f(\tau, x(h(\tau)))\Delta\tau.$$

Hence we have

$$z(t) - z(T_0) = \int_{T_0}^{t}\int_{\tau}^{\infty} f(s, x(h(s)))\Delta s\Delta\tau,$$

which, together with (4.24) and (4.26), implies

$$\int_{T_0}^{\infty}\int_{\tau}^{\infty} f(s, h(s))\Delta s\Delta\tau = \infty.$$

Note that since

$$\begin{aligned}\int_{t_0}^{\infty}(\sigma(\tau) - t_0)f(\tau, h(\tau))\Delta\tau &= \int_{t_0}^{\infty}\int_{\tau}^{\infty} f(s, h(s))\Delta s\Delta\tau \\ &\geq \int_{T_0}^{\infty}\int_{\tau}^{\infty} f(s, h(s))\Delta s\Delta\tau,\end{aligned}$$

it is clear that (4.21) holds.

Next we demonstrate the sufficiency. We first consider the case $0 \leq p_0 < 1$. By the same steps as in the proof of Theorem 4.1, we take p_1, T_0 and T_1 with $T_1 > T_0$ such that (4.6)–(4.8) hold for $K = 1$, and

$$p(t)t \leq \frac{7p_1 - 3}{8}, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

where $p_1 > 3/7$. Let $\lambda = 1$. Define the Banach space as in (2.1) and let

$$X = \{x \in BC[T_0, \infty)_{\mathbb{T}} : \ 1 \leq x(t) \leq t\}.$$

Define two operators U and S by

$$(Ux)(t) = \begin{cases} \dfrac{3p_1}{4} - \dfrac{p(T_1)x(g(T_1))t}{T_1}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3p_1}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}} \end{cases}$$

and

$$(Sx)(t) = \begin{cases} \dfrac{3}{4}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3}{4} + \displaystyle\int_{T_1}^{t} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta \tau, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Then, similar to the proof of Theorem 4.1, there exists $x \in X$ such that

$$x(t) = \frac{3(1+p_1)}{4} - p(t)x(g(t)) + \int_{T_1}^{t} \int_{\tau}^{\infty} f(s, x(h(s)))\Delta s \Delta \tau, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{4.28}$$

From (4.22) and (4.23), equation (4.28) implies that $\lim_{t\to\infty} x(t)/t = 0$ and $\lim_{t\to\infty} x(t) = \infty$. That says (1.1) has a solution in $A_1(\infty, 0)$ when $0 \le p_0 < 1$. When $-1 < p_0 < 0$, we replace U by

$$(Ux)(t) = \begin{cases} -\dfrac{3p_1}{4} - \dfrac{p(T_1)x(g(T_1))t}{T_1}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ -\dfrac{3p_1}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

and the rest is the same as above. So we skip the proof. The proof is complete. ■

Theorem 4.4. If there exists $T_0 \in \mathbb{T}$ with $T_0 > 0$ such that

$$-p(t)e^{-g(t)} > e^{-t}, \quad t \in [T_0, \infty)_{\mathbb{T}} \tag{4.29}$$

and

$$\int_{t}^{\infty} (\sigma(\tau) - t) f(\tau, \frac{1}{h(\tau)})\Delta \tau \le \left(-p(t)e^{t-g(t)} - 1\right) e^{-t}, \quad t \in [T_0, \infty)_{\mathbb{T}}, \tag{4.30}$$

then equation (1.1) has a solution in $A_0(0)$.

Proof. By the assumptions (H1)–(H2), we obtain that $\lim_{t\to\infty} p(t)t/g(t) = p_0$. Now take $T_1 \in \mathbb{T}$ with $T_1 > T_0$ such that

$$g(t) \ge T_0, \quad h(t) \ge T_0, \quad t \in [T_1, \infty)_{\mathbb{T}}$$

and

$$-\frac{p(t)t}{g(t)} \le 1, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{4.31}$$

Let $\lambda = 0$ and define the Banach space $BC[T_0, \infty)_{\mathbb{T}}$ as in (2.1). Let

$$X = \left\{ x \in BC[T_0, \infty)_{\mathbb{T}} : \ e^{-t} \le x(t) \le \frac{1}{t} \text{ for } t \in [T_1, \infty)_{\mathbb{T}} \text{ and } e^{-T_1} \le x(t) \le \frac{1}{t} \text{ for } t \in [T_0, T_1]_{\mathbb{T}} \right\}.$$

Define an operator S on X by

$$(Sx)(t) = \begin{cases} -p(T_1)x(g(T_1)) - \displaystyle\int_{T_1}^{\infty}\int_{\tau}^{\infty} f(\tau, x(h(\tau)))\Delta s\Delta\tau, & t \in [T_0, T_1]_{\mathbb{T}}, \\ -p(t)x(g(t)) - \displaystyle\int_{t}^{\infty}\int_{\tau}^{\infty} f(\tau, x(h(\tau)))\Delta s\Delta\tau, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Note that $p(t) \le 0$ on $[T_0, \infty)_{\mathbb{T}}$ and Lemma 2.2 implies that

$$\int_t^{\infty} (\sigma(\tau) - t) f\left(\tau, \frac{1}{h(\tau)}\right)\Delta\tau = \int_t^{\infty}\int_{\tau}^{\infty} f\left(s, \frac{1}{h(s)}\right)\Delta s\Delta\tau.$$

Then, from (4.29)–(4.31), we have when $t \in [T_1, \infty)_{\mathbb{T}}$,

$$(Sx)(t) \ge -p(t)e^{-g(t)} - \left(-p(t)e^{t-g(t)} - 1\right)e^{-t} \ge e^{-t}$$

and

$$(Sx)(t) \le -\frac{p(t)t}{g(t)} \cdot \frac{1}{t} \le \frac{1}{t}.$$

It is obvious that $e^{-t} \le x(t) \le 1/t$ when $t \in [T_0, T_1]_{\mathbb{T}}$. Meanwhile, by the same ways as in the proof of Theorem 4.1, we can show that S is completely continuous. Hence, by Corollary 2.5, there exists $x \in X$ such that

$$x(t) = -p(t)x(g(t)) - \int_t^{\infty}\int_{\tau}^{\infty} f(\tau, x(h(\tau)))\Delta s\Delta\tau, \quad t \in [T_1, \infty)_{\mathbb{T}},$$

which means that x solves (1.1) on $[T_1, \infty)_{\mathbb{T}}$. In addition, we have $\lim_{t\to\infty} x(t) = 0$ by the definition of X. The proof is complete. ■

Note that the conclusion in Theorem 4.4 is based on the assumption that p is eventually negative. In case p is eventually nonnegative, we assert that (1.1) has no eventually positive solutions in $A_0(0)$. Otherwise, suppose that x is an eventually positive solution of (1.1) satisfying

$$\lim_{t\to\infty} x(t) = 0.$$

Then, $\lim_{t\to\infty} z(t) = 0$. On the other hand, by the proof of Theorem 3.1 we learn that $z^{\Delta}(t) > 0$ and $z^{\Delta\Delta}(t) < 0$ eventually. Hence

$$\lim_{t\to\infty} z^{\Delta}(t) = 0.$$

We now set

$$x(g(t)) > 0, \quad x(h(t)) > 0, \quad p(t) \ge 0, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

Then from (3.2) we have

$$x(t) = -p(t)x(g(t)) - \int_t^\infty \int_\tau^\infty f(s, x(h(s)))\Delta s \Delta\tau,$$

which means that $x(t) < 0$ on $[T_0, \infty)_{\mathbb{T}}$, and this contradicts our assumption. The following result is now clear.

Theorem 4.5. If p is eventually nonnegative, then equation (1.1) has no eventually positive solutions in $A_0(0)$.

5. Examples

Example 5.1. Let $c > 0$ and $\mathbb{T} = \{c\} + \bigcup_{n=1}^{\infty}[2nc, (2n+1)c]$. Let ρ be the backward jump operator on $\mathbb{T}$ defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}.$$

Consider the equation

$$\left[x(t) + \frac{t-2}{2t}x(g(t))\right]^{\Delta\Delta} + \frac{x(\sigma(t))}{\sigma^3(t)t} = 0, \quad t \in \mathbb{T}, \tag{5.1}$$

where $p(t) = (t-2)/(2t)$, $h(t) = \sigma(t)$, $f(t, x) = x/(\sigma^3(t)t)$, and

$$g(t) = \begin{cases} t - c, & \text{when} \quad t > \rho(t) \text{ or } t < \sigma(t), \\ t, & \text{otherwise.} \end{cases}$$

Then it is easy to verify that all the assumptions (H1)–(H3) are satisfied. Note that due to the improper integration (see [4, Chapter 5])

$$\int_c^\infty (\sigma(\tau) - c) f(\tau, h(\tau))\Delta\tau \le \int_c^\infty \frac{\Delta\tau}{\sigma(\tau)\tau} = \frac{1}{c},$$

we see by Theorem 4.3 that (5.1) has no eventually positive solutions in $A_1(\infty, 0)$. In addition, we have

$$\int_c^\infty f(\tau, h(\tau))\Delta\tau \le \frac{1}{c}.$$

Hence (5.1) has an eventually positive solution in $A_1(\infty, a)$ by Theorem 4.1. Also, since

$$\int_c^\infty (\sigma(\tau) - c) f(\tau, 1)\Delta\tau \le \frac{1}{c},$$

by Theorem 4.2, (5.1) has an eventually positive solution in $A_1(a, 0)$.

Example 5.2. Let $\mathbb{T} = [0, \infty) \subset \mathbb{R}$ and consider the equation

$$\left[x(t) - \frac{2}{e^{1-\frac{1}{t}}} x(t-1)\right]^{\Delta\Delta} + x(t)e^{-t} = 0, \quad t \geq 0, \tag{5.2}$$

where $p(t) = -2/e^{1-1/t}$, $g(t) = t - 1$, $h(t) = t$, and $f(t, x) = xe^{-t}$. Then the assumptions (H1)–(H3) are all satisfied. Furthermore, we have

$$-p(t)e^{-g(t)} \geq 2e^{-t}, \quad \left[-p(t)e^{t-g(t)} - 1\right]e^{-t} \geq e^{-t}, \quad t \geq 0$$

and

$$\int_c^\infty (\sigma(\tau) - c) f\left(\tau, \frac{1}{h(\tau)}\right) \Delta\tau \leq e^{-t}.$$

Hence, by Theorem 4.4, (5.2) has an eventually positive solution in $A_0(0)$.

As a final word, we remark that the techniques as above can be taken to study eventually positive solutions of higher order neutral dynamic equations on time scales. We will discuss those elsewhere.

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