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On Similarity Solutions for a Class of Nonlinear Diffusion Equations with Convection[1]

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Abstract

This paper deals with a class of nonlinear N-diffusion equations with convection. The unique self-similar solution of a free boundary problem is constructed utilizing the unique solution of a related singular nonlinear two-point boundary value problem. To obtain the main result, shooting methods and certain integral representations of solutions are used.

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1. Introduction

In this paper we consider a free boundary problem of the form

$$\begin{cases} u_t = x^{1-M}\left[x^{M-1}k(u)|u_x|^{N-1}u_x\right]_x + x^{1-M}[p(u)]_x t^{\frac{M-N-1}{N+1}}, & (x,t)\in D, \\ u\Big|_{x=\varphi(t)} = A, \quad t>0, \\ g\left(t^{\frac{N-M+1}{N+1}}x^{M-1}k(u)|u_x|^{N-1}u_x\right)\Big|_{x=\varphi(t)} = (\varphi^{N+1}(t))', \quad t>0, \\ u\Big|_{t=0} = B > A, \quad x>\varphi(0)=0, \end{cases} \tag{1.1}$$

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in which $D := \{(x,t) : x > \varphi(t),\ t > 0\}$, and $\varphi(t)$ is unknown a priori and must be determined as part of the solution. Here we make the following hypotheses:

(H_1) $k, p, p' : (0,\infty) \to \mathbb{R}$ are continuous with

$$k(u) > 0, \quad p(u) > 0, \quad p(s) \le p(A) \quad \text{for all} \quad s \in [A,B];$$

(H_2) $M, N \in \mathbb{R}$ with $1 < M \le N+1$;

(H_3) $g : (0,\beta_0] \to [0,\infty)$ is continuous and strictly decreasing, where $\beta_0 > L(B-A)$ for some suitable $L > 0$,

$$\lim_{\beta\to 0^+} g(\beta) \ge \left((N+1)\frac{p(A)-p(B)}{B-A}\right)^{\frac{N+1}{M}} \quad \text{and} \quad g(\beta_0) = 0.$$

The generalized diffusion equation mentioned above is of considerable interest in mathematical physics. In some special cases it can be used to model physical situations in fields involving convection-diffusion processes. The free boundary problem describes certain phase-change processes as does a common Stefan problem. Since a model for certain generalized diffusion processes was suggested by Philip [6], many people devoted their research to diffusion equations which are similar to the first equation in (1.1), with different initial or boundary conditions [1–3, 5]. For example, Wang Junyu and others presented many results about the free boundary problem of certain diffusion equations [7–9]. This paper is inspired by what they have done.

2. Conversion Process

We look for a self-similar solution of the form

$$(\varphi(t), u(x,t)) = (\xi_A t^{1/(N+1)}, f(\eta)), \quad \eta = \frac{x}{t^{1/(N+1)}}.$$

After some calculations we arrive at the following free boundary problem for an ordinary differential equation:

$$\begin{cases} \left[\eta^{M-1}k(f(\eta))|f'(\eta)|^{N-1}f'(\eta)\right]' = -\dfrac{1}{N+1}\eta^M f'(\eta) - p'(f(\eta))f'(\eta), \quad \eta > \xi_A, \\ f(\xi_A) = A, \\ g\left(\eta^{M-1}k(f(\eta))|f'(\eta)|^{N-1}f'(\eta)\right)\Big|_{\eta=\xi_A} = \xi_A{}^{N+1}, \\ f(\infty) = B. \end{cases} \tag{2.1}$$

Suppose $s = f(\eta)$ is strictly increasing in η. Then there exists an inverse function $\eta = z(s)$ satisfying

$$f'(\eta) = \frac{1}{z'(s)}. \quad \text{Let} \quad w := k\frac{z^{M-1}}{(z')^N}.$$

Then (2.1) is converted to a singular nonlinear two-point boundary value problem of the form

$$\begin{cases} w'(s) = -\dfrac{1}{N+1} z^M(s) - p'(s), \quad s \in [A, B), \\ z'(s) = \left(\dfrac{k(s)}{w(s)}\right)^{\frac{1}{N}} z^{\frac{M-1}{N}}(s), \quad s \in [A, B), \\ z(A) = [g(w(A))]^{\frac{1}{N+1}}, \quad w(B) = 0. \end{cases} \tag{2.2}$$

3. Two-point Boundary Value Problem

In order to establish the existence and uniqueness of the solution of the singular nonlinear two-point boundary value problem (2.2), in this section we consider a more general two-point boundary value problem of the form

$$\begin{cases} w'(s) = -\dfrac{1}{N+1} z^M(s) - p'(s), \quad s \in [A, B), \\ z'(s) = y(s, w(s)) z^{\frac{M-1}{N}}(s), \quad s \in [A, B), \\ z(A) = [g(w(A))]^{\frac{1}{N+1}}, \quad w(B) = 0, \end{cases} \tag{3.1}$$

where (H_1)–(H_3) are all satisfied and the following additional hypothesis is adopted:

(H_4) $y : [A, B) \times (0, \infty) \to (0, \infty)$ is continuous, and strictly decreasing and Lipschitz-continuous in the second variable.

Obviously, the two-point boundary value problem (2.2) is a particular case of the problem (3.1).

Lemma 3.1. Assume (H_1)–(H_4). For each fixed $S \in [A, B]$, $W > 0$ and $Z \geq 0$, the initial value problem

$$\begin{cases} w'(s) = -\dfrac{1}{N+1} z^M(s) - p'(s), \quad s \in [A, B), \\ z'(s) = y(s, w(s)) z^{\frac{M-1}{N}}(s), \quad s \in [A, B), \\ w\big|_{s=S} = W, \quad z\,|_{s=S} = Z \end{cases}$$

has a unique solution (w, z), which can be represented by

$$w(s) = W + p(S) - p(s) - \frac{1}{N+1} \int_S^s z^M(t)\,dt, \tag{3.2}$$

$$z(s) = \begin{cases} \left[Z^{\frac{N-M+1}{N}} + \dfrac{N-M+1}{N} \displaystyle\int_S^s y(t, w(t))\,dt \right]^{\frac{N}{N-M+1}} \\ \qquad \text{if} \quad M < N+1, \\ Z \exp\left(\displaystyle\int_S^s y(t, w(t))\,dt \right) \\ \qquad \text{if} \quad M = N+1 \end{cases} \tag{3.3}$$

and depends continuously on S, W, Z. If the maximal interval of existence of the solution is denoted by (S_1, S_2), then either $S_1 = A$ or $\lim_{s \to S_1^+} w(s) = 0$, and either $S_2 = B$ or $\lim_{s \to S_2^-} w(s) = 0$.

Proof. Since the proof is similar to the proof of [8, Lemma 2.1], we omit it here. ■

Lemma 3.2. Assume (H_1)–(H_4). For $\beta > 0$, let $(w(\cdot;\beta), z(\cdot;\beta))$ be the unique solution of the initial value problem

$$\begin{cases} w'(s) = -\dfrac{1}{N+1} z^M(s) - p'(s), \quad s \in [A, B), \\ z'(s) = y(s, w(s)) z^{\frac{M-1}{N}}(s), \quad s \in [A, B), \\ w\big|_{s=A} = \beta, \quad z\big|_{s=A} = (g(\beta))^{\frac{1}{N+1}}. \end{cases}$$

If $\beta_1 > \beta_2 > 0$, then

$$w(s;\beta_1) > w(s;\beta_2) \quad \text{and} \quad z(s;\beta_1) < z(s;\beta_2) \quad \text{for all} \quad s \in I_{\beta_2},$$

where I_{β_2} is the maximal interval of existence of the solution $(w(\cdot;\beta_2), z(\cdot;\beta_2))$.

Proof. Let $\beta_1 > \beta_2 > 0$. Assume that the first assertion is not true. Since

$$w(A;\beta_1) = \beta_1 > \beta_2 = w(A;\beta_2),$$

there exists $S \in I_{\beta_2}$ such that

$$w(s;\beta_1) > w(s;\beta_2) \quad \text{for all} \quad s \in [A, S) \quad \text{but} \quad w(S;\beta_1) = w(S;\beta_2).$$

Now note that for all $s \in [A, S)$ we have by (3.3) that

$$z(s;\beta) = \begin{cases} \left[(g(\beta))^{\frac{N-M+1}{N(N+1)}} + \dfrac{N-M+1}{N} \displaystyle\int_A^s y(t, w(t;\beta))dt \right]^{\frac{N}{N-M+1}} \\ \qquad \text{if} \quad M < N+1, \\ (g(\beta))^{\frac{1}{N+1}} \exp\left(\displaystyle\int_A^s y(t, w(t;\beta))dt \right) \\ \qquad \text{if} \quad M = N+1. \end{cases} \tag{3.4}$$

Thus, due to (H_2)–(H_4),

$$z(s;\beta_1) < z(s;\beta_2) \quad \text{for all} \quad s \in [A, S).$$

From (3.2) we get

$$\begin{aligned}0 < \beta_1 - \beta_2 &= w(S;\beta_1) + \frac{1}{N+1}\int_A^S z^M(s;\beta_1)ds - w(S;\beta_2)\\ &\quad -\frac{1}{N+1}\int_A^S z^M(s;\beta_2)ds\\ &= \frac{1}{N+1}\int_A^S \left[z^M(s;\beta_1) - z^M(s;\beta_2)\right]ds \le 0,\end{aligned}$$

which is a contradiction and hence proves the first assertion. The second assertion follows from (H_2)–(H_4) and (3.4). ■

Lemma 3.3. Assume (H_1)–(H_4). Then (3.1) has a positive solution.

Proof. Define the set $E := \{\beta > 0 : w(B;\beta) > 0\}$. By ($H_3$), there exists a number β_0 such that $g(\beta_0) = 0$. Then $\beta_0 \in E$, i.e., $E \neq \emptyset$, since, according to (3.2) and (3.3), for all $s \in [A, B]$, when $M < N + 1$,

$$\begin{aligned}w(s;\beta_0) &= \beta_0 + p(A) - p(s)\\ &\quad -\frac{1}{N+1}\int_A^s \left[\frac{N-M+1}{N}\int_A^t y(r, w(r;\beta_0))dr\right]^{\frac{MN}{N-M+1}} dt\\ &> \beta_0 + p(A) - p(s) - L(B-A) > 0,\end{aligned}$$

where L is some suitable positive number as y is positive and continuous in $[A, B) \times (0, \infty)$, and when $M = N+1$, then $z(s;\beta_0) \equiv 0$ and $w(s;\beta_0) = p(A) - p(s) + \beta_0 > 0$. Now we claim that $\beta^* := \inf E > 0$. If not so, then $\beta^* = 0$ and hence $w(B; 0) \ge 0$. Moreover, because of (H_3), we get that for all $s \in [A, B]$,

$$z^M(s; 0) > (N+1)\frac{p(A) - p(B)}{B - A}.$$

Hence it follows by (3.2) that

$$0 \le w(B; 0) = p(A) - p(B) - \frac{1}{N+1}\int_A^B z^M(s; 0)ds < 0,$$

which is impossible.

We prove that the solution $(w(\cdot;\beta^*), z(\cdot;\beta^*))$ is a positive solution of the two-point boundary value problem (3.1). Clearly, it is enough to show that $w(B;\beta^*) = 0$. If $w(B;\beta^*) > 0$, then there will be a number $\beta \in (0, \beta^*)$ such that $w(B;\beta) = w(B;\beta^*)/2$, by Lemma 3.1 and Lemma 3.2. I.e., $\beta \in E$, which contradicts the definition of β^*. The proof is complete. ■

Lemma 3.4. Assume (H_1)–(H_4). Let both (w_1, z_1) and (w_2, z_2) be solutions of the first two equations in (3.1) defined on $[a, b] \subset [A, B]$. If

$$w_1(a) = w_2(a) \quad \text{and} \quad w_1(b) = w_2(b),$$

then $w_1 = w_2$ on $[a, b]$.

Proof. Assume that the statement is not true. Then we may assume without loss of generality $w_1(s) < w_2(s)$ for all $s \in (a, b)$. Thus $w_1'(a) \le w_2'(a)$. Now it follows from the first equation in (3.1) that $z_1(a) \ge z_2(a)$, and thus, by (H_2)–(H_4) and (3.3),

$$\begin{aligned} z_1(s) \quad &= \begin{cases} \left[(z_1(a))^{\frac{N-M+1}{N}} + \dfrac{N-M+1}{N}\displaystyle\int_a^s y(t, w_1(t))dt\right]^{\frac{N}{N-M+1}} \\ \qquad \text{if} \quad M < N+1, \\ z_1(a)\exp\left(\displaystyle\int_a^s y(t, w_1(t))dt\right) \\ \qquad \text{if} \quad M = N+1 \end{cases} \\ &\ge \begin{cases} \left[(z_2(a))^{\frac{N-M+1}{N}} + \dfrac{N-M+1}{N}\displaystyle\int_a^s y(t, w_2(t))dt\right]^{\frac{N}{N-M+1}} \\ \qquad \text{if} \quad M < N+1, \\ z_2(a)\exp\left(\displaystyle\int_a^s y(t, w_2(t))dt\right) \\ \qquad \text{if} \quad M = N+1 \end{cases} \\ &= \quad z_2(s) \end{aligned}$$

for all $s \in [a, b]$. Therefore, by (3.2),

$$0 > w_1(s) - w_2(s) = \frac{1}{N+1}\int_s^b \left[z_1^M(t) - z_2^M(t)\right] dt \ge 0, \quad s \in (a, b),$$

which is a contradiction and hence finishes the proof. ■

Now we can summarize the above results in the following statement.

Theorem 3.5. Assume (H_1)–(H_4). Then (3.1) has a unique positive solution.

Proof. The existence is shown in Lemma 3.3. Now assume (w_1, z_1) and (w_2, z_2) are solutions of (3.1). Then $w_1(B) = 0 = w_2(B)$ and $z_1(A) = z_2(A)$. Hence $w_1(A) = w_2(A)$ due to (H_3). By Lemma 3.4, $w_1 = w_2$. Thus $w_1' = w_2'$ and therefore $z_1 = z_2$. ■

Theorem 3.6. In addition to (H_1)–(H_3), suppose

$$y(s, w) = \left(\frac{k(s)}{w}\right)^{\frac{1}{N}}.$$

Then

$$z(B^-)\begin{cases} = \infty & \text{if} \quad N \le 1 \\ < \infty & \text{if} \quad N > 1. \end{cases}$$

Proof. Suppose first $N \le 1$. Assume $z(B^-) < \infty$. Then, by (3.1), $w'(B^-)$ is finite. Thus there exists $\theta > 0$ such that $|w'(s)| < \theta$ for all $s \in [A, B)$. Then

$$\begin{aligned} w(s) \le |w(s)| = |w(B) - w(s)| = \left| \int_s^B w'(r)dr \right| \le \int_s^B |w'(r)|dr \\ \le \int_s^B \theta dr = \theta(B - s) \end{aligned}$$

for all $s \in [A, B]$. Therefore, by (3.3),

$$z(B^-) = \begin{cases} \left[(g(w(A)))^{\frac{N-M+1}{N(N+1)}} + \dfrac{N-M+1}{N} \displaystyle\int_A^B \left(\frac{k(t)}{w(t)} \right)^{\frac{1}{N}} dt \right]^{\frac{N}{N-M+1}} \\ \qquad \text{if } M < N + 1, \\ ((g(w(A)))^{\frac{1}{N}} \exp \left(\displaystyle\int_A^B \left(\frac{k(t)}{w(t)} \right)^{\frac{1}{N}} dt \right) \\ \qquad \text{if } M = N + 1 \end{cases}$$

is finite and thus

$$\begin{aligned} \infty \quad &> \quad \int_A^B \left(\frac{k(t)}{w(t)} \right)^{\frac{1}{N}} dt \ge \int_A^B \left(\frac{k(t)}{\theta(B-t)} \right)^{\frac{1}{N}} dt \ge \frac{1}{\theta} \int_A^B \frac{k(t)}{B-t} dt \\ &\ge \quad \frac{1}{\theta} \int_A^B \frac{L_1}{B-t} dt = \frac{L_1}{\theta} \int_A^B \frac{dt}{B-t} = \infty, \end{aligned}$$

where $0 < L_1 \le k(t) \le L_2$ for all $t \in [A, B]$ due to (H_1), a contradiction.

Now suppose $N > 1$. Assume $z(B^-) = \infty$. Then, by (3.1), $w'(B^-) = \infty$. Thus there exists $S \in [A, B)$ such that $w'(s) < -1$ for all $s \in [S, B)$. Then

$$w(s) = w(s) - w(B) = -\int_s^B w'(r)dr > B - s$$

for all $s \in [S, B]$ and therefore by (H_1)

$$\int_S^s \left(\frac{k(r)}{w(r)} \right)^{\frac{1}{N}} dr \le \int_S^s \left(\frac{k(r)}{B-r} \right)^{\frac{1}{N}} dr < \infty \quad \text{for all} \quad s \in [S, B],$$

which implies $z(B^-) < \infty$, a contradiction. ∎

4. Free Boundary Problem

In this section we construct a self-similar solution of the free boundary problem (1.1), utilizing the unique positive solution of the two-point boundary value problem (2.2).

By a solution of the free boundary problem (1.1), we mean a pair (φ, u) satisfying the following conditions:

(a) $\varphi : [0, \infty) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) = 0$;

(b) $u : \overline{D} \setminus \{(0, 0)\} \to \mathbb{R}$ and $Lu : D \to \mathbb{R}$ are continuously differentiable, where

$$Lu(x) := x^{M-1}k(u)|u_x|^{N-1}u_x;$$

(c) the pair (φ, u) satisfies (1.1).

Similarly, we call the pair (ξ_A, f) a solution of the free boundary problem (2.1), if it satisfies the following conditions:

(a) $\xi_A > 0$;

(b) $f : [\xi_A, \infty) \to \mathbb{R}$ is increasing and continuously differentiable;

(c) $Mf : [\xi_A, \infty) \to \mathbb{R}$ is continuously differentiable, where

$$Mf(\eta) := \eta^{M-1}k(f(\eta))|f'(\eta)|^{N-1}f'(\eta);$$

(d) the pair (ξ_A, f) satisfies (2.1).

Theorem 3.5 asserts that under the hypotheses (H_1)–(H_4), the two-point boundary value problem (2.2) has a unique positive solution (w, z), in which z is strictly increasing. Consequently, the function $s = f(\eta)$ inverse to $\eta = z(s)$ exists in (ξ_A, ξ_B), where $\xi_A := z(A)$ and $\xi_B := z(B^-)$. Theorem 3.6 tells us that $\xi_B = \infty$ when $N \leq 1$ and $\xi_B < \infty$ when $N > 1$. When $N > 1$, it is stipulated that $f(\eta) = B$ for all $\eta \geq \xi_B$. Clearly,

$$f(\xi_A) = A, \quad \lim_{\eta \to \xi_B} f(\eta) = B, \quad \lim_{\eta \to \xi_B} f'(\eta) = 0$$

since for all $\eta \in [\xi_A, \xi_B)$,

$$\eta = z(f(\eta)), \quad f'(\eta) = \frac{1}{z'(s)} > 0.$$

This shows that $f : [\xi_A, \infty)$ is continuously differentiable and strictly increasing on $[\xi_A, \xi_B]$. Next we prove that the pair (ξ_A, f) is a solution of the free boundary problem (2.1). Substituting $s = f(\eta)$ into (2.2) yields

$$w'(f(\eta)) = -p'(f(\eta)) - \frac{1}{N+1}\eta^M, \quad \eta \in [\xi_A, \xi_B), \tag{4.1}$$

$$w(f(\eta)) = \eta^{M-1}k(f(\eta))|f'(\eta)|^{N-1}f'(\eta), \quad \eta \in [\xi_A, \xi_B), \tag{4.2}$$

and hence

$$\begin{aligned}\left(\eta^{M-1}k(f(\eta))|f'(\eta)|^{N-1}f'(\eta)\right)' &= w'(f(\eta))f'(\eta)\\ &= \left(-p'(f(\eta)) - \frac{1}{N+1}\eta^M\right)f'(\eta) \qquad (4.3)\end{aligned}$$

for all $\eta \in [\xi_A, \xi_B)$. When $N > 1$, the above equations read all $0 = 0$ for all $\eta \geq \xi_B$. The equations (4.2) and (4.3) show that $Mf : [\xi_A, \infty) \to \mathbb{R}$ is continuously differentiable. From the last condition in (2.2), (4.1) and (4.2), we have

$$g\left(\eta^{M-1}k(f(\eta))|f'(\eta)|^{N-1}f'(\eta)\right)\big|_{\eta=\xi_A} = {\xi_A}^{N+1}.$$

To sum up, the pair (ξ_A, f) is a solution of the free boundary problem (2.1). Finally, let us define the pair (φ, u) by

$$\varphi(t) = \xi_A t^{1/(N+1)}, \quad u(x,t)) = f\left(\frac{x}{t^{1/(N+1)}}\right),$$

where (ξ_A, f) is a solution of the free boundary problem (2.1). It is easy to verify that the pair (φ, u) is a self-similar solution of the free boundary problem (1.1).

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