On Finite-time Blow-up for a Nonlocal Quasi-linear Parabolic Problem Describing Shear Bands in Metals

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Abstract

In this paper we investigate results on finite-time blow-up of solutions to a nonlocal quasi-linear parabolic problem. We extend some known results to higher dimensions.

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1. Introduction

In this paper we investigate results on finite-time blow-up of solutions to the nonlocal mixed problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
  u_t = \Delta_p u + \frac{\delta e^u}{(\int_{\Omega} e^u)^q}, \quad x \in \Omega, \quad 0 < t < T, \\
  u(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T, \\
  u(x, 0) = u_0(x) \geq 0, \quad x \in \Omega,
\end{array} \right. \\
\end{aligned}
\]

(1.1)

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where $\triangle_p u = \text{div}(\nabla u|^{p-2} \nabla u)$, $p \geq 2$, $\delta > 0$, $0 < q < p - 1$, $\Omega \in \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$.

System (1.1) is the classical reaction-diffusion system of Fujita type for $p = 2$. If $p \neq 2$, (1.1) appears in the theory of non-Newtonian fluids [4, 21] and in nonlinear filtration theory [12]. In the non-Newtonian fluids theory, the quantity $p$ is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

In (1.1), $u = u(x, t) = u(x, t, \delta)$ stands for the dimensionless temperature of a conductor when an electric current flows through it. The parameter $\delta$ is positive and represents mainly, in the physical problem, the square of the electric current or of the potential difference in the electric circuit, depending also upon the “size” of the conductor [13, 16, 22]. By the general theory of parabolic equations it is known that for any sufficiently regular $u_0$, say, in $C^\alpha(\overline{\Omega}) \cap C_0(\overline{\Omega})$, (1.1) admits a unique maximal solution $u$ in $\overline{\Omega} \times [0, T)$, $T \leq \infty$, positive in $\Omega \times [0, T)$ which blows up in finite time, i.e.,

$$\sup_{\Omega} u(\cdot, t) \to \infty \text{ as } t \to T,$$

when $T$ is finite.

Problem (1.1) has a simple appearance but rich mathematical structure. The motivation for studying this kind of problem is that, for all $q > 0$, (1.1) has important implications for a variety of physical situations and technological processes, which arises, for example, in thermistors (see [1–3, 13] and the references therein), fuse wires, electric arcs and fluorescent lights, in the analytical study of phenomena associated with the occurrence of shear bands in metals being deformed at high strain rates (see [5–7] and the references therein), in modelling Ohmic heating phenomena [16, 17], in the theory of gravitational equilibrium of polytropic stars [15], in the investigation of the fully turbulent behavior of flows, using invariant measures for the Euler equation [9], and in modelling aggregation of cells via interaction with a chemical substance (chemotaxis) see [23]. For $p = 2$, the equation arises by reducing the system of two equations

$$u_t = \nabla (k(u) \nabla u) + \sigma(u) |\nabla \phi|^2, \quad (1.2)$$

$$\nabla (\sigma(u) \nabla \phi) = 0, \quad (1.3)$$

to a simple, but still realistic equation. More precisely, (1.2) is a parabolic equation while (1.3) is elliptic, $u$ represents the temperature produced by an electric current flowing through a conductor, $\phi = \phi(x; t)$ is the electric potential, $k(u)$ is the thermal conductivity and $\sigma(u)$ is the electrical conductivity. The conductivity $\sigma$ may be either decreasing or increasing in $u$ depending upon the nature of the conductor.

Some questions concerning the steady problem to (1.2) and (1.3) were investigated by Cimatti [11], see also [1]. A similar problem to (1.2) and (1.3) with radial symmetry, Robin boundary conditions of the form $u_n + \beta u = 0$ and conductivity $\sigma(u) = \exp(-f(u)/\varepsilon)$ was discussed by Fowler et al. [13]. Some numerical results are also given for small $\beta$. See also Howison [14] for how the steady problem to (1.2) and (1.3) may be reduced to a nonlinear ordinary differential equation and Laplace’s
equation. Carrillo [10] has looked at the bifurcation diagram of the nonlocal elliptic problem with decreasing nonlinearity and Dirichlet boundary conditions, in $\Omega \subset \mathbb{R}^N$. See [6] for a similar study where $\Omega$ is the unit ball in $\mathbb{R}^N$. For an extended study of the structure of solutions of the nonlocal elliptic problem see [20].

As in [5, 26], in this paper we pay close attention to the following question: For what values $\delta$ does any maximal solution of (1.1) blow up in finite time? In particular, blow-up of the solution describes the formation of shear bands in materials due to the insufficiency of diffusion compared to the heat generated by high strain rates. Here $u$ stands for the temperature. When $q = 0$ in (1.1), the problem becomes local, and is studied rather thoroughly before.

In [18] the local reaction-diffusion problem

$$\begin{cases}
U_t = \Delta U + \lambda f(U) & x \in \Omega \subset \mathbb{R}^N, \quad N \geq 2, \quad t > 0 \\
U(x, t) = 0, & x \in \partial \Omega, \quad t > 0 \\
U(x, 0) = U_0(x), & x \in \Omega
\end{cases}$$

has been studied, where $\Omega$ is radially symmetric. Under specific conditions on $f$, there exists a $\lambda^*$ such that for each $0 < \lambda < \lambda^*$ there corresponds a unique steady-state solution and $U = U(x, t, \lambda)$ is a global in time-bounded solution, which tends to the unique steady-state solution as $t \to \infty$. Whereas for $\lambda \geq \lambda^*$ there is no steady-state and if $\lambda > \lambda^*$, then $U$ blows up globally.

The work of this paper is motivated by some recent results. The mathematical theory of the nonlocal problem (1.1) when $p = 2$ was initiated and studied in [5–7, 26]. The following results are known: Consider the stationary-state of the following problem with $0 < q < 1$

$$\begin{cases}
0 = \delta e^u - \left(\int_{\Omega} e^u\right)^q, & x \in \Omega \\
u(x, t) = 0, & x \in \partial \Omega.
\end{cases} \tag{1.4}
$$

Let

$$\delta(q) = \inf\{\delta' : (1.4) \text{ has no solutions for all } \delta \geq \delta'\} \quad \tag{1.5}$$

and then

(i) $\delta(q)$ is finite for any strictly star-shaped domain, where $\delta(q) \geq \frac{2n|\partial \Omega|}{\sigma|\Omega|^{1-q}}$;

(ii) on any $n$-dimensional bounded domain for $n \geq 1$, if $\delta(q)$ is finite, then for any $\delta > \delta(q)$, all maximal solutions of (1.4) blow up in finite time.

Although there are many similarities between the nonlocal problem and the local problem, full results have not been recovered. This is mainly due to the fact that the comparison principle, which is essential in the study of the local problem, is not available for the present situation. Equations of the type (1.1) have been studied by many authors when $p = 2$ (see, for example, [1–3, 13, 16, 17, 22] and the references therein). To do so, the authors always use the nice properties of $\Delta$, such as maximum principles and comparison
principles and so on. But most of the properties of $\triangle$ cannot be found in $\triangle_p$ due to the lack of linearity and nondegeneracy (see [19, 24, 25]).

In this paper we shall establish some results extending those to degenerate parabolic operators. To formulate them it is necessary to describe a weak notion of a stationary solution, whose motivation will become clear in the proof in Section 2. Similar results can be found in [19]. Specifically, a function $u$ in $W^{1,p}_0(\Omega)$ is called a weak stationary solution of (1.1) if there exists a sequence $\{u_j\}$ in $C^1(\bar{\Omega}) \cap C^0(\Omega)$ satisfying

\[
\begin{cases}
  u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}_0(\Omega), \\
  u_j \to u \quad \text{a.e.}, \\
  e^{u_j} \to e^u \quad \text{in } L^1(\Omega),
\end{cases}
\]

and

\[
\triangle_p u_j + \frac{\delta e^{u_j}}{(\int_\Omega e^{u_j})^q} \to 0 \quad \text{in } L^p(\Omega) \quad \text{as } j \to \infty.
\]

It follows that any weak stationary solution $u$ satisfies

\[
-\int \nabla \phi |\nabla u|^{p-2} \nabla u + \int \delta \frac{e^u \phi}{(\int_\Omega e^u)^q} = 0 \quad \text{for all } \phi \in C^1_0(\Omega).
\]

We also set $\delta(q)$ as in (1.5). Our main results are the following.

**Theorem 1.1.** On any $n$-dimensional bounded domain for $n \geq 1$, if $\delta(q)$ is finite, then all maximal solutions of (1.1), $\delta > \delta(q)$, blow up in finite time.

For the classical parabolic problem, there is an earlier result relating weak steady-state solutions to finite time blow-up in [8]. The next result gives some sufficient conditions on the finiteness of $\delta(q)$. Recall that $\Omega$ is strictly star-shaped if there exists some $\sigma > 0$ such that

\[
x \cdot \nu(x) \geq \sigma \quad \text{on } \partial \Omega,
\]

where $\nu(x)$ is the unit outer normal at $x$.

**Theorem 1.2.** $\delta(q)$ is finite if $\Omega$ is strictly star-shaped in $\mathbb{R}^n$, $n \geq 3$. Moreover,

\[
\delta(q) \geq \left( \frac{pn}{\sigma} |\partial \Omega| \right)^{\frac{1}{p-1}} |\Omega|^q - 1.
\]

This paper is organized as follows. In the next section, we prove that the maximal solution blows up in finite time. In Section 3, we will give the lower bound of $\delta(q)$.

2. **Universal Finite-time Blow-up**

Problem (1.1) has a Liapunov functional called its energy, which is given by

\[
E(u) = \frac{1}{p} \int |\nabla u|^p dx - \frac{\delta}{p-1-q} \left( \int e^u \right)^{p-1-q}.
\]
We have the dissipation relation

\[ E(u(\cdot, t)) + \int_t^{t'} \int u_t^2(x, t) dx dt = E(u(\cdot, t')), \quad t' \leq t. \]

From the above equality, we can see that the energy is decreasing in time.

**Proof of Theorem 1.1.** We shall assume \( u \) is a global solution of (1.1), \( \delta > \bar{\delta}(q) \), and draw a contradiction. We begin with a standard argument. Let \( \xi(t) = \left( \int e^u \right)^q \). We have

\[
\frac{1}{2} \frac{d}{dt} \int u^2 = \int u (\Delta_p u + \frac{\delta}{\xi(t)} e^u) = \int u \Delta_p u + \frac{\delta}{\xi(t)} \int ue^u \\
= -\int |\nabla u|^p + \frac{\delta}{\xi(t)} \int ue^u \\
= -pE(u) - \frac{p\delta}{(p - 1 - q)\xi(t)} \int e^u + \frac{\delta}{\xi(t)} \int ue^u \\
= -pE(u) + \frac{\delta}{\xi(t)} \left( \int ue^u - \frac{p}{p - 1 - q} \int e^u \right) \\
\geq -pE(u) - C_0 + \frac{\delta}{p - 1 - q} \left( \int e^u \right)^{p-1-q}
\]

for some \( C_0 \) depending only on \( \delta, p, q \) and \( |\Omega| \). We fix \( m \in \mathbb{N} \) such that \( m(p-1-q) > p \). Then

\[
\left( \int e^u \right)^{p-1-q} \geq \left( \frac{1}{m!} \right)^{p-1-q} \left( \int u^m \right)^{p-1-q} \\
\geq \left( \frac{1}{m!} \right)^{p-1-q} |\Omega|^{\frac{p-m}{p}} (p-1-q) \left( \int u^p \right)^{m(p-1-q)}.
\]

Hölder’s inequality implies

\[
\int u^p \geq |\Omega|^\frac{2-p}{2} \left( \int u^2 \right)^\frac{p}{2}.
\]

Then (2.1) becomes

\[
\left( \int e^u \right)^{p-1-q} \geq \left( \frac{1}{m!} \right)^{p-1-q} |\Omega|^{\frac{2-p}{2}} (p-1-q) \left( \int u^2 \right)^\frac{m(p-1-q)}{2}.
\]

Using the energy decreasing property, if, at some \( t_0 \geq 0 \),

\[ E(u(\cdot, t_0)) \leq -\frac{C_0}{p}, \]
then
\[
\frac{1}{2} \frac{d}{dt} \int u^2 \geq \frac{\delta}{p-1-q} \left( \frac{1}{m!} \right)^{p-1-q} |\Omega| \frac{m-p(p-1-q)}{2} \left( \int u^2 \right)^{\frac{m(p-1-q)}{2}}
\]
for all \( t \geq t_0 \), so \( \int u^2 \) blows up in finite time. This is a contradiction. This shows that the energy of any global solution is bounded below by \(-C_0/p\).

However, when the energy has a uniform lower bounded for all time, using the energy dissipation relation, we can find a sequence \( \{t_j\} \), \( t_j \to \infty \), such that
\[
\Delta_p u(\cdot, t_j) + \frac{\delta}{\xi(t)} e^{u(\cdot, t_j)} = u_t(\cdot, t_j) \to 0 \quad \text{in} \quad L^p(\Omega).
\]

Multiplying (1.1) with \( u_j \equiv u(\cdot, t_j) \) and then integrating over \( \Omega \), we have
\[
-pE(u_j) - \frac{p\delta}{(p-1-q)\xi(t)} \int e^{u_j} + \frac{\delta}{\xi(t)} \int u_j e^{u_j} = \int u_j u_t(\cdot, t_j).
\]

Note
\[
\delta \int u_j e^{u_j} \leq \frac{p\delta}{p-1-q} \int e^{u_j} + \frac{\delta}{\xi(t)} \left( \int u_j^2 \right)^{\frac{1}{2}} \left( \int u_t(\cdot, t_j)^2 \right)^{\frac{1}{2}}.
\]

Fix \( q', q < q' < 1 \). We can find a constant \( C_1 \) such that
\[
\int u_j^2 \leq C_1 \left( \int e^{u_j} \right)^{2(q'-q)}.
\]
So
\[
\delta \int u_j e^{u_j} \leq pE(u_0) \left( \int e^{u_j} \right)^{q} + \frac{p\delta}{p-1-q} \int e^{u_j} + C_1 \left( \int u_t(\cdot, t_j)^2 \right)^{\frac{1}{2}} \left( \int e^{u_j} \right)^{q'}. \]

As \( u_t(\cdot, t_j) \to 0 \) in \( L^p(\Omega) \), we conclude that
\[
\int u_j e^{u_j} \leq C_2
\]
for some \( C_2 \) independent of \( j \). From the expression for the energy we also get
\[
\int |\nabla u_j|^p \leq C_3
\]
for some \( C_3 \) independent of \( j \). By passing to a subsequence if necessary, we may assume further that
\[
\begin{cases}
  u_j \rightharpoonup u \\
  u_j \to u 
\end{cases}
\quad \text{in} \quad W^{1,p}_0(\Omega), \quad \text{a.e.}
\]
for some $u$. If we can show that

$$e^{u_j} \to e^u \quad \text{in} \quad L^1(\Omega),$$

(2.2)

then $u$ is a weak stationary solution, whose existence is impossible for $\delta > \delta(q)$. This contradiction shows that any maximal solution blows up in finite time.

Now we shall prove (2.2). We observe that for any $\varepsilon > 0$, there exists a large $M$ such that

$$\int_{u_j \geq M} e^{u_j} \leq \frac{1}{M} \int_{u_j \geq M} u_j e^{u_j} \leq \frac{C_2}{M} \leq \varepsilon/2.$$

For any measurable set $E \subset \Omega$,

$$\int_E e^{u_j} \leq e^M |E| + \varepsilon/2.$$

Hence

$$\int_E e^{u_j} < \varepsilon$$

for all $j$ whenever $|E| < \frac{\varepsilon}{2e^M}$. We have shown that $\{e^{u_j}\}$ is uniformly integrable. Since $\{e^{u_j}\}$ also converges to $e^u$ a.e., we conclude from the Vitali convergence theorem that (2.2) holds. The proof is complete. ■

3. Nonexistence of Stationary Solutions

In this section we will prove Theorem 1.2. In [6] it is shown that no stationary solution exists when (1.6) holds for large $\delta$. Here we demonstrate that the same argument can be adapted to weak stationary solutions.

**Proof of Theorem 1.2.** Let $u$ be a weak stationary solution and let $\{u_j\}$ be an “approximation sequence” described in its definition. We set

$$h_j = \Delta_p u_j + \frac{\delta}{\xi_j(t)} e^{u_j}, \quad \xi_j(t) = \left(\int e^{u_j}\right)^q.$$ (3.1)

By performing integrating by parts, we have

$$\int x \cdot \nabla u_j h_j = \int x \cdot \nabla u_j \left(\Delta_p u_j + \frac{\delta}{\xi_j(t)} e^{u_j}\right)$$

$$= -\int \text{div} \left( |\nabla u_j|^{p-2} \nabla u_j (x \cdot \nabla u_j) - x |\nabla u_j|^p - x \frac{\delta (e^{u_j} - 1)}{\xi_j}\right)$$

$$+ \int \frac{p-n}{p} |\nabla u_j|^p - \frac{n \delta}{\xi_j} \int (e^{u_j} - 1)$$

$$= \frac{1}{p} \int_{\partial \Omega} \left| \frac{\partial u_j}{\partial \nu} \right| x \cdot \nu + \frac{p-n}{p} \int |\nabla u_j|^p - \frac{n \delta}{\xi_j} \int (e^{u_j} - 1).$$
From (3.1), we have

$$\int |\nabla u_j|^p = - \int u_j h_j + \frac{\delta}{\xi_j} \int u_j e^{u_j}.$$ 

Therefore,

$$\frac{\sigma}{p} \int_{\partial\Omega} \left| \frac{\partial u_j}{\partial v} \right|^p \leq \int x \cdot \nabla u_j h_j + \frac{n - p}{p} \left( \int u_j h_j - \frac{\delta}{\xi_j} \int u_j e^{u_j} \right) + \frac{n \delta}{\xi_j} \int (e^{u_j} - 1)$$

$$\leq C \| u_j \|_{H^1} \| h_j \|_{L^2} + \frac{n \delta}{\xi_j} \int e^{u_j},$$

where \( C \) depends on \( \text{diam } \Omega \) and \( n \). By integrating (3.1), we have

$$\frac{\delta}{\xi_j} \int e^{u_j} - \int h_j = \int_{\partial\Omega} \left| \frac{\partial u_j}{\partial v} \right|^{p-1} \leq \left( \int_{\partial\Omega} \left| \frac{\partial u_j}{\partial v} \right|^p \right)^{\frac{p-1}{p}} \left| \partial\Omega \right|^{\frac{1}{p}}.$$

Putting this into the above inequality yields

$$\frac{\sigma}{p} \left( \frac{\delta}{\xi_j} \int e^{u_j} - \xi_j \int h_j \right)^p \leq \left| \partial\Omega \right| \left( C \xi_j^p \| u_j \|_{H^1} \| h_j \|_{L^2} + \delta n \xi_j^{p-1} \int e^{u_j} \right).$$

Letting \( j \to \infty \), we conclude

$$\delta^{p-1} \leq \frac{pn |\partial\Omega|}{\sigma} \left( \int e^{u_j} \right)^{(p-1)(q-1)} \leq \frac{pn |\partial\Omega|}{\sigma} |\Omega|^{(p-1)(q-1)},$$

which means that

$$\delta \leq \left( \frac{pn |\partial\Omega|}{\sigma} \right)^{\frac{1}{p-1}} |\Omega|^{q-1}.$$

So we have

$$\overline{\delta}(q) \geq \left( \frac{pn |\partial\Omega|}{\sigma} \right)^{\frac{1}{p-1}} |\Omega|^{q-1}.$$

This completes the proof. \( \blacksquare \)

References


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