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Existence of a Solution for a Class of Parabolic Equations with Three Unbounded Nonlinearities

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Abstract

We give an existence result of a renormalized solution for a class of nonlinear parabolic equations $\dfrac{\partial b(x,u)}{\partial t} - \operatorname{div}\,(a(x,u,\nabla u) + \Phi(u)) = f$, where the right-hand side belongs to $L^1((0,T)\times\Omega)$ and where $b(x,u)$ is an unbounded function of u and where $-\operatorname{div}(a(t,x,u,\nabla u) + \Phi(u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u, but without any growth assumption on u.

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1. Introduction

In the present paper we establish an existence result of a renormalized solution for a class of nonlinear parabolic equations of the type

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \operatorname{div}\,(a(x,t,u,\nabla u) + \Phi(u)) = f \quad \text{in} \quad \Omega\times(0,T), \\ b(x,u)(t=0) = b(x,u_0) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega\times(0,T). \end{cases} \tag{1.1}$$

In Problem (1.1), the framework is the following: Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, T is a positive real number while the data f and $b(x,u_0)$ are in $L^1(\Omega\times(0,T))$ and $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator which is coercive

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and which grows like $|\nabla u|^{p-1}$ with respect to ∇u, but which is not restricted by any growth condition with respect to u (see assumptions (2.3), (2.4) and (2.5) of Section 2). The function Φ is just assumed to be continuous on $\mathbb{R}$.

When Problem (1.1) is investigated, the difficulty is due to the facts that the data f and $b(x, u_0)$ only belong to L^1 and the functions $a(x, t, u, \nabla u)$ and $\Phi(u)$ do not belong to $(L^1_{\text{loc}}((0, T) \times \Omega))^N$ in general, so that proving existence of a weak solution (i.e., in the distribution meaning) seems to be an arduous task. To overcome this difficulty we use in this paper the framework of renormalized solutions. This notion was introduced by P.-L. Lions and Di Perna [16] for the study of Boltzmann equation (see also P.-L. Lions [20] for a few applications to fluid mechanics models). This notion was then adapted to elliptic versions of (1.1) in Boccardo, J.-L. Diaz, D. Giachetti, F. Murat [9], in P.-L. Lions and F. Murat [21] and F. Murat [21,22]. At the same time, the equivalent notion of entropy solutions has been developed independently by Bénilan and et al. [1] for the study of nonlinear elliptic problems.

As far as the parabolic equation case (1.1) is concerned, the existence and uniqueness of renormalized solutions has been proved in D. Blanchard, F. Murat and H. Redwane [4] (see also A. Porretta [23]) in the case where $b(x, u) = u$ and where f is replaced by $f + \text{div}(g)$ (where $g \in (L^{p'}(Q))^N$). In the case where $a(x, t, s, \xi)$ is independent of s, $\Phi = 0$ and $g = 0$, existence and uniqueness have been established in D. Blanchard [2], D. Blanchard and F. Murat [3], and in the case where $b(x, u) = b(u)$ (where b is a strictly increasing function of u that can possibly blow up for some finite r_0) and $a(x, t, s, \xi)$ is independent of s and linear with respect to ξ, existence and uniqueness have been established in D. Blanchard and H. Redwane [7], and in the case where $b(x, u) = b(u)$ (where b is a maximal monotone graph on $\mathbb{R}$) and $a(x, t, s, \xi)$ is independent of t, existence and uniqueness have been established in D. Blanchard and A. Porretta [6] (see also J. Carrillo [12], J. Carrillo and P. Wittbold [13, 14]).

With respect to the previous ones, the originality of the present work lies on the noncontrolled growth of the function $a(x, t, s, \xi)$ with respect to s, and the function Φ is just assumed to be continuous on $\mathbb{R}$, and $f,\ b(x, u_0)$ are just assumed to belong to L^1.

The paper is organized as follows: Section 2 is devoted to specify the assumptions on b, $a(x, t, s, \xi)$, Φ, f and u_0 needed in the present study and gives the definition of a renormalized solution of (1.1). In Section 3 (Theorem 3.1) we establish the existence of such a solution. In Section 4 (appendix) we give the proof of Lemma 3.2.

2. Assumptions on Data and Definition of a Renormalized Solution

Throughout the paper, we assume that the following assumptions hold true.

Assumption 2.1. Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0, T)$,

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function} \tag{2.1}$$

such that for every $x \in \Omega$, $b(x, \cdot)$ is a strictly increasing C^1-function with $b(x, 0) = 0$. Next, for any $K > 0$, there exists $\lambda_K > 0$ and functions $A_K \in L^1(\Omega)$ and $B_K \in L^p(\Omega)$

such that

$$\lambda_K \leq \frac{\partial b(x,s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \left| \nabla_x \left(\frac{\partial b(x,s)}{\partial s} \right) \right| \leq B_K(x) \tag{2.2}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq K$. Also,

$$a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function,} \tag{2.3}$$

$$a(x,t,s,\xi)\xi \geq \alpha |\xi|^p \tag{2.4}$$

for almost every $(x,t) \in Q$, for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ is a given real number. Next, for any $K > 0$, there exist $\beta_K > 0$ and a function $C_K \in L^{p'}(\Omega)$ such that

$$|a(x,t,s,\xi)| \leq C_K(x,t) + \beta_K |\xi|^{p-1} \tag{2.5}$$

for almost every $(t,x) \in Q$, for every s such that $|s| \leq K$, and for every $\xi \in \mathbb{R}^N$,

$$[a(x,t,s,\xi) - a(x,t,s,\xi')][\xi - \xi'] \geq 0 \tag{2.6}$$

for any $s \in \mathbb{R}$, for any $(\xi, \xi') \in \mathbb{R}^{2N}$, and for almost every $(x,t) \in Q$. Finally,

$$\Phi : \mathbb{R} \to \mathbb{R}^N \quad \text{is a continuous function,} \tag{2.7}$$

$$f \in L^1(Q), \tag{2.8}$$

$$u_0 \quad \text{is a measurable function defined on } \Omega \text{ such that } b(\cdot, u_0) \in L^1(\Omega). \tag{2.9}$$

Remark 2.2. In (2.2), we denote by $\nabla_x \left(\frac{\partial b(x,s)}{\partial s} \right)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions.

As already mentioned in the introduction, Problem (1.1) does not admit a weak solution under assumptions (2.1)–(2.9) (even when $b(x,u) = u$, $f = 0$ and $u_0 = 0$) since the growths of $a(x,t,u,\nabla u)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when $u \in L^p(0,T; W_0^{1,p}(\Omega))$).

Throughout, for any $K \geq 0$, we denote by $T_K(r) = \min\{K, \max\{r, -K\}\}$ the truncation function at height K. The definition of a renormalized solution for Problem (1.1) can be stated as follows.

Definition 2.3. A measurable function u defined on Q is a renormalized solution of Problem (1.1) if

$$T_K(u) \in L^p(0,T; W_0^{1,p}(\Omega)) \text{ for any } K \geq 0 \text{ and } b(x,u) \in L^\infty(0,T; L^1(\Omega)), \tag{2.10}$$

$$\int_{\{(t,x) \in Q : n \leq |u(x,t)| \leq n+1\}} a(x,t,u,\nabla u) \nabla u \, dx \, dt \to 0 \quad \text{as} \quad n \to \infty, \tag{2.11}$$

and if, for every increasing function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has compact support, we have

$$\begin{aligned}\frac{\partial b_S(x,u)}{\partial t} &- \operatorname{div}(S'(u)a(x,t,u,\nabla u)) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &+ \operatorname{div}(S'(u)\Phi(u)) - S''(u)\Phi(u)\nabla u = fS'(u) \quad \text{in} \quad D'(Q)\end{aligned} \tag{2.12}$$

and

$$b_S(x,u)(t=0) = b_S(x,u_0) \quad \text{in} \quad \Omega, \tag{2.13}$$

where $b_S(x,r) = \displaystyle\int_0^r \frac{\partial b(x,s)}{\partial s} S'(s)ds$.

The following remarks are concerned with a few comments on Definition 2.3.

Remark 2.4. Equation (2.12) is formally obtained through pointwise multiplication of the first equation in (1.1) by $S'(u)$. Note that due to (2.10), each term in (2.12) has a meaning in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. Indeed for the sake of simplicity, we denote $a(u,\nabla u) = a(x,t,u,\nabla u)$, and if K is such that supp $S' \subset [-K,K]$, the following identifications are made in (2.12):

- $S(u) \in L^\infty(Q)$ since S is a bounded function.
- $S'(u)a(u,\nabla u)$ is identified with $S'(u)a(T_K(u),\nabla T_K(u))$ a.e. in Q. Since indeed $|T_K(u)| \le K$ a.e. in Q, assumptions (2.3) and (2.5) imply that

  $$|a(x,t,T_K(u),\nabla T_K(u))| \le C_K(x,t) + \beta_K|\nabla T_K(u)|^{p-1} \quad \text{a.e. in} \quad Q.$$

  As a consequence of (2.10) and of $S'(u) \in L^\infty(Q)$, it follows that

  $$S'(u)a(T_K(u),\nabla T_K(u)) \in L^{p'}(Q)^N.$$

- $S''(u)a(u,\nabla u)\nabla u$ is identified with $S''(u)a(T_K(u),\nabla T_K(u))\nabla T_K(u)$, and due to (2.3), (2.5), and (2.10), one has

  $$S''(u)a(T_K(u),\nabla T_K(u))\nabla T_K(u) \in L^1(Q).$$

- $S'(u)\Phi(u)$ is identified with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(u)\nabla u$ is identified with $S''(u)\Phi(T_K(u))\nabla T_K(u)$. Due to the properties of S and (2.7), the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (2.10) implies that

  $$S'(u)\Phi(T_K(u)) \in (L^\infty(Q))^N \quad \text{and} \quad S''(u)\Phi(T_K(u))\nabla T_K(u) \in L^p(Q).$$

The above shows that (2.12) holds in $D'(Q)$ and that

$$\frac{\partial b_S(x,u)}{\partial t} \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Due to the properties of S and (2.2), $\dfrac{\partial S(u)}{\partial t} \in L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))$, which implies that $S(u) \in C^0([0, T]; L^1(\Omega))$ (for a proof of this trace result see [23]), so that the initial condition (2.13) makes sense, since, due to the properties of S (increasing) and (2.2), we have

$$\left|b_S(x, r) - b_S(x, r')\right| \leq A_K(x)|S(r) - S(r')| \quad \text{for all} \quad r, r' \in \mathbb{R}. \tag{2.14}$$

3. Existence Result

In this section we establish the following existence theorem.

Theorem 3.1. Under assumptions (2.1)–(2.9) there exists at least one renormalized solution u of Problem (1.1).

Proof. The proof is divided into 9 steps. In Step 1, we introduce an approximate problem. Step 2 is devoted to establish a few a priori estimates. In Step 3, the limit u of the approximate solutions u^ε is introduced and it is shown that $b(x, u) \in L^\infty(0, T; L^1(\Omega))$ and that (2.10) holds. In Step 4, we define a time regularization of the field $T_K(u)$ and we establish Lemma 3.2, which a allows us to control the parabolic contribution that arises in the monotonicity method when passing to the limit. Step 5 is devoted to prove an energy estimate (Lemma 3.3) which is a key point for the monotonicity arguments that are developed in Step 6 and Step 7. In Step 8, we prove that u satisfies (2.11). At last, Step 9 is devoted to prove that u satisfies (2.12) and (2.13) of Definition 2.3. ■

Step 1

Let us introduce the following regularization of the data: For $\varepsilon > 0$ fixed,

$$b_\varepsilon(x, s) = b(x, T_{1/\varepsilon}(s)) + \varepsilon s \quad \text{a.e. in} \quad \Omega, \quad s \in \mathbb{R}, \tag{3.1}$$

$$a_\varepsilon(x, t, s, \xi) = a(x, t, T_{1/\varepsilon}(s), \xi) \quad \text{a.e. in} \quad Q, \quad s \in \mathbb{R}, \quad \xi \in \mathbb{R}^N, \tag{3.2}$$

$$\Phi_\varepsilon : \mathbb{R} \to \mathbb{R}^N \quad \text{is a Lipschitz continuous bounded function} \tag{3.3}$$

such that Φ_ε uniformly converges to Φ on any compact subset of $\mathbb{R}$ as $\varepsilon \to 0$, and

$$f^\varepsilon \in L^{p'}(Q) \quad \text{satisfies} \quad f^\varepsilon \to f \quad \text{in} \quad L^1(Q) \quad \text{as} \quad \varepsilon \to 0, \tag{3.4}$$

$$u_0^\varepsilon \in C_0^\infty(\Omega) \quad \text{satisfies} \quad b_\varepsilon(x, u_0^\varepsilon) \to b(x, u_0) \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad \varepsilon \to 0. \tag{3.5}$$

Let us now consider the regularized problem

$$\begin{cases} \dfrac{\partial b_\varepsilon(x, u^\varepsilon)}{\partial t} - \operatorname{div}\left(a_\varepsilon(x, t, u^\varepsilon, \nabla u^\varepsilon) + \Phi_\varepsilon(u^\varepsilon)\right) = f^\varepsilon \quad \text{in} \quad Q, \\ u^\varepsilon = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \\ b_\varepsilon(x, u^\varepsilon)(t = 0) = b_\varepsilon(x, u^\varepsilon) \quad \text{in} \quad \Omega. \end{cases} \tag{3.6}$$

In view of (3.1), b_ε satisfies (2.1) and (2.2), and due to (2.2), there exist $\lambda_\varepsilon > 0$ and functions $A_\varepsilon \in L^1(\Omega)$ and $B_\varepsilon \in L^P(\Omega)$ such that

$$\lambda_\varepsilon \le \frac{\partial b_\varepsilon(x,s)}{\partial s} \le A_\varepsilon(x) \quad \text{and} \quad \left|\nabla_x \left(\frac{\partial b_\varepsilon(x,s)}{\partial s}\right)\right| \le B_\varepsilon(x) \quad \text{a.e. in} \quad \Omega, \quad s \in \mathbb{R}.$$

In view of (3.2), a_ε satisfies (2.3), (2.4), and (2.6), and due to (2.5), there exist $\beta_\varepsilon > 0$ and a function $C_\varepsilon \in L^{p'}(\Omega)$ such that

$$|a_\varepsilon(x,t,s,\xi)| \le C_\varepsilon(x,t) + \beta_\varepsilon|\xi|^{p-1} \quad \text{a.e. in} \quad Q, \quad s \in \mathbb{R}, \quad \xi \in \mathbb{R}^N.$$

For the sake of simplicity, we denote $a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon) = a_\varepsilon(x,t,u^\varepsilon,\nabla u^\varepsilon)$.

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ of (3.6) is an easy task (see e.g., [19]).

Step 2

The estimates derived in this step rely on usual techniques for problems of type (3.6), and we just sketch the proofs of them (the reader is referred to [2–5,7,8] or to [9,21,22] for elliptic versions of (3.6)).

Using $T_K(u^\varepsilon)$ as a test function in (3.6) leads to

$$\begin{aligned}&\int_\Omega b_K^\varepsilon(x,u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u^\varepsilon)\,dx\,ds\\&+\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)\nabla T_K(u^\varepsilon)\,dx\,ds = \int_0^t\int_\Omega f^\varepsilon T_K(u^\varepsilon)\,dx\,ds + \int_\Omega b_K^\varepsilon(x,u^\varepsilon)\,dx \end{aligned}\tag{3.7}$$

for almost every $t \in (0,T)$, and where $b_K^\varepsilon(x,r) = \displaystyle\int_0^r T_K(s)\frac{\partial b_\varepsilon(x,s)}{\partial s}\,ds$.

The Lipschitz character of Φ_ε and Stokes' formula together with the boundary condition in (3.6) make it possible to obtain

$$\int_0^t\int_\Omega \Phi_\varepsilon(u^\varepsilon)\nabla T_K(u^\varepsilon)\,dx\,ds = 0 \quad \text{for almost any} \quad t \in (0,T).$$

Due to the definition of b_K^ε we have

$$0 \le b_K^\varepsilon(x,u_0^\varepsilon) \le K|b_\varepsilon(x,u_0^\varepsilon)| \quad \text{a.e. in} \quad \Omega,$$

so that

$$0 \le \int_\Omega b_K^\varepsilon(x,u_0^\varepsilon)\,dx \le K\int_\Omega |b_\varepsilon(x,u_0^\varepsilon)|\,dx.$$

Since a_ε satisfies (2.4), the properties of f^ε and $b_\varepsilon(u_0^\varepsilon)$ permit to deduce from (3.7) that

$$T_K(u^\varepsilon) \quad \text{is bounded in} \quad L^p(0,T;W_0^{1,p}(\Omega)),\tag{3.8}$$

independently of ε for any $K \geq 0$.

Proceeding as in [3,4,7], we have for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support ($\operatorname{supp} S' \subset [-K, K]$)

$$S(u^\varepsilon) \quad \text{is bounded in} \quad L^p(0, T; W_0^{1,p}(\Omega)) \tag{3.9}$$

and

$$\frac{\partial S(u^\varepsilon)}{\partial t} \quad \text{is bounded in} \quad L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega)), \tag{3.10}$$

independently of ε. As a first consequence we have

$$\nabla S(u^\varepsilon) = S'(u^\varepsilon)\nabla T_K(u^\varepsilon) \quad \text{a.e. in} \quad Q. \tag{3.11}$$

As a consequence of (3.8) and (3.11), we then obtain (3.9). To show that (3.10) holds true, we multiply the equation for u^ε in (3.6) by $S'(u^\varepsilon)$ to obtain

$$\begin{aligned} \frac{\partial b_S^\varepsilon(x, u^\varepsilon)}{\partial t} &= \operatorname{div}\left(S'(u^\varepsilon)a_\varepsilon(t, x, u^\varepsilon, \nabla u^\varepsilon)\right) \\ &- S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon + \operatorname{div}(\Phi_\varepsilon(u^\varepsilon))S'(u^\varepsilon) + f^\varepsilon S'(u^\varepsilon) \quad \text{in} \quad D'(Q), \end{aligned} \tag{3.12}$$

where $b_S^\varepsilon(x, r) = \displaystyle\int_0^r S'(s)\frac{\partial b_\varepsilon(x, s)}{\partial s}\, ds$. Since $\operatorname{supp} S'$ and $\operatorname{supp} S''$ are both included in $[-K, K]$, u^ε may be replaced by $T_K(u^\varepsilon)$ in each of these terms. Thus each term on the right-hand side of (3.12) is bounded either in $L^{p'}(0, T; W^{-1,p'}(\Omega))$ or in $L^1(Q)$. As a consequence of (2.2) and (3.12), we then obtain (3.10).

Now for fixed $K > 0$, $a_\varepsilon(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)) = a(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon))$ a.e. in Q as soon as $\varepsilon < 1/K$, while assumption (3.2) gives

$$\left|a_\varepsilon(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon))\right| \leq C_K(t, x) + \beta_K |\nabla T_K(u^\varepsilon)|^{p-1},$$

where $\beta_K > 0$ and $C_K \in L^{p'}(Q)$. In view of (3.8), we find

$$a_\varepsilon\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \quad \text{is bounded in} \quad (L^{p'}(Q))^N,$$

independently of ε for $\varepsilon < 1/K$.

For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined by

$$\theta_n(r) = T_{n+1}(r) - T_n(r).$$

Remark that $\|\theta_n\|_{L^\infty(\mathbb{R})} \leq 1$ for any $n \geq 1$ and that $\theta_n(r) \to 0$ for any r when $n \to \infty$. Using this admissible test function $\theta_n(u^\varepsilon)$ in (3.6) leads to

$$\begin{aligned} &\int_\Omega b_{\varepsilon,n}(x, u^\varepsilon)(t)\, dx + \int_0^t \int_\Omega a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla\theta_n(u^\varepsilon)\, dx\, ds \\ &+ \int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon)\nabla\theta_n(u^\varepsilon)\, dx\, ds = \int_0^t \int_\Omega f^\varepsilon \theta_n(u^\varepsilon)\, dx\, ds + \int_\Omega b_{\varepsilon,n}(x, u_0^\varepsilon)\, dx \end{aligned} \tag{3.13}$$

for almost any $t \in (0, T)$, and where $b_{\varepsilon,n}(x, r) = \displaystyle\int_0^r \frac{\partial b_\varepsilon(x, s)}{\partial s}\theta_n(s)\, ds$.

The Lipschitz character of Φ_ε and Stokes' formula together with the boundary condition in (3.6) allow to obtain

$$\int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon)\nabla\theta_n(u^\varepsilon)\, dx\, ds = 0. \tag{3.14}$$

Since $b_{\varepsilon,n}(x, r) \geq 0$ and

$$a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla\theta_n(u^\varepsilon) = a(u^\varepsilon, \nabla u^\varepsilon)\nabla\theta_n(u^\varepsilon) \quad \text{a.e. in} \quad Q \quad \text{for} \quad \varepsilon < \frac{1}{n+1},$$

equality (3.13) implies that

$$\int_0^t \int_\Omega a(u^\varepsilon, \nabla u^\varepsilon)\nabla\theta_n(u^\varepsilon)\, dx\, ds \leq \int_0^t \int_\Omega f^\varepsilon\theta_n(u^\varepsilon)\, dx\, ds + \int_\Omega b_{\varepsilon,n}(x, u_0^\varepsilon)\, dx \tag{3.15}$$

for almost every $t \in (0, T)$ and for $\varepsilon < 1/(n + 1)$.

Step 3

Arguing again as in [3–5, 7], estimates (3.9) and (3.10) imply that, for a subsequence still indexed by ε,

$$u^\varepsilon \to u \quad \text{a.e. in} \quad Q, \tag{3.16}$$

$$b_\varepsilon(x, u^\varepsilon) \to b(x, u) \quad \text{a.e. in} \quad Q, \tag{3.17}$$

and with the help of (3.8),

$$T_K(u^\varepsilon) \rightharpoonup T_K(u) \quad \text{weakly in} \quad L^p(0, T; W_0^{1,p}(\Omega)), \tag{3.18}$$

$$\theta_n(u^\varepsilon) \rightharpoonup \theta_n(u) \quad \text{weakly in} \quad L^p(0, T; W_0^{1,p}(\Omega)), \tag{3.19}$$

$$a_\varepsilon\left(T_K(u^\varepsilon), DT_K(u^\varepsilon)\right) \rightharpoonup \sigma_K \quad \text{weakly in} \quad (L^{p'}(Q))^N \tag{3.20}$$

as $\varepsilon \to 0$ for any $K > 0$ and any $n \geq 1$, and where for any $K > 0$, $\sigma_K \in (L^{p'}(Q))^N$.

We now establish that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$. Indeed using $T_\sigma(u^\varepsilon)/\sigma$ as a test function in (3.6) leads to

$$\begin{aligned}\int_\Omega \frac{1}{\sigma} b_\sigma^\varepsilon(x, u^\varepsilon)(t)\, dx &+ \int_0^t \int_\Omega \frac{1}{\sigma} a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla T_\sigma(u^\varepsilon)\, dx\, ds \\ &+ \int_0^t \int_\Omega \frac{1}{\sigma}\Phi_\varepsilon(u^\varepsilon)\nabla T_\sigma(u^\varepsilon)\, dx\, ds \leq \left(\|f\|_{L^1(Q)} + \|b(x, u_0)\|_{L^1(\Omega)}\right)\end{aligned}$$

for almost every $t \in (0, T)$, and where $b_\sigma^\varepsilon(x, r) = \displaystyle\int_0^r T_\sigma(s)\frac{\partial b_\varepsilon(x, s)}{\partial s}\, ds$.

As usual, the divergence formula shows that the third term on the left-hand side of the above inequality is equal to zero, while the second term is nonnegative. Letting $\sigma \to 0$, it follows that

$$\int_\Omega |b_\varepsilon(x, u^\varepsilon)|(t)\,dx \le \left(\|f^\varepsilon\|_{L^1(Q)} + \|b_\varepsilon(x, u_0^\varepsilon)\|_{L^1(\Omega)}\right) \quad \text{a.e. in} \quad (0, T).$$

With (3.4) and (3.5), we have $b(x, u) \in L^\infty(0, T; L^1(\Omega))$.

We are now in a position to exploit (3.15). Due to the definition of θ_n we have

$$a(u^\varepsilon, \nabla u^\varepsilon)\nabla\theta_n(u^\varepsilon) = a(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon \chi_{\{n \le |u^\varepsilon| \le n+1\}} \quad \text{a.e. in} \quad Q.$$

Inequality (3.14), the pointwise convergence of u^ε to u and $b_\varepsilon(x, u_0^\varepsilon)$ to $b(x, u_0)$ then imply that

$$\overline{\lim_{\varepsilon\to 0}} \int_{\{n \le |u^\varepsilon| \le n+1\}} a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt \le \int_Q f\theta_n(u)\,dx\,dt + \int_\Omega b_n(x, u_0)\,dx.$$

Since θ_n and $\overline{\theta_n}$ both converge to zero everywhere as $n \to \infty$, Lebesgue's convergence theorem permits to conclude that

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \int_{\{n \le |u^\varepsilon| \le n+1\}} a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt = 0. \tag{3.21}$$

Step 4

This step is devoted to introduce for fixed $K \ge 0$ a time regularization of the function $T_K(u)$ in order to perform the monotonicity method which will be developed in Step 5 and Step 6. This kind of regularization has been first introduced by R. Landes (see [18, Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231]). More recently, it has been exploited in [11,15] to solve a few nonlinear evolution problems with L^1 or measure data.

This specific time regularization of $T_K(u)$ (for fixed $K \ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \quad \text{for all} \quad \mu > 0,$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \le K \quad \text{for all} \quad \mu > 0,$$

and

$$v_0^\mu \to T_K(u_0) \quad \text{a.e. in} \quad \Omega \quad \text{and} \quad \frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad \mu \to \infty. \tag{3.22}$$

Existence of such a subsequence $(v_0^\mu)_\mu$ is easy to establish (see e.g., [17]). For fixed $K \geq 0$ and $\mu > 0$, let us consider the unique solution $T_K(u)_\mu \in L^\infty(Q) \cap L^p(0, T; W_0^{1,p}(\Omega))$ of the monotone problem

$$\frac{\partial T_K(u)_\mu}{\partial t} + \mu \left(T_K(u)_\mu - T_K(u)\right) = 0 \quad \text{in} \quad D'(Q), \tag{3.23}$$

$$T_K(u)_\mu(t=0) = v_0^\mu \quad \text{in} \quad \Omega. \tag{3.24}$$

Remark that due to (3.23), we have for $\mu > 0$ and $K \geq 0$,

$$\frac{\partial T_K(u)_\mu}{\partial t} \in L^p(0, T; W_0^{1,p}(\Omega)).$$

The behavior of $T_K(u)_\mu$ as $\mu \to \infty$ is investigated in [18] (see also [15, 17]), and we just recall here that (3.22) and (3.23) imply that

$$T_K(u)_\mu \to T_K(u) \quad \text{a.e. in} \quad Q \tag{3.25}$$

and in $L^\infty(Q)$ weak star and strongly in $L^p(0, T; W_0^{1,p}(\Omega))$ as $\mu \to \infty$, and

$$\|T_K(u)_\mu\|_{L^\infty(Q)} \leq \max\left\{\|T_K(u)\|_{L^\infty(Q)},\ \|v_0^\mu\|_{L^\infty(\Omega)}\right\} \leq K \tag{3.26}$$

for any μ and any $K \geq 0$.

The main estimate is the following.

Lemma 3.2. Suppose $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$, and h has compact support. Then

$$\varliminf_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^s \left\langle \frac{\partial b_\varepsilon(x, u^\varepsilon)}{\partial t},\ h(u^\varepsilon)\left(T_K(u^\varepsilon) - (T_K(u))_\mu\right)\right\rangle dt\, ds \geq 0,$$

where $\langle , \rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.

Proof of Lemma 3.2. See appendix. ∎

Step 5

In this step, we prove the following lemma, which is the key point in the monotonicity arguments that will be developed in Step 6.

Lemma 3.3. The subsequence of u^ε defined in Step 3 satisfies for any $K \geq 0$

$$\begin{aligned}\varlimsup_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega & a(u^\varepsilon, \nabla T_K(u^\varepsilon))\nabla T_K(u^\varepsilon)\, dx\, ds\, dt \\ &\leq \int_0^T \int_0^t \int_\Omega \sigma_K \nabla T_K(u)\, dx\, ds\, dt.\end{aligned} \tag{3.27}$$

Proof. We first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that for any $n \geq 1$

$$S_n(r) = r \quad \text{for} \quad |r| \leq n, \quad \operatorname{supp} S_n' \subset [-(n+1), (n+1)], \quad \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1. \tag{3.28}$$

We use the sequence $T_K(u)_\mu$ of approximations of $T_K(u)$ defined by (3.23), (3.24) of Step 4, and plug the test function $S_n'(u^\varepsilon)(T_K(u^\varepsilon) - T_K(u)_\mu)$ (for $\varepsilon > 0$ and $\mu > 0$) in (3.6). Through setting, for fixed $K \geq 0$,

$$W_\mu^\varepsilon = (T_K(u^\varepsilon) - T_K(u)_\mu), \tag{3.29}$$

we obtain upon integration over $(0, t)$ and then over $(0, T)$:

$$\begin{aligned}
&\int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x, u^\varepsilon)}{\partial t}, \; S_n'(u^\varepsilon) W_\mu^\varepsilon \right\rangle ds\, dt \\
&\quad + \int_0^T \int_0^t \int_\Omega S_n'(u^\varepsilon) a_\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla W_\mu^\varepsilon \, dx\, ds\, dt \\
&\quad + \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon) W_\mu^\varepsilon a_\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla u^\varepsilon \, dx\, ds\, dt \\
&\quad + \int_0^T \int_0^t \int_\Omega \Phi_\varepsilon(u^\varepsilon) S_n'(u^\varepsilon) \nabla W_\mu^\varepsilon \, dx\, ds\, dt \\
&\quad + \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon) W_\mu^\varepsilon \Phi_\varepsilon(u^\varepsilon) \nabla u^\varepsilon \, dx\, ds\, dt \\
&= \int_0^T \int_0^t \int_\Omega f^\varepsilon S_n'(u^\varepsilon) W_\mu^\varepsilon \, dx\, ds\, dt.
\end{aligned} \tag{3.30}$$

In the following we pass to the limit in (3.30) as $\varepsilon \to 0$, then $\mu \to \infty$, and then $n \to \infty$, the real number $K \geq 0$ kept being fixed. In order to perform this task, we prove below the following results for fixed $K \geq 0$:

$$\varliminf_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x, u^\varepsilon)}{\partial t}, \; S_n'(u^\varepsilon) W_\mu^\varepsilon \right\rangle ds\, dt \geq 0 \quad \text{for any} \quad n \geq K, \tag{3.31}$$

$$\lim_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S_n'(u^\varepsilon) \Phi_\varepsilon(u^\varepsilon) \nabla W_\mu^\varepsilon \, dx\, ds\, dt = 0 \quad \text{for any} \quad n \geq 1, \tag{3.32}$$

$$\lim_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon) W_\mu^\varepsilon \Phi_\varepsilon(u^\varepsilon) \nabla u^\varepsilon \, dx\, ds\, dt = 0 \quad \text{for any} \quad n, \tag{3.33}$$

$$\lim_{n\to\infty} \varlimsup_{\mu\to\infty} \varlimsup_{\varepsilon\to 0} \left| \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon) W_\mu^\varepsilon a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon) \nabla u^\varepsilon \, dx\, ds\, dt \right| = 0, \tag{3.34}$$

and

$$\lim_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega f^\varepsilon S_n'(u^\varepsilon) W_\mu^\varepsilon \, dx\, ds\, dt = 0 \quad \text{for any} \quad n \geq 1. \tag{3.35}$$

Proof of (3.31). In view of the definition (3.29) of W_μ^ε, Lemma 3.2 applies with $h = S_n$ for fixed $n \geq K$. As a consequence, (3.31) holds true. ■

Proof of (3.32). For fixed $n \geq 1$, we have

$$S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla W_\mu^\varepsilon = S_n'(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))\nabla W_\mu^\varepsilon \tag{3.36}$$

a.e. in Q and for all $\varepsilon \leq 1/(n+1)$, and where supp $S_n' \subset [-(n+1), n+1]$. Since S_n' is smooth and bounded, (3.3) and (3.16) lead to

$$S_n'(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon)) \to S_n'(u)\Phi(T_{n+1}(u)) \tag{3.37}$$

a.e. in Q and in $L^\infty(Q)$ weak star as $\varepsilon \to 0$. For fixed $\mu > 0$, we have

$$W_\mu^\varepsilon \rightharpoonup (T_K(u) - T_K(u)_\mu) \quad \text{weakly in} \quad L^p(0, T; W_0^{1,p}(\Omega)) \tag{3.38}$$

and a.e. in Q and in $L^\infty(Q)$ weak star as $\varepsilon \to 0$. As a consequence of (3.36), (3.37), and (3.38), we deduce that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^T \int_0^t \int_\Omega S_n'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla W_\mu^\varepsilon \, dx\, ds\, dt \\ = \int_0^T \int_0^t \int_\Omega S_n'(u)\Phi(u)\nabla \left[T_K(u) - T_K(u)_\mu\right] dx\, ds\, dt \end{aligned} \tag{3.39}$$

for any $\mu > 0$. Appealing now to (3.25) and passing to the limit as $\mu \to \infty$ in (3.39) allows to conclude that (3.32) holds true. ■

Proof of (3.33). For fixed $n \geq 1$, and by the same arguments than those that lead to (3.36), we have

$$S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon = S_n''(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))\nabla T_{n+1}(u^\varepsilon) W_\mu^\varepsilon \quad \text{a.e. in} \quad Q.$$

From (3.3) and (3.16), it follows that for any $\mu > 0$

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon \, dx\, ds\, dt \\ = \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon)\Phi_\varepsilon(T_{n+1}(u^\varepsilon))\nabla T_{n+1}(u^\varepsilon) W_\mu^\varepsilon \, dx\, ds\, dt. \end{aligned}$$

With the help of (3.25) and passing to the limit as $\mu \to \infty$, the above equality leads to (3.33). ■

Proof of (3.34). For any fixed $n \geq 1$, we have supp $S_n'' \subset [-(n+1), -n] \cup [n, n+1]$. As a consequence,

$$\begin{aligned} &\left| \int_0^T \int_0^t \int_\Omega S_n''(u^\varepsilon) a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon \, dx\, ds\, dt \right| \\ &\quad \leq T \|S_n''\|_{L^\infty(\mathbb{R})} \|W_\mu^\varepsilon\|_{L^\infty(Q)} \int_{\{n \leq |u^\varepsilon| \leq n+1\}} a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon \, dx\, dt \end{aligned}$$

for any $n \geq 1$ and any $\mu > 0$. The above inequality together with (3.26) and (3.28) make it possible to obtain

$$\begin{aligned}
&\overline{\lim_{\mu\to\infty}}\,\overline{\lim_{\varepsilon\to 0}}\left|\int_0^T\int_0^t\int_\Omega S_n''(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon W_\mu^\varepsilon\,dx\,ds\,dt\right| \qquad (3.40)\\
&\quad\leq C\,\overline{\lim_{\varepsilon\to 0}}\int_{\{n\leq|u^\varepsilon|\leq n+1\}} a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\,dt
\end{aligned}$$

for any $n \geq 1$, where C is a constant independent of n. Appealing now to (3.21) permits to pass to the limit as $n \to \infty$ in (3.40) and to establish (3.34). ■

Proof of (3.35). For fixed $n \geq 1$, and in view of (3.4), (3.16), and (3.28), Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega f^\varepsilon S_n'(u^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt\\
&\quad=\int_0^T\int_0^t\int_\Omega f S_n'(u)\left(T_K(u)-T_K(u)_\mu\right)\,dx\,ds\,dt.
\end{aligned}$$

Now for fixed $n \geq 1$, using (3.25) permits to pass to the limit as $\mu \to \infty$ in the above equality to obtain (3.35). ■

We now return to the proof of Lemma 3.3. Due to (3.30)–(3.35), we are in a position to pass to the lim sup when $\varepsilon \to 0$, then to the lim sup when $\mu \to \infty$, and then to the limit as $n \to \infty$ in (3.30). We obtain using the definition of W_μ^ε that for any $K \geq 0$,

$$\lim_{n\to\infty}\overline{\lim_{\mu\to\infty}}\,\overline{\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla\left(T_K(u^\varepsilon)-T_K(u)_\mu\right)\,dx\,ds\,dt \leq 0.$$

Since $S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u^\varepsilon) = a(u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u^\varepsilon)$ for $\varepsilon \leq 1/K$ and $K \leq n$, the above inequality implies that for $K \leq n$,

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u^\varepsilon)\,dx\,ds\,dt \qquad (3.41)\\
&\quad\leq \lim_{n\to\infty}\overline{\lim_{\mu\to\infty}}\,\overline{\lim_{\varepsilon\to 0}}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u)_\mu\,dx\,ds\,dt.
\end{aligned}$$

The right-hand side of (3.41) is computed as follows. In view of (3.2) and (3.28), we have for $\varepsilon \leq 1/(n+1)$,

$$S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon) = S_n'(u^\varepsilon)a\left(T_{n+1}(u^\varepsilon),\nabla T_{n+1}(u^\varepsilon)\right) \quad \text{a.e. in} \quad Q.$$

Due to (3.20), it follows that for fixed $n \geq 1$,

$$S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon) \rightharpoonup S_n'(u)\sigma_{n+1} \quad \text{weakly in} \quad (L^{p'}(Q))^N$$

when $\varepsilon \to 0$. The strong convergence of $T_K(u)_\mu$ to $T_K(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\mu \to \infty$ then allows to conclude that

$$\lim_{\mu\to\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n'(u^\varepsilon)a_\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(u)_\mu\,dx\,ds\,dt \tag{3.42}$$
$$=\int_0^T\int_0^t\int_\Omega S_n'(u)\sigma_{n+1}\nabla T_K(u)\,dx\,ds\,dt=\int_0^T\int_0^t\int_\Omega \sigma_{n+1}\nabla T_K(u)\,dx\,ds\,dt$$

as long as $K \le n$, since $S_n'(r) = 1$ for $|r| \le n$. Now for $K \le n$, we have

$$a\left(T_{n+1}(u^\varepsilon),\nabla T_{n+1}(u^\varepsilon)\right)\chi_{\{|u^\varepsilon|<K\}} = a\left(T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)\right)\chi_{\{|u^\varepsilon|<K\}}\quad \text{a.e. in}\quad Q.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\sigma_{n+1}\chi_{\{|u|<K\}} = \sigma_K\chi_{\{|u|<K\}}\quad \text{a.e. in}\quad Q\setminus\{|u|=K\}\quad\text{for}\quad K\le n. \tag{3.43}$$

As a consequence of (3.43), we have for $K \le n$,

$$\sigma_{n+1}DT_K(u) = \sigma_K DT_K(u)\quad \text{a.e. in}\quad Q. \tag{3.44}$$

Recalling (3.41), (3.42), and (3.44) allows to conclude that (3.27) holds true, and the proof of Lemma 3.3 is complete. ■

Step 6

In this step, we prove the following monotonicity estimate.

Lemma 3.4. The subsequence of u^ε defined in Step 3 satisfies for any $K \ge 0$,

$$\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega\left[a(T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) - a(T_K(u^\varepsilon),\nabla T_K(u))\right]\times \tag{3.45}$$
$$\times\left[\nabla T_K(u^\varepsilon)-\nabla T_K(u)\right]\,dx\,ds\,dt = 0.$$

Proof. Let $K \ge 0$ be fixed. The monotone character (2.6) of $a(s,\xi)$ with respect to ξ implies that

$$\int_0^T\int_0^t\int_\Omega\left[a(T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) - a(T_K(u^\varepsilon),\nabla T_K(u))\right]\times \tag{3.46}$$
$$\times\left[\nabla T_K(u^\varepsilon)-\nabla T_K(u)\right]\,dx\,ds\,dt \ge 0.$$

To pass to the lim sup as $\varepsilon \to 0$ in (3.46), let us remark that (2.3), (2.4), and (3.16) imply that

$$a(T_K(u^\varepsilon),\nabla T_K(u)) \to a(T_K(u),\nabla T_K(u))\quad \text{a.e. in}\quad Q$$

as $\varepsilon \to 0$, and that

$$\left|a(T_K(u^\varepsilon),\nabla T_K(u))\right| \le C_K(t,x) + \beta_K|\nabla T_K(u)|^{p-1}\quad \text{a.e. in}\quad Q,$$

uniformly with respect to ε. It follows that when $\varepsilon \to 0$,

$$a\left(T_K(u^\varepsilon), \nabla T_K(u)\right) \to a\left(T_K(u), \nabla T_K(u)\right) \quad \text{strongly in} \quad (L^{p'}(Q))^N. \tag{3.47}$$

Inequality (3.46) is split into

$$A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon \geq 0, \tag{3.48}$$

where

$$A_1^\varepsilon = \int_0^T \int_0^t \int_\Omega a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \nabla T_K(u^\varepsilon)\, dx\, ds\, dt,$$

$$A_2^\varepsilon = -\int_0^T \int_0^t \int_\Omega a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \nabla T_K(u)\, dx\, ds\, dt,$$

and

$$A_3^\varepsilon = \int_0^T \int_0^t \int_\Omega a\left(T_K(u^\varepsilon), \nabla T_K(u)\right)\left(\nabla T_K(u^\varepsilon) - \nabla T_K(u)\right)\, dx\, ds\, dt.$$

Using (3.27) of Lemma 3.3, we obtain

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}} A_1^\varepsilon &= \overline{\lim_{\varepsilon \to 0}} \int_0^T \int_0^t \int_\Omega a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \nabla T_K(u^\varepsilon)\, dx\, ds\, dt \\ &\leq \int_0^T \int_0^t \int_\Omega \sigma_K \nabla T_K(u)\, dx\, ds\, dt. \end{aligned} \tag{3.49}$$

In view of (3.20), we have

$$\lim_{\varepsilon \to 0} A_2^\varepsilon = -\int_0^T \int_0^t \int_\Omega \sigma_K \nabla T_K(u)\, dx\, ds\, dt. \tag{3.50}$$

Due to (3.18), we conclude that

$$\left(\nabla T_K(u^\varepsilon) - \nabla T_K(u)\right) \rightharpoonup 0 \quad \text{weakly in} \quad (L^p(Q))^N \quad \text{as} \quad \varepsilon \to 0. \tag{3.51}$$

As a consequence of (3.47) and (3.51), we have for all $K > 0$,

$$\lim_{\varepsilon \to 0} A_3^\varepsilon = 0. \tag{3.52}$$

Now (3.49), (3.50), and (3.52) allow to pass to the lim sup as $\varepsilon \to 0$ in (3.48) and to obtain (3.45) of Lemma 3.4. ■

Step 7

In this step, we identify the weak limit σ_K and we prove the weak L^1 convergence of the "truncated" energy $a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \nabla T_K(u^\varepsilon)$ as $\varepsilon \to 0$.

Lemma 3.5. For fixed $K \geq 0$, we have as $\varepsilon \to 0$,

$$\sigma_K = a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \quad \text{a.e. in} \quad Q \tag{3.53}$$

and

$$a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \nabla T_K(u^\varepsilon) \rightharpoonup a\left(T_K(u), \nabla T_K(u)\right) \nabla T_K(u) \quad \text{weakly in} \quad L^1(Q). \tag{3.54}$$

Proof. The proof is standard once we remark that for any $K > 0$, any $0 < \varepsilon < 1/K$ and any $\xi \in \mathbb{R}^N$,

$$a_\varepsilon(T_K(u^\varepsilon), \xi) = a(T_K(u^\varepsilon), \xi) \quad \text{a.e. in} \quad Q$$

which with (3.18) and (3.47) makes it possible to obtain from (3.45) of Lemma 3.4,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_0^T \int_0^t \int_\Omega a_{1/K}\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) T_K(u^\varepsilon)\, dx\, ds\, dt \\ &\quad = \int_0^T \int_0^t \int_\Omega \sigma_K \nabla T_K(u)\, dx\, ds\, dt. \end{aligned} \tag{3.55}$$

Since, for fixed $K > 0$, the function $a_{1/K}(s, \xi)$ is continuous and bounded with respect to s, the usual Minty argument applies in view of (3.18), (3.20), and (3.55). It follows that (3.53) holds true.

Using the above convergence results that for any $K \geq 0$ and any $T' < T$,

$$\left[a(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)) - a(T_K(u^\varepsilon), \nabla T_K(u))\right]\left[\nabla T_K(u^\varepsilon) - \nabla T_K(u)\right] \to 0$$

strongly in $L^1((0, T') \times \Omega)$ as $\varepsilon \to 0$ because of Lemma 3.4. This shows that for any $K \geq 0$ and any $T' < T$,

$$a\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \nabla T_K(u^\varepsilon) \rightharpoonup a\left(T_K(u), \nabla T_K(u)\right) \nabla T_K(u) \tag{3.56}$$

weakly in $L^1((0, T') \times \Omega)$ as $\varepsilon \to 0$.

At the possible expense of extending the functions $a(t, x, s, \xi)$ and f on a time interval $(0, \overline{T})$ with $\overline{T} > T$ in such a way that (2.3)–(2.8) hold true with $\overline{T}$ in place of T, we can show that the convergence (3.56) is still weakly in $L^1(Q)$, namely that (3.54) holds true. ■

Step 8

In this step, we prove that u satisfies (2.11). To this end, remark that for any fixed $n \geq 0$ one has

$$\begin{aligned}
&\int_{\{(t,x)/\ n\leq|u^\varepsilon|\leq n+1\}} a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon \, dx\, dt \\
&\quad = \int_Q a_\varepsilon(u^\varepsilon, Du^\varepsilon)\left[\nabla T_{n+1}(u^\varepsilon) - \nabla T_{n+1}(u^\varepsilon)\right] dx\, dt \\
&\quad = \int_Q a\left(T_{n+1}(u^\varepsilon), \nabla T_{n+1}(u^\varepsilon)\right)\nabla T_{n+1}(u^\varepsilon)\, dx\, dt \\
&\qquad - \int_Q a\left(T_n(u^\varepsilon), \nabla T_n(u^\varepsilon)\right)\nabla T_n(u^\varepsilon)\, dx\, dt
\end{aligned}$$

for $\varepsilon < 1/(n+1)$. According to (3.54), one may pass to the limit as $\varepsilon \to 0$ for fixed $n \geq 0$ and obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\{(t,x)/\ n\leq|u^\varepsilon|\leq n+1\}} a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon \, dx\, dt \\
&\quad = \int_Q a\left(T_{n+1}(u), \nabla T_{n+1}(u)\right)\nabla T_{n+1}(u)\, dx\, dt \\
&\qquad - \int_Q a\left(T_n(u), \nabla T_n(u)\right)\nabla T_n(u)\, dx\, dt \\
&\quad = \int_{\{(t,x)/\ n\leq|u|\leq n+1\}} a(u, \nabla u)\nabla u \, dx\, dt.
\end{aligned} \tag{3.57}$$

Taking the limit as $n \to \infty$ in (3.57) and using the estimate (3.21) show that u satisfies (2.11).

Step 9

In this step, u is shown to satisfies (2.12) and (2.13). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has compact support. Let $K > 0$ be such that supp $S' \subset [-K, K]$. Pointwise multiplication of the approximate equation (3.6) by $S'(u^\varepsilon)$ leads to

$$\begin{aligned}
&\frac{\partial b_S^\varepsilon(x, u^\varepsilon)}{\partial t} - \operatorname{div}\left(S'(u^\varepsilon)a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\right) + S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)Du^\varepsilon \\
&\quad - \operatorname{div}\left(S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\right) + S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla u^\varepsilon = f^\varepsilon S'(u^\varepsilon) \quad \text{in} \quad D'(Q),
\end{aligned} \tag{3.58}$$

where $b_S^\varepsilon(x, r) = \displaystyle\int_0^r \frac{\partial b_\varepsilon(x, s)}{\partial s} S'(s)\, ds$. In what follows we pass to the limit as $\varepsilon \to 0$ in each term of (3.58).

(i) Since S is bounded and $b_S^\varepsilon(x, u^\varepsilon) \to b_S(x, u)$ a.e. in Q and weak star in $L^\infty(Q)$,

$$\frac{\partial b_S^\varepsilon(x, u^\varepsilon)}{\partial t} \to \frac{\partial b_S(x, u)}{\partial t} \quad \text{in} \quad D'(Q) \quad \text{as} \quad \varepsilon \to 0.$$

(ii) Since supp $S' \subset [-K, K]$, we have for $\varepsilon < 1/K$,

$$S'(u^\varepsilon)a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon) = S'(u^\varepsilon)a_\varepsilon\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \quad \text{a.e. in} \quad Q.$$

The pointwise convergence $u^\varepsilon \to u$ as $\varepsilon \to 0$, the bounded character of S, (3.20), and (3.53) of Lemma 3.5 imply that

$$S'(u^\varepsilon)a_\varepsilon\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right) \rightharpoonup S'(u)a\left(T_K(u), \nabla T_K(u)\right) \text{ weakly in } (L^{p'}(Q))^N$$

as $\varepsilon \to 0$, because $S'(u) = 0$ for $|u| \geq K$ a.e. in Q, and hence

$$S'(u)a\left(T_K(u), \nabla T_K(u)\right) = S'(u)a(u, \nabla u) \quad \text{a.e. in} \quad Q.$$

(iii) Since supp $S'' \subset [-K, K]$, we have for $\varepsilon \leq 1/K$,

$$S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon = S''(u^\varepsilon)a_\varepsilon\left(T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\right)\nabla T_K(u^\varepsilon) \quad \text{a.e. in} \quad Q.$$

The pointwise convergence $S''(u^\varepsilon) \to S''(u)$ as $\varepsilon \to 0$, the bounded character of S'', and (3.54) of Lemma 3.5 allow to conclude that

$$S''(u^\varepsilon)a_\varepsilon(u^\varepsilon, \nabla u^\varepsilon)\nabla u^\varepsilon \rightharpoonup S''(u)a\left(T_K(u), \nabla T_K(u)\right)\nabla T_K(u) \text{ weakly in } L^1(Q)$$

as $\varepsilon \to 0$ and

$$S''(u)a\left(T_K(u), \nabla T_K(u)\right)\nabla T_K(u) = S''(u)a(u, \nabla u)\nabla u \quad \text{a.e. in} \quad Q.$$

(iv) Since supp $S' \subset [-K, K]$, we have for $\varepsilon \leq 1/K$,

$$S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) = S'(u^\varepsilon)\Phi_\varepsilon(T_K(u^\varepsilon)) \quad \text{a.e. in} \quad Q.$$

As a consequence of (3.3) and (3.16), it follows that for any $1 \leq q < \infty$,

$$S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon) \to S'(u)\Phi(T_K(u)) \quad \text{strongly in} \quad L^q(Q)$$

as $\varepsilon \to 0$. The term $S'(u)\Phi(T_K(u))$ is denoted by $S'(u)\Phi(u)$.

(v) Since $S' \in W^{1,\infty}(\mathbb{R})$ with supp $S' \subset [-K, K]$, we have

$$S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla u^\varepsilon = \Phi_\varepsilon(T_K(u^\varepsilon))\nabla S'(u^\varepsilon) \quad \text{a.e. in} \quad Q.$$

Thus $\nabla S'(u^\varepsilon) \to \nabla S'(u)$ weakly in $L^p(Q)^N$ as $\varepsilon \to 0$, while $\Phi_\varepsilon(T_K(u^\varepsilon))$ is uniformly bounded with respect to ε and converges a.e. in Q to $\Phi(T_K(u))$ as $\varepsilon \to 0$. Therefore

$$S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)\nabla u^\varepsilon \rightharpoonup \Phi(T_K(u))\nabla S'(u) \quad \text{weakly in} \quad L^p(Q).$$

(vi) Due to (3.4) and (3.16), we have

$$f^\varepsilon S'(u^\varepsilon) \to f S'(u) \quad \text{strongly in} \quad L^1(Q) \quad \text{as} \quad \varepsilon \to 0.$$

As a consequence of the above convergence result, we are in a position to pass to the limit as $\varepsilon \to 0$ in equation (3.58) and to conclude that u satisfies (2.12). It remains to show that $b_S(x,u)$ satisfies the initial condition (2.13). To this end, firstly remark that, S being bounded, $S(u^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (3.58) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega))$. Due to (2.2), we deduce that $\dfrac{\partial S(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T; W^{-1,p'}(\Omega))$. As a consequence, an Aubin type lemma (see e.g., [24, Corollary 4]) implies that $S(u^\varepsilon)$ lies in a compact set of $C^0([0,T]; W^{-1,s}(\Omega))$ for any $s < \inf(p', N/(N-1))$. It follows that, on one hand, $S(u^\varepsilon)(t=0) \to S(u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. On the other hand, the smoothness of S implies that $S(u^\varepsilon)(t=0) \to S(u)(t=0)$ strongly in $L^q(\Omega)$ for all $q<\infty$. Due to (2.14) and (3.5), we conclude that $b_S^\varepsilon(x,u^\varepsilon)(t=0) = b_S^\varepsilon(x,u_0^\varepsilon) \to b_S(x,u)(t=0)$ strongly in $L^q(\Omega)$. Thus

$$b_S(x,u)(t=0) = b_S(x,u_0) \quad \text{in} \quad \Omega.$$

By Step 3, Step 8, and Step 9, the proof of Theorem 3.1 is complete. ■

4. Appendix

In this appendix we give the proof of Lemma 3.2.

Proof of Lemma 3.2. Suppose $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$, and h has compact support. By the integration by parts formula, using the properties of $T_K(u)_\mu$, we have

$$\begin{aligned}
&\int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon)\big(T_K(u^\varepsilon) - (T_K(u))_\mu\big)\right\rangle ds\,dt \qquad (4.1)\\
&= \int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon)T_K(u^\varepsilon)\right\rangle ds\,dt\\
&\quad - \int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon)(T_K(u))_\mu\right\rangle ds\,dt\\
&= I_1^\varepsilon + I_2^{\varepsilon,\mu}.
\end{aligned}$$

We denote

$$B_{h,K}^\varepsilon(x,z) = \int_0^z \frac{\partial b_\varepsilon(x,s)}{\partial s} h(s) T_K(s)\,ds \quad \text{and} \quad B_h^\varepsilon(x,z) = \int_0^z \frac{\partial b_\varepsilon(x,s)}{\partial s} h(s)\,ds.$$

Upon applying [10, Lemma 2.4] to the first term on the right-hand side of (4.1), we obtain

$$\begin{aligned} I_1^\varepsilon &= \int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon) T_K(u^\varepsilon) \right\rangle \\ &= \int_\Omega \int_0^T \big(B^\varepsilon_{h,K}(x,u^\varepsilon) - B^\varepsilon_{h,K}(x,u_0^\varepsilon) \big)\, dt\, dx. \end{aligned} \tag{4.2}$$

Passing to the limit in (4.2) as $\varepsilon \to 0$, we first observe that

$$\frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial s} h(u^\varepsilon) = \left(\frac{\partial b(x, T_n(u^\varepsilon))}{\partial s} + \varepsilon \right) h(u^\varepsilon) \text{ for } \varepsilon < \frac{1}{n} \text{ with supp } h \subset [-n, n].$$

In view of (2.1), (2.2), (3.5), and (3.17), passing to the limit in (4.2) leads to

$$\lim_{\varepsilon \to 0} I_1^\varepsilon = \int_\Omega \int_0^T \big(B_{h,K}(x,u) - B_{h,K}(x,u_0) \big)\, dt\, dx. \tag{4.3}$$

The second term on the right-hand side of (4.1) can be rewritten as

$$\begin{aligned} I_2^{\varepsilon,\mu} &= -\int_0^T \int_0^t \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon) (T_K(u))_\mu \right\rangle dt\, ds \qquad (4.4) \\ &= -\int_0^T \int_0^t \left\langle \frac{\partial B_h^\varepsilon(x,u^\varepsilon)}{\partial t}, (T_K(u))_\mu \right\rangle dt\, ds \\ &= -\int_\Omega \int_0^T \big(B_h^\varepsilon(x,u^\varepsilon)(T_K(u))_\mu - B_h^\varepsilon(x,u_0^\varepsilon)(T_K(u))_\mu(0) \big)\, dt\, dx \\ &\quad + \int_0^T \int_0^t B_h^\varepsilon(x,u^\varepsilon) \frac{\partial (T_K(u))_\mu}{\partial t}\, dt\, ds \\ &= -\int_\Omega \int_0^T \big(B_h^\varepsilon(x,u^\varepsilon)(T_K(u))_\mu - B_h^\varepsilon(x,u_0^\varepsilon)(T_K(u))_\mu(0) \big)\, dt\, dx \\ &\quad + \mu \int_0^T \int_0^t B_h^\varepsilon(x,u^\varepsilon) \big(T_K(u) - T_K(u)_\mu \big)\, dt\, ds. \end{aligned}$$

In view of (3.5), (3.16), and (3.17), passing to the limit as $\varepsilon \to 0$ in (4.4) is an easy task and leads to

$$\begin{aligned} \lim_{\varepsilon \to 0} I_2^{\varepsilon,\mu} = &-\int_\Omega \int_0^T \big(B_h(x,u)(T_K(u))_\mu - B_h(x,u_0)(T_K(u))_\mu(0) \big)\, dt\, dx \\ &+ \mu \int_0^T \int_0^t B_h(x,u) \big(T_K(u) - T_K(u)_\mu \big)\, dt\, ds \end{aligned} \tag{4.5}$$

for any $\mu > 0$. In order to pass to the lim inf as $\mu \to \infty$ in (4.5), we now use the definition of $T_K(u)_\mu$. In view of (3.22), (3.25), passing to the limit as $\mu \to \infty$ yields

$$\begin{aligned} \lim_{\mu\to\infty} \int_\Omega \int_0^T \big(B_h(x,u)(T_K(u))_\mu - B_h(x,u_0)T_K(u)_\mu(0)\big)\, dt\, dx \\ = \int_\Omega \int_0^T (B_h(x,u)T_K(u) - B_h(x,u_0)T_K(u_0))\, dt\, dx. \end{aligned} \tag{4.6}$$

The right-hand side of (4.5) can be rewritten as

$$\begin{aligned} &\mu \int_0^T \int_0^t B_h(x,u)\big(T_K(u) - T_K(u)_\mu\big)\, dt\, ds \\ &= \mu \int_0^T \int_0^t (B_h(x,u) - B_h(x,T_K(u)))\big(T_K(u) - T_K(u)_\mu\big)\, dt\, ds \\ &\quad + \mu \int_0^T \int_0^t \big(B_h(x,T_K(u)) - B_h(x,(T_K(u))_\mu)\big)\big(T_K(u) - T_K(u)_\mu\big)\, dt\, ds \\ &\quad + \mu \int_0^T \int_0^t B_h(x,(T_K(u))_\mu)\big(T_K(u) - T_K(u)_\mu\big)\, dt\, ds \\ &= II_1^\mu + II_2^\mu + II_3^\mu, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} II_1^\mu =& \mu \int_\Omega \int_0^T \int_0^t (B_h(x,u) - B_h(x,T_K(u)))\big(T_K(u) - T_K(u)_\mu\big)\, dt\, ds\, dx \\ =& \mu \int_{\{u>K\}} \int_0^T \int_0^t (B_h(x,u) - B_h(x,K))\big(K - T_K(u)_\mu\big)\, dt\, ds\, dx \\ &+ \mu \int_{\{u<-K\}} \int_0^T \int_0^t (B_h(x,u) - B_h(x,-K))\big(-K - T_K(u)_\mu\big)\, dt\, ds\, dx \ge 0 \end{aligned}$$

as $B_h(x,s)$ is nondecreasing for s and $-K \le (T_K(u))_\mu \le K$. It also follows that $II_2^\mu \ge 0$. Next,

$$\begin{aligned} II_3^\mu =& \mu \int_\Omega \int_0^T \int_0^t B_h(x,(T_K(u))_\mu)\big(T_K(u) - T_K(u)_\mu\big)\, dt\, ds\, dx \\ =& \int_\Omega \int_0^T \int_0^t B_h(x,(T_K(u))_\mu)\frac{\partial T_K(u)_\mu}{\partial t}\, dt\, ds\, dx \\ =& \int_\Omega \int_0^T \big(\overline{B_h}(x,(T_K(u))_\mu) - \overline{B_h}(x,(T_K(u))_\mu(0))\big)\, dt\, dx, \end{aligned}$$

where $\overline{B_h}(x,z) = \displaystyle\int_0^z B_h(x,s)ds$. Since $T_K(u)_\mu \to T_K(u)$ a.e. in Q and $|T_K(u)_\mu| \le K$, Lebesgue's convergence theorem shows that

$$\lim_{\mu\to\infty} II_3^\mu = \int_\Omega \int_0^T \big(\overline{B_h}(x,T_K(u)) - \overline{B_h}(x,(T_K(u_0)))\big)\, dt\, dx.$$

As a consequence of (4.5), (4.6), and (4.7), we deduce that

$$\begin{aligned}\varliminf_{\mu\to\infty}\lim_{\varepsilon\to 0} I_2^{\varepsilon,\mu} \geq &\int_\Omega\int_0^T \left(B_h(x,u)(T_K(u)) - B_h(x,u_0)T_K(u_0)\right)\,dt\,dx\\ &+\int_\Omega\int_0^T \left(\overline{B_h}(x,T_K(u)) - \overline{B_h}(x,(T_K(u_0)))\right)\,dt\,dx. \qquad (4.8)\end{aligned}$$

Due to (4.1), (4.3), and (4.8), we deduce that

$$\begin{aligned}&\varliminf_{\mu\to\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^s \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon)\left(T_K(u^\varepsilon) - (T_K(u))_\mu\right)\right\rangle dt\,ds\\ &\geq \int_\Omega\int_0^T \left(B_{h,K}(x,u) - B_{h,K}(x,u_0)\right)\,dt\,dx\\ &\quad -\int_\Omega\int_0^T \left(B_h(x,u)T_K(u) - B_h(x,u_0)T_K(u_0)\right)\,dt\,dx\\ &\quad +\int_\Omega\int_0^T \left(\overline{B_h}(x,T_K(u)) - \overline{B_h}(x,T_K(u_0))\right)\,dt\,dx = 0,\end{aligned}$$

where the last equality holds since for any $z \in \mathbb{R}$ and for almost every $x \in \Omega$, we have

$$\overline{B_h}(x,T_K(z)) = B_h(x,z)T_K(z) - B_{h,K}(x,z).$$

Indeed,

$$\begin{aligned}\overline{B_h}(x,T_K(z)) &= \int_0^{T_K(z)} B_h(x,s)\,ds = \int_0^{T_K(z)}\int_0^s \frac{\partial b(x,w)}{\partial w}h(w)\,dw\,ds\\ &= \int_0^{T_K(z)} (T_K(z) - w)\frac{\partial b(x,w)}{\partial w}h(w)\,dw\\ &= \int_0^{T_K(z)} (T_K(z) - T_K(w))\frac{\partial b(x,w)}{\partial w}h(w)\,dw\\ &= \int_0^{T_K(z)} (T_K(z) - T_K(w))\frac{\partial b(x,w)}{\partial w}h(w)\,dw.\end{aligned}$$

This concludes the proof. ∎

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