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On a Class of Nondensely Defined Hermitian Contractions

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Abstract

We give a functional characterization of a nondensely defined Hermitian contractive operator A and its self-adjoint extension $\hat{A}$ with the same norm which are unitarily equivalent to their linear-fractional transformations. Special attention is paid to the case when the orthogonal complement of the domain of A is one-dimensional. An example of such an operator A is considered.

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1. Introduction

In this article we consider contractive operators which are defined on a proper subspaces of a Hilbert space and which have the Hermitian property. In addition we assume that any such operator is unitarily equivalent to its linear-fractional transformation (the precise meaning of this property is given in Definition 3.1). We call such operators nondensely defined invariant Hermitian contractions.

M. G. Kreĭn proved [18] that any nondensely defined Hermitian contraction A admits a self-adjoint extension $\hat{A}$ with the same norm. In our article [6] it was proved that any invariant Hermitian contraction A admits a self-adjoint extension $\hat{A}$ with the same norm which is also unitarily equivalent to its linear-fractional transformation (an invariant extension). In particular, it was proved that the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ are invariant.

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In this article we give a functional characterization of an invariant pair (a nondensely defined Hermitian contraction A and its self-adjoint extension $\hat{A}$). This characterization is given in terms of the $\mathcal{N}$-resolvent of $\hat{A}$, the restriction of the resolvent of $\hat{A}$ on the orthogonal complement of the domain of A. The corresponding result is formulated in Theorem 3.6. In Theorem 3.11 we parametrize all invariant self-adjoint extensions $\hat{A}$ in terms of solutions of a Riccati equation. As a consequence, we obtain, that if the orthogonal complement to the domain of A is one-dimensional, then extreme extensions are only invariant self-adjoint extensions of A.

In Section 4 we consider the case when the orthogonal complement to the domain of A is one-dimensional, and the operator A is invariant with respect to a continuous group of linear-fractional transformations. In this case it is possible to solve a functional equation for the $\mathcal{N}$-resolvent of the invariant extension $\hat{A}$. It turns out there exists a one-parametric family of nondensely defined Hermitian contractions which are invariant with respect to a continuous group of linear-fractional transformations. Any nondensely defined Hermitian contraction with this property is unitarily equivalent to a member of this family.

In Section 5 we consider an example of a nondensely defined invariant Hermitian contraction. This example gives a universal model of the Hermitian contraction considered in Section 4.

2. Preliminaries

Let $\mathfrak{D} = \overline{\mathfrak{D}}$ be a proper subspace of a Hilbert space $\mathfrak{H}$, and let A be an operator defined on $\mathfrak{D}$ which possesses the following properties

1. $(Ah_1, h_2) = (h_1, Ah_2)$, $h_1, h_2 \in \mathfrak{D}$, Hermitian property;

2. $\|Ah\| \leq \|h\|$, $h \in \mathfrak{D}$.

Then, the operator A is called a nondensely defined Hermitian contractive operator, or just a nondensely defined Hermitian contraction. For a nondensely defined Hermitian contraction A we define $\Delta(A)$ as the set of all self-adjoint operators $\hat{A}$ which are extensions of A and have the same norm, that is,

$$\Delta(A) = \{\hat{A} : \hat{A}f = Af,\ f \in \mathfrak{D},\ \hat{A}^* = \hat{A},\ \|\hat{A}\| = \|A\|\}.$$

In [18] M. G. Kreĭn proved that $\Delta(A) \neq \emptyset$. Moreover in [18] it was also proved that the set $\Delta(A)$ contains the minimal element $\hat{A}_\mu$ and the maximal element $\hat{A}_M$.

A description of the set $\Delta(A)$ was originally obtained by M. G. Kreĭn [18] and is also presented in [1]. The article [19] among other important and interesting results contains a description of the resolvents of operators $\hat{A} \in \Delta(A)$. Other proofs of such type of results as well as further generalizations can be found in [2, 3, 12, 21]. The last two articles also contain extensive lists of references.

In the form that we use in the present article, the description of the set $\Delta(A)$ was obtained or can be easily extracted from results of articles [4, 8, 17, 23].

Let $\mathfrak{N}$ be the orthogonal complement of $\mathfrak{D}$ in $\mathfrak{H}$, $\mathfrak{N} = \mathfrak{H}\ominus\mathfrak{D}$, let $P_{\mathfrak{D}}$ be the orthogonal projection onto $\mathfrak{D}$, and let $P_{\mathfrak{N}} = I - P_{\mathfrak{D}}$ be the orthogonal projection onto $\mathfrak{N}$. Define the operators B and C by

$$B = P_{\mathfrak{D}}A, \quad C = P_{\mathfrak{N}}A.$$

The operator C maps $\mathfrak{D}$ into $\mathfrak{N}$, and B is a self-adjoint operator on $\mathfrak{D}$, $B^* = B$.

Using these notations, the operator $A : \mathfrak{D} \mapsto \mathfrak{H}$ can be represented as a block operator matrix

$$A = \begin{bmatrix} B \\ C \end{bmatrix},$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{D} \oplus \mathfrak{N}$, and any operator $\hat{A} \in \Delta(A)$ can be represented as a block operator matrix

$$\hat{A} = \begin{bmatrix} B & C^* \\ C & \mathcal{E} \end{bmatrix}, \tag{2.1}$$

where $\mathcal{E} : \mathfrak{N} \mapsto \mathfrak{N}$ satisfies $\mathcal{E}^* = \mathcal{E}$.

The condition $\|A\| \leq 1$ implies that for any $f \in \mathfrak{D}$ we have

$$((I_{\mathfrak{D}} - B^2)f, f) \geq (C^*Cf, f).$$

The last inequality means that there exists an operator $X : \mathfrak{D} \to \mathfrak{N}$ such that $\|X\| \leq 1$ and

$$C = X(I_{\mathfrak{D}} - B^2)^{1/2}.$$

Initially the operator X is defined on $\overline{\mathfrak{R}(I_{\mathfrak{D}} - B^2)}$ (the closure of the range of $I_{\mathfrak{D}} - B^2$) and then defined as zero operator on $\mathfrak{D} \ominus \mathfrak{R}(I_{\mathfrak{D}} - B^2)$. In such a way the operator X is defined uniquely. The operator $\mathcal{E}$ is given by the following formulas (see above mentioned references):

$$\mathcal{E} = O + R^{1/2}ZR^{1/2},$$

where

$$O = -XBX^*, \quad R = I_{\mathfrak{N}} - XX^*, \tag{2.2}$$

$I_{\mathfrak{D}}$ and $I_{\mathfrak{N}}$ are identity operators in $\mathfrak{D}$ and $\mathfrak{N}$ respectively, and Z is an arbitrary self-adjoint contraction ($Z = Z^*$, $\|Z\| \leq 1$) on $\mathfrak{N}$. In particular, the set $\Delta(A)$ contains only one element if and only if $R = 0$, or $XX^* = I_{\mathfrak{N}}$, that is, if and only if the operator X is a coisometry.

We will need the following theorem of M. G. Kreĭn [18, Theorem 4] that gives a characterization of the extensions $\hat{A}_\mu$ and $\hat{A}_M$.

Theorem 2.1. Let $\hat{A} \in \Delta(A)$ and let

$$\hat{A}f = \int_{-1}^{1} \lambda dE(\lambda)f$$

be its spectral decomposition. Then in order that $\hat{A} = \hat{A}_\mu$ (respectively $\hat{A} = \hat{A}_M$), it is necessary and sufficient that the integral

$$J(\varphi, \hat{A}) = \int_{-1}^{1} \frac{d\|E(\lambda)\varphi\|^2}{1+\lambda} \quad \text{respectively} \quad J(\varphi, -\hat{A}) = \int_{-1}^{1} \frac{d\|E(\lambda)\varphi\|^2}{1-\lambda}$$

is equal to ∞ for any $\varphi \in \mathfrak{N}$. The operator $\hat{A}$ is the unique self-adjoint extension of A with norm ≤ 1 if and only if simultaneously

$$J(\varphi, \hat{A}) = J(\varphi, -\hat{A}) = \infty.$$

Remark 2.2. The theorem is cited exactly as it is formulated in [18]. Of course, it is assumed that $\varphi \neq 0$.

Let $\hat{A} \in \Delta(A)$. Denote by $E(\lambda)$ the resolution of identity of the operator $\hat{A}$. $E(\lambda) = 0$ for $\lambda < 0$, and $E(\lambda) = I$ for $\lambda > 1$. We normalize $E(\lambda)$ in such a way that $E(\lambda) = (E(\lambda - 0) + E(\lambda + 0))/2$. We also denote by $\sigma(\lambda)$ a nondecreasing function, whose values are operators in $\mathfrak{N}$ defined as

$$\sigma(\lambda) = P_{\mathfrak{N}} E(\lambda)|\mathfrak{N}.$$

The function $\sigma(\lambda)$ also satisfies $\sigma(\lambda) = (\sigma(\lambda - 0) + \sigma(\lambda + 0))/2$ and $\sigma(1 + 0) = I_{\mathfrak{N}}$.

The $\mathfrak{N}$-resolvent $\mathcal{N}(z)$ of the operator $\hat{A}$ is defined by the expression

$$\mathcal{N}(z) = P_{\mathfrak{N}} R(z)|\mathfrak{N}, \tag{2.3}$$

where $R(z) = (\hat{A} - zI)^{-1}$ is the resolvent of the operator $\hat{A}$. The function $\mathcal{N}(z)$ is analytic on $\mathbb{C} \setminus (-1, 1)$. Using the spectral representation of $\hat{A}$, one can rewrite (2.3) in the form

$$\mathcal{N}(z) = \int_{-1}^{1} \frac{d\sigma(\lambda)}{\lambda - z}. \tag{2.4}$$

The function $\mathcal{N}$ has the following properties:

1. $\mathcal{N}(\bar{z}) = [\mathcal{N}(z)]^*$;
2. $\frac{1}{2i}\{\mathcal{N}(z) - [\mathcal{N}(z)]^*\} \geq 0$, $\Im z > 0$;
3. $\lim_{z\to\infty} \mathcal{N}(z) = 0$;
4. $\lim_{z\to\infty} z\mathcal{N}(z) = -I_{\mathfrak{N}}$.

(The last property follows from the fact that $\sigma(1 + 0) = I_{\mathfrak{N}}$). Properties 1 and 2 above imply the function $\mathcal{N}(z)$ is of Herglotz–Nevanlinna class.

Recall that a nondensely defined Hermitian contraction A on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}$ is said to be *simple* if $\mathfrak{D}$ does not contain any (nonzero) subspace that is invariant with respect to A. Since A is a bounded operator, without loss of generality it is possible to assume that such invariant subspace is closed.

In the remainder of this paper we always assume that all nondensely defined Hermitian contractions are simple unless the opposite is stated explicitly.

The $\mathfrak{N}$-resolvent $\mathcal{N}(z)$ defines the pair $(A, \hat{A})$ uniquely up to unitary equivalence.

The next theorem is obvious.

Theorem 2.3. Let A and A' be nondensely defined Hermitian contractions in Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$ with domains $\mathfrak{D} \subset \mathfrak{H}$ and $\mathfrak{D}' \subset \mathfrak{H}'$ respectively, and suppose that $\hat{A} \in \Delta(A)$, $\hat{A}' \in \Delta(A')$. Let $U : \mathfrak{H} \to \mathfrak{H}'$ be a unitary operator such that $U\mathfrak{D} = \mathfrak{D}'$, and

$$UA = A'U, \quad U\hat{A} = \hat{A}'U.$$

Then there exists a unitary operator $U_0 : \mathfrak{N} \to \mathfrak{N}'$ such that

$$U_0\mathcal{N}(z) = \mathcal{N}(z)U_0.$$

Theorem 2.4. Let $\mathcal{N}(z)$ be a function of the Herglotz–Nevanlinna class with values in the set of bounded operators on some Hilbert space $\mathfrak{N}$, which admits the integral representation (2.4), where $\int_{-1}^{1} d\sigma(\lambda) = I_{\mathfrak{N}}$. Then there exist a Hilbert space $\mathfrak{H}$ which contains $\mathfrak{N}$ as a proper subspace, a simple nondensely defined Hermitian contraction A, and its contractive self-adjoint extension $\hat{A}$ such that $\mathfrak{D} = \mathfrak{H} \ominus \mathfrak{N}$ is the domain of A, and the $\mathfrak{N}$-resolvent of $\hat{A}$ coincides with $\mathcal{N}$. The pair $(A, \hat{A})$ is defined uniquely up to unitary equivalence.

Theorem 2.4 is well known and, in fact, is a simplified version of one of the results from [19]. It immediately follows from the theorem of M. A. Najmark [1].

Let $\hat{A} \in \Delta(A)$ and $z \in \rho(\hat{A}) \cap \rho(\hat{A}_\mu)$, in particular, $z \notin \Omega$. With respect to the decomposition $\mathfrak{H} = \mathfrak{D} \oplus \mathfrak{N}$ we have

$$(\hat{A} - zI) - (\hat{A}_\mu - zI) = \begin{bmatrix} 0 & 0 \\ 0 & \Delta\mathcal{E} \end{bmatrix},$$

where $\Delta\mathcal{E} = \mathcal{E} - \mathcal{E}_\mu$ is a self-adjoint operator on $\mathfrak{N}$ which satisfies $0 \le \Delta\mathcal{E} \le 2R$ (see formula (2.2)), and $\mathcal{E}$ is the right bottom element of the block representation (2.1) of the operator $\hat{A}$. From the last expression we obtain that

$$R_{\hat{A}}(z) = \left[I + R_\mu(z) \begin{bmatrix} 0 & 0 \\ 0 & \Delta\mathcal{E} \end{bmatrix} \right]^{-1} R_\mu(z),$$

where R_μ is the resolvent of the operator $\hat{A}_\mu$. Representing $R_{\hat{A}}(z)$ and $R_\mu(z)$ as block operator matrices, we have

$$\mathcal{N}(z) = \left[I_{\mathfrak{N}} + \mathcal{N}_\mu(z)\Delta\mathcal{E} \right]^{-1} \mathcal{N}_\mu(z), \tag{2.5}$$

where $\mathcal{N}_\mu$ is the $\mathfrak{N}$-resolvent of $\hat{A}_\mu$. Along with (2.5) we have

$$\mathcal{N}(z) = \mathcal{N}_\mu(z)\left[I_{\mathfrak{N}} + \Delta\mathcal{E}\mathcal{N}_\mu(z)\right]^{-1}.$$

A general description of the resolvents of nondensely defined Hermitian contractions was obtained in [19].

3. Invariant Contractions

Fix a number $t > 0$ and put $\kappa = \tanh t$. Then $0 < \kappa < 1$. Denote by g a linear-fractional transformation of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ onto itself defined as

$$g(z) = \frac{z - \kappa}{1 - \kappa z}, \tag{3.1}$$

and let $G = \{g^n, n = 0, \pm 1, \pm 2, \ldots, \}$ be the group of linear fractional transformations generated by g. Each transformation g^n from G is of the form

$$g^n : z \mapsto \frac{z - \kappa_n}{1 - \kappa_n z},$$

where

$$\kappa_n = \tanh nt, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Let U be a unitary operator on a Hilbert space $\mathfrak{H}$ and put $U_n = U^n$, $n = 0, \pm 1, \pm 2, \ldots$.

Definition 3.1. Let A be a nondensely defined Hermitian contraction on a Hilbert space $\mathfrak{H}$. The operator A is said to be (g, U)-invariant (or just invariant) if

$$U^n A U^{*n} = g^n(A) = (A - \kappa_n I_{\mathfrak{D}})(I_{\mathfrak{D}} - \kappa_n A)^{-1}, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Denote by $\mathfrak{M}_z$, $z \in \mathbb{C}$, the range of the operator $A - zI_{\mathfrak{D}}$. For $z \notin (-1, 1)$, the set $\mathfrak{M}_z$ is a closed subspace of $\mathfrak{H}$.

Definition 3.1 is understood in the following sense: The unitary operator U^n maps the subspace $\mathfrak{D}$ onto $\mathfrak{M}_{1/\kappa_n}$ and $\mathfrak{M}_0$ onto $\mathfrak{M}_{\kappa_n}$. In other words, for any $h \in \mathfrak{D}$ there exists $h' \in \mathfrak{D}$ such that

$$U^n h = h' - \kappa_n A h'$$

and

$$U^n A h = A h' - \kappa_n h'.$$

Definition 3.2. An operator $\hat{A} \in \Delta(A)$ is called (g, U)-invariant if

$$U^n \hat{A} U^{*n} = g^n(\hat{A}), \quad n = 0, \pm 1, \pm 2, \ldots. \tag{3.2}$$

Remark 3.3. For any $\hat{A} \in \Delta(A)$ the expression

$$g^n(\hat{A}) = (\hat{A} - \kappa_n I)(I - \kappa_n \hat{A})^{-1}$$

is a well defined self-adjoint contractive operator.

Definition 3.4. Let A be a nondensely defined Hermitian contraction, and let $\hat{A} \in \Delta(A)$. The pair $(A, \hat{A})$ is said to be (g, U)-invariant if both A and $\hat{A}$ are (g, U)-invariant.

The following theorem was proved in [6].

Theorem 3.5. Let A be a nondensely defined (g, U)-invariant Hermitian contraction. Then it admits a (g, U)-invariant self-adjoint contractive extension $\hat{A} \in \Delta(A)$. In particular, the extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ are (g, U)-invariant.

Our next goal is to give a functional characterization of a (g, U)-invariant pair $(A, \hat{A})$. From (3.2) it follows that for all integers n

$$U^n \int_{-1}^{1} \lambda dE(\lambda) U^{*n} = \int_{-1}^{1} \frac{\lambda - \kappa_n}{1 - \kappa_n \lambda} dE(\lambda) = \int_{-1}^{1} g^n(\lambda) dE(\lambda).$$

From this expression we deduce that for any Borel set $\delta \subset [-1, 1]$

$$U^n E(\delta) U^{*n} = E(g^{-n}(\delta)). \tag{3.3}$$

It is also clear that if a resolution of identity $E(\lambda)$ satisfies (3.3), then the corresponding self-adjoint operator $\hat{A}$ is (g, U)-invariant. Therefore we have the following.

A self-adjoint operator $\hat{A} \in \Delta(A)$ is (g, U)-invariant if and only if the resolution of identity $E(\lambda)$ associated with $\hat{A}$ satisfies condition (3.3).

Observe that for a (g, U)-invariant $\hat{A}$ and for $\varphi \in \mathfrak{N}$, the vector $(I - \kappa \hat{A}) U \varphi$ is in $\mathfrak{N}$. Indeed, according to Definition 3.1, the operator U maps the subspace $\mathfrak{D}$ onto subspace $\mathfrak{M}_{1/\kappa} = (A - 1/\kappa I_{\mathfrak{D}})\mathfrak{D}$. Therefore, the operator U maps $\mathfrak{N} = \mathfrak{D}^\perp$ onto $\mathfrak{N}_{1/\kappa} = \mathfrak{M}_{1/\kappa}^\perp$. Thus the vector $U\varphi$ is in $\mathfrak{N}_{1/\kappa}$. Now for any $h \in \mathfrak{D}$ we have

$$(h, (I - \kappa \hat{A}) U \varphi) = ((I - \kappa \hat{A}) h, U \varphi) = ((I_{\mathfrak{D}} - \kappa A), h, U\varphi) = 0,$$

because $\hat{A} \in \Delta(A)$.

Define an operator $\mathcal{B}$ on $\mathfrak{N}$ by the formula

$$\mathcal{B}\varphi = \frac{1}{\sqrt{1 - \kappa^2}} (I - \kappa \hat{A}) U \varphi, \quad \varphi \in \mathfrak{N}. \tag{3.4}$$

It is clear that $\mathcal{B}$ is an invertible operator and

$$\mathcal{B}^{-1}\varphi = \sqrt{1 - \kappa^2} U^* (I - \kappa \hat{A}))^{-1} \varphi, \quad \varphi \in \mathfrak{N}.$$

Theorem 3.6. Let A be a nondensely defined Hermitian contraction defined on a proper subspace $\mathfrak{D} = \overline{\mathfrak{D}}$ of a Hilbert space $\mathfrak{H}$ with $\dim \mathfrak{N} = \dim \mathfrak{D}^\perp < \infty$, and let $\hat{A} \in \Delta(A)$. Then the pair $(A, \hat{A})$ is (g, U)-invariant if and only if there exists an invertible operator

$\mathcal{B}$ on the subspace $\mathfrak{N}$ such that the $\mathfrak{N}$-resolvent $\mathcal{N}(z)$ of the operator $\hat{A}$ satisfies the equation

$$\mathcal{N}(g(z)) - \mathcal{N}(g(\zeta)) = \mathcal{B}^* [\mathcal{N}(z) - \mathcal{N}(\zeta)] \mathcal{B}, \quad z, \zeta \notin (-1, 1). \tag{3.5}$$

Operators $\hat{A}$, U, and $\mathcal{B}$ are related by the formula (3.4).

Proof. Let $(A, \hat{A})$ be a (g, U)-invariant pair. Then from (3.2) (with $n = 1$) it follows that for any $z \notin (-1, 1)$

$$UR(z)U^* = \frac{1}{1+\kappa z} R(g^{-1}(z))(I - \kappa\hat{A}).$$

Therefore, for $\varphi, \psi \in \mathfrak{N}$ we have

$$\begin{aligned}
(\mathcal{N}(z)\varphi, \psi) &= \frac{1}{1+\kappa z}(R(g^{-1}(z))(I - \kappa\hat{A})U\varphi, U\psi) \\
&= \frac{1-\kappa^2}{1+\kappa z}((I - \kappa\hat{A})^{-1}R(g^{-1}(z))\mathcal{B}\varphi, \mathcal{B}\psi) \\
&= -\frac{1-\kappa^2}{\kappa(1+\kappa z)}\left(R\left(\frac{1}{\kappa}\right)R(g^{-1}(z))\mathcal{B}\varphi, \mathcal{B}\psi\right) \\
&= \frac{1-\kappa^2}{\kappa(1+\kappa z)}\frac{1}{\frac{1}{\kappa} - g^{-1}(z)}\left(\left[R(g^{-1}(z)) - R\left(\frac{1}{\kappa}\right)\right]\mathcal{B}\varphi, \mathcal{B}\psi\right) \\
&= (\mathcal{B}^*\mathcal{N}(g^{-1}(z))\mathcal{B}\varphi, \psi) - \left(\mathcal{B}^*\mathcal{N}\left(\frac{1}{\kappa}\right)\mathcal{B}\varphi, \psi\right),
\end{aligned}$$

where (3.4) and the resolvent identity were used. Thus we have proved that

$$\mathcal{N}(z) = \mathcal{B}^*\left[\mathcal{N}(g^{-1}(z)) - \mathcal{N}\left(\frac{1}{\kappa}\right)\right]\mathcal{B} \tag{3.6}$$

from which (3.5) follows. Formula (3.6) can be written in the form

$$\mathcal{N}(g(z)) = \mathcal{B}^*\mathcal{N}(z)\mathcal{B} + T, \tag{3.7}$$

where

$$T = -\mathcal{B}^*\int_{-1}^{1}\frac{d\sigma(\lambda)}{\lambda - 1/\kappa}\mathcal{B} = -\mathcal{B}^*\mathcal{N}\left(\frac{1}{\kappa}\right)\mathcal{B} \tag{3.8}$$

is a self-adjoint operator on $\mathfrak{N}$. Since we assume that $0 < \kappa < 1$, operator T is positive.

Assume now that the $\mathfrak{N}$-resolvent $\mathcal{N}(z)$ of the operator $\hat{A}$ satisfies equation (3.5) or equivalently, equation (3.7). The expression for $\mathcal{N}(g(z))$ can be written in the form

$$\mathcal{N}(g(z)) = (1-\kappa z)\int_{-1}^{1}\frac{1}{(\lambda+\kappa)/(1+\kappa\lambda) - z}\frac{d\sigma(\lambda)}{1+\kappa\lambda}.$$

In the last expression we substitute λ by $g(\lambda)$. Then formula (3.7) takes the form

$$\frac{1}{1-\kappa^2}\int_{-1}^{1}\frac{d\tilde{\sigma}(\lambda)}{\lambda - z} = \int_{-1}^{1}\frac{d\tilde{\tilde{\sigma}}(\lambda)}{\lambda - z},$$

where $d\tilde{\sigma}(\lambda) = d\sigma(g(\lambda))(1-\kappa\lambda)$ and $d\tilde{\tilde{\sigma}}(\lambda) = d[\mathcal{B}^*\sigma(\lambda)\mathcal{B}]/(1-\kappa\lambda)$. Therefore,

$$\mathcal{B}^* d\sigma(\lambda)\mathcal{B} = \frac{(1-\kappa\lambda)^2}{1-\kappa^2} d\sigma(g(\lambda)). \tag{3.9}$$

In the space $L^2([-1,1], d\sigma)$ consider an operator U defined as

$$(Uf)(\lambda) = \frac{\sqrt{1-\kappa^2}}{1-\kappa\lambda}(\mathcal{B}f)(g(\lambda)), \tag{3.10}$$

that is, a weighted composition operator. It is clear that U is an invertible operator, and it is easy to check that U^{-1} is given by the formula

$$(U^{-1}f)(\lambda) = \frac{\sqrt{1-\kappa^2}}{1+\kappa\lambda}(\mathcal{B}^{-1}f)(g^{-1}(\lambda)).$$

Now we show that U is a unitary operator:

$$\begin{aligned}
\|Uf\|^2 &= \int_{-1}^{1}(d\sigma(\lambda)(Uf)(\lambda), (Uf)(\lambda)) \\
&= (1-\kappa^2)\int_{-1}^{1}\frac{(\mathcal{B}^* d\sigma(\lambda)\mathcal{B}f(g(\lambda)), f(g(\lambda)))}{(1-\kappa\lambda)^2} \\
&= \int_{-1}^{1}(d\sigma(g(\lambda))f(g(\lambda)), f(g(\lambda))) = \int_{-1}^{1}(d\sigma(\lambda)f(\lambda), f(\lambda)) = \|f\|^2,
\end{aligned}$$

where (3.9) and (3.10) were used. Thus U is a unitary operator.

Let $\hat{A}_0$ be an operator of multiplication by λ in the space $L^2([-1,1], d\sigma)$. Clearly this operator is bounded and self-adjoint. For $f \in L^2([-1,1], d\sigma)$ we have

$$\begin{aligned}
(U\hat{A}_0U^*f) &= U\left[\lambda\frac{(1-\kappa^2)^{1/2}}{1+\kappa\lambda}(\mathcal{B}^{-1}f)(g^{-1}(\lambda))\right] \\
&= \frac{\lambda-\kappa}{1-\kappa\lambda}f(\lambda) = g(\hat{A}_0)f(\lambda).
\end{aligned}$$

Thus the operator $\hat{A}_0$ is (g, U)-invariant. Let $\mathfrak{D}$ be the subspace of $L^2([-1,1], d\sigma)$ defined as

$$\mathfrak{D} = \left\{ f \in L^2([-1,1], d\sigma) : \int_{-1}^{1} d\sigma(\lambda)f(\lambda) = 0 \right\},$$

and let A_0 be a restriction of operator $\hat{A}_0$ onto $\mathfrak{D}$. It is obvious that A_0 is a non-densely defined Hermitian contraction and $\hat{A}_0 \in \Delta(A_0)$. We need to show that A_0 is (g, U)-invariant. Indeed, for $f \in \mathfrak{D}$, let h be a vector from $L^2([-1, 1], d\sigma)$ defined as $(Uf)(\lambda) = (1 - \kappa\lambda)h(\lambda)$, that is,

$$h(\lambda) = \frac{\sqrt{1-\kappa^2}}{(1-\kappa\lambda)^2}(Bf)(g(\lambda))$$

(see formula (3.10)). For the vector h we have

$$\begin{aligned} \int_{-1}^{1} d\sigma(\lambda)h(\lambda) &= \sqrt{1-\kappa^2}\int_{-1}^{1}\frac{d\sigma(\lambda)\mathcal{B}f(g(\lambda))}{(1-\kappa\lambda)^2} \\ &= \frac{1}{\sqrt{1-\kappa^2}}(\mathcal{B}^*)^{-1}\int_{-1}^{1} d\sigma(g(\lambda))f(g(\lambda)) = 0, \end{aligned}$$

because $\mathcal{B}$ is a continuous operator and (3.9) was used. Therefore, for any $f \in \mathfrak{D}$ the vector Uf is in $(I - \kappa A_0)\mathfrak{D}$.

Conversely, let $h \in (I_{\mathfrak{D}} - \kappa A_0)\mathfrak{D}$, that is, $h(\lambda) = (1 - \kappa\lambda)f(\lambda)$, where $f \in \mathfrak{D}$. We need to show that there exists $\varphi \in \mathfrak{D}$ such that $U\varphi = h$. Put

$$\varphi(\lambda) = \frac{(1-\kappa g^{-1}(\lambda))^2}{\sqrt{1-\kappa^2}}(\mathcal{B}^{-1}f)(g^{-1}(\lambda)).$$

From (3.10) it follows that $(U\varphi)(\lambda) = (1 - \kappa\lambda)f(\lambda)$. Moreover,

$$\begin{aligned} \int_{-1}^{1} \delta\sigma(l)\varphi(\lambda) &= \int_{-1}^{1} d\sigma(\lambda)\frac{(1-\kappa g^{-1}(\lambda))^2}{\sqrt{1-\kappa^2}}(\mathcal{B}^{-1}f)(g^{-1}(\lambda)) \\ &= \int_{-1}^{1} d\sigma(g(\lambda))\frac{(1-\kappa\lambda)^2}{\sqrt{1-\kappa^2}}(\mathcal{B}^{-1}f)(\lambda) \\ &= \sqrt{1-\kappa^2}\mathcal{B}^*\int_{-1}^{1} d\sigma(\lambda)f(\lambda) = 0, \end{aligned}$$

where (3.9) was used. Thus, $U\mathfrak{D} = (I_{\mathfrak{D}} - \kappa A_0)\mathfrak{D}$. Now

$$\begin{aligned} U(A_0 f) &= \frac{\lambda-\kappa}{1-\kappa\lambda}\frac{\sqrt{1-\kappa^2}}{1-\kappa\lambda}\mathcal{B}f(g(\lambda)) \\ &= (\lambda-\kappa)h(\lambda) = (A_0 - \kappa I)h, \end{aligned}$$

which proves that A_0 is (g, U)-invariant. Therefore $(A_0, \hat{A}_0)$ is a (g, U)-invariant pair. Since the $\mathfrak{N}$-resolvent $\mathcal{N}(z)$ of $\hat{A}_0$ is given by $\int d\sigma(\lambda)/(\lambda - z)$, the pair $(A_0, \hat{A}_0)$ is unitarily equivalent to the given pair $(A, \hat{A})$. This completes the proof. ∎

According to Theorem 3.6, to each $\hat{A} \in \Delta(A)$, such that the pair $(A, \hat{A})$ is (g, U)-invariant, there corresponds an operator $\mathcal{B} = \mathcal{B}(\hat{A})$ defined by (3.4), such that the equation (3.5) is fulfilled. From (3.4) it follows that for any such operator B its spectrum $\sigma(\mathcal{B})$ lies in the annulus

$$\sigma(\mathcal{B}) \subset \left\{ z : \sqrt{\frac{1-\kappa}{1+\kappa}} \le |z| \le \sqrt{\frac{1+\kappa}{1-\kappa}} \right\}. \tag{3.11}$$

Let $\mathcal{B}_\mu$ (respectively $\mathcal{B}_M$) be the operator defined by (3.4) with $\hat{A} = \hat{A}_\mu$ (respectively $\hat{A} = \hat{A}_M$).

Theorem 3.7. Let $(A, \hat{A})$ be a (g, U)-invariant pair. Then the spectrum $\sigma(\mathcal{B}_\mu)$ of the operator $\mathcal{B}_\mu$ satisfies the condition

$$\sigma(\mathcal{B}_\mu) \subset \left\{ 1 \le |z| \le \sqrt{\frac{1+\kappa}{1-\kappa}} \right\}, \tag{3.12}$$

while the spectrum $\sigma(\mathcal{B}_M)$ of the operator $\mathcal{B}_M$ satisfies the condition

$$\sigma(\mathcal{B}_M) \subset \left\{ \sqrt{\frac{1-\kappa}{1+\kappa}} \le |z| \le 1 \right\}. \tag{3.13}$$

Proof. The estimate for the upper bound in (3.12) and lower bound in (3.13) follow from (3.11).

In order to prove that for any eigenvalue ρ of $\mathcal{B}_\mu$ the inequality $|\rho| \ge 1$ is fulfilled, we observe at first that from (3.6) it follows that

$$\mathcal{N}(g^n(z)) = \mathcal{B}^{*n}\mathcal{N}(z)\mathcal{B}^n + \sum_{k=0}^{n-1} \mathcal{B}^{*k} T \mathcal{B}^k. \tag{3.14}$$

For $z \ne 1$ the sequence $\{g^n(z)\}$, $n = 0, 1, 2, \dots$ converges to -1 as $n \to \infty$. Let z_0 be a real number, $z_0 < -1$. Then from the monotone convergence theorem it follows that

$$\int_{-1}^{1} \frac{d(\sigma(\lambda) f, f)}{\lambda + 1} = \lim_{n\to\infty} \int_{-1}^{1} \frac{d(\sigma(\lambda) f, f)}{\lambda - g^n(z_0)} = \lim_{n\to\infty} (\mathcal{N}(g^n(z_0)) f, f). \tag{3.15}$$

Applying (3.15) to an eigenvector f of operator B_μ which corresponds to an eigenvalue ρ and taking into account (3.14), we obtain

$$\int_{-1}^{1} \frac{d(\sigma_\mu(\lambda) f, f)}{\lambda + 1} = \lim_{n\to\infty} \left[|\rho|^{2n} (\mathcal{N}_\mu(z_0) f, f) + (T_\mu f, f) \sum_{k=0}^{n-1} |\rho|^{2k} \right].$$

According to Theorem 2.1, the integral in the left-hand side diverges, which is possible if and only if $|\rho| \ge 1$.

Similar arguments show that any eigenvalue of the operator $\mathcal{B}_M$ is in absolute value greater than or equal to one if we observe that for $z \neq -1$ the sequence $\{g^{-n}(z)\}$ converges to 1. ■

If $\dim \mathfrak{N} = 1$, then the operator $\mathcal{B}$ in (3.5) is an operator of multiplication by a complex number. We denote this number also by $\mathcal{B}$. The following statement was proved in [6] by different arguments.

Corollary 3.8. Let A be a nondensely defined (g, U)-invariant Hermitian contraction such that $\dim \mathfrak{D}^{\perp} = 1$. Suppose that $\hat{A}_{\mu} \neq \hat{A}_M$. Then $\hat{A}_{\mu}$ and $\hat{A}_M$ are the only invariant self-adjoint extensions of A with norm ≤ 1.

Proof. Let $\hat{A} \in \Delta(A)$ be an invariant self-adjoint contractive extension of A. Denote by $E(\lambda)$ the resolution of identity of $\hat{A}$, and let $\mathcal{B}$ be a complex number from equation (3.7) for the $\mathcal{N}$-resolvent of $\hat{A}$. For any (nonzero) vector $\varphi \in \mathfrak{N}$ and a point $z_0 < -1$

$$J(\varphi, \hat{A}) = \int_{-1}^{1} \frac{d\|E(\lambda)\varphi\|^2}{1+\lambda} = \lim_{n\to\infty} \left[|\mathcal{B}|^{2n} (\mathcal{N}(z_0)\varphi, \varphi) + (T\varphi, \varphi) \sum_{k=0}^{n-1} |\mathcal{B}|^{2k} \right]. \tag{3.16}$$

Similarly,

$$\begin{aligned} J(\varphi, -\hat{A}) &= \int_{-1}^{1} \frac{d\|E(\lambda)\varphi\|^2}{1-\lambda} \\ &= \lim_{n\to\infty} \left[-|\mathcal{B}|^{-2n} (\mathcal{N}(z_0)\varphi, \varphi) + (T\varphi, \varphi) \sum_{k=0}^{n-1} |\mathcal{B}|^{-2k} \right], \end{aligned} \tag{3.17}$$

where $z_0 > 1$ is arbitrary and, consequently, $(\mathcal{N}(z_0)\varphi, \varphi) < 0$.

From (3.16) it follows that the conditions $|\mathcal{B}| \geq 1$ and $J(\varphi, \hat{A}) = \infty$ are equivalent. From Theorem 2.1 it follows that $\hat{A} = \hat{A}_{\mu}$ if and only if $|\mathcal{B}| \geq 1$. In the same way from (3.17) it follows that conditions $|\mathcal{B}| \leq 1$ and $J(\varphi, -\hat{A}) = \infty$ are equivalent. Therefore according to the same theorem of M. G. Kreĭn (Theorem 2.1), $\hat{A} = \hat{A}_M$ if and only if $|\mathcal{B}| \leq 1$.

Thus, if $\dim \mathfrak{N} = 1$, any invariant extension is either $\hat{A}_{\mu}$ or $\hat{A}_M$. Since $\hat{A}_{\mu}$ and $\hat{A}_M$ are always invariant, it proves the corollary. ■

Remark 3.9. It is necessary to emphasize that the arguments above, in particular, formulas (3.16) and (3.17), are essentially based on the assumption that $\kappa > 0$.

Corollary 3.10. Let A be a nondensely defined (g, U)-invariant Hermitian contraction such that $\dim \mathfrak{D}^{\perp} = 1$. Then the following three conditions are equivalent:

1. $\hat{A}_{\mu} = \hat{A}_M$, that is, the set $\Delta(A)$ contains only one element;

2. $|\mathcal{B}_M| = 1$;

3. $|\mathcal{B}_\mu| = 1$.

Proof. If $\hat{A}_\mu = \hat{A}_M$, then $\mathcal{B}_\mu = \mathcal{B}_M$. According to Theorem 3.7, $1 \le |\mathcal{B}_M| = |\mathcal{B}_\mu| \le 1$. Thus, conditions *2* and *3* are fulfilled.

Assume now that $|\mathcal{B}_M| = 1$. Then $J(\varphi, -\hat{A}_M) = \infty$. From (3.16) with $\mathcal{B}_M$ instead of $\mathcal{B}$ it follows that also $J(\varphi, \hat{A}_M) = \infty$. Theorem 2.1 gives that $\hat{A}_\mu = \hat{A}_M$ and $\mathcal{B}_\mu = \mathcal{B}_M$, that is, from *2* follows *1* and *3*. In case $|\mathcal{B}_\mu| = 1$, the same arguments and equation (3.17) are used.

Let $\hat{A} \in \Delta(A)$ and let $\mathcal{N}(z)$ be the $\mathfrak{N}$-resolvent of $\hat{A}$. Since operator $\hat{A}_\mu$ is (g, U)-invariant, from (2.5) and (3.6) it follows that

$$\begin{aligned}
&\mathcal{N}(g(z)) \qquad (3.18)\\
&= \left\{\mathcal{N}(z)\left[\mathcal{B}_\mu\Delta\mathcal{E} - \Delta\mathcal{E}\mathcal{B}_\mu^{*-1} - \Delta\mathcal{E}\mathcal{B}_\mu^{*-1}T_\mu\Delta\mathcal{E}\right] + \mathcal{B}^{*-1}\left[I_\mathfrak{N} + T_\mu\Delta\mathcal{E}\right]\right\}^{-1} \times\\
&\quad \times \left\{\mathcal{N}(z)\left[\mathcal{B}_\mu - \Delta\mathcal{E}\mathcal{B}_\mu^{*-1}T_\mu\right] + \mathcal{B}_\mu^{*-1}T_\mu\right\}\\
&= \left\{\left[I_\mathfrak{N} + T_\mu\Delta\mathcal{E}\right]^{-1} B_\mu^*\mathcal{N}(z)\left[\mathcal{B}_\mu\Delta\mathcal{E} - \Delta\mathcal{E}\mathcal{B}_\mu^{*-1} - \Delta\mathcal{E}B_\mu^{*-1}T_\mu\Delta\mathcal{E}\right] + I_\mathfrak{N}\right\}^{-1} \times\\
&\quad \times \left\{\left[I_\mathfrak{N} + T_\mu\Delta\mathcal{E}\right]^{-1} \mathcal{B}_\mu^*\mathcal{N}(z)\left[\mathcal{B}_\mu - \Delta\mathcal{E}\mathcal{B}_\mu^{*-1}T_\mu\right] + \left[I_\mathfrak{N} + T_\mu\Delta\mathcal{E}\right]^{-1} T_\mu\right\}.
\end{aligned}$$

Similarly, using (2.5) and (3.6) we obtain that

$$\begin{aligned}
\mathcal{N}(g(z)) &= \left\{\left[\mathcal{B}_\mu^* - \mathcal{B}_\mu^{-1}\Delta\mathcal{E}\right]\mathcal{N}(z) + T_\mu\mathcal{B}_\mu^{-1}\right\} \times \qquad (3.19)\\
&\quad \times \left\{\left[\mathcal{B}_\mu^{-1}\Delta\mathcal{E} + \Delta\mathcal{E}\mathcal{B}_\mu^* - \Delta\mathcal{E}T_\mu\mathcal{B}_\mu^{-1}\Delta\mathcal{E}\right]\mathcal{N}(z) + \left[I_\mathfrak{N} + \Delta\mathcal{E}T_\mu\right]\mathcal{B}_\mu^{-1}\right\}^{-1}\\
&= \left\{\left[\mathcal{B}_\mu^* - B_\mu^{-1}\Delta\mathcal{E}\right]\mathcal{N}(z)\mathcal{B}_\mu\left[I_\mathfrak{N} + \Delta\mathcal{E}T_\mu\right]^{-1} + T_\mu\left[I_\mathfrak{N} + \Delta\mathcal{E}T_\mu\right]^{-1}\right\} \times\\
&\quad \times \left\{\left[\mathcal{B}_\mu^{-1}\Delta\mathcal{E} + \Delta\mathcal{E}\mathcal{B}_\mu^* - \Delta\mathcal{E}T_\mu\mathcal{B}_\mu^{-1}\Delta\mathcal{E}\right]\mathcal{N}(z)B_\mu\left[I_\mathfrak{N} + \Delta\mathcal{E}T_\mu\right]^{-1} + I_\mathfrak{N}\right\}^{-1}.
\end{aligned}$$

From (3.6) it follows that $T_\mu = \mathcal{N}(-1/\kappa)$ (since $g(\infty) = -1/\kappa$ and $\mathcal{N}(\infty) = 0$). Now (2.5) and (3.6) give that $[I_\mathfrak{N} + T_\mu\Delta\mathcal{E}]^{-1}T_\mu = T_\mu[I_\mathfrak{N} + \Delta\mathcal{E}T_\mu]^{-1} = \mathcal{N}(-1/\kappa)$.

For any extension $\hat{A}$ (invariant or not), formula (3.4) can be written as

$$\begin{aligned}
\mathcal{B} &= \frac{1}{\sqrt{1-\kappa^2}}(I - \kappa\hat{A})U|\mathfrak{N}\\
&= \frac{1}{\sqrt{1-\kappa^2}}\left[I - P_\mathfrak{N}\Delta\mathcal{E}P_\mathfrak{N}\kappa(I - \kappa\hat{A}_\mu)^{-1})\right](I - \kappa\hat{A}_\mu)U|\mathfrak{N}\\
&= \left[I_\mathfrak{N} + \Delta\mathcal{E}\mathcal{N}\left(\frac{1}{\kappa}\right)\right]\mathcal{B}_\mu = \mathcal{B}_\mu - \Delta\mathcal{E}B_\mu^{*-1}T_\mu,
\end{aligned}$$

where (3.8) was used. ∎

Theorem 3.11. Suppose that $\hat{A} \in \Delta(A)$. Then operator $\hat{A}$ is (g, U)-invariant if and only if the operator $\mathcal{E}$ from the block representation (2.1) of $\hat{A}$ satisfies the equation

$$\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E} + \Delta\mathcal{E}\mathcal{B}_{\mu}^{*} - \Delta\mathcal{E}T_{\mu}\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E} = 0, \tag{3.20}$$

where $\Delta\mathcal{E} = \mathcal{E} - \mathcal{E}_{\mu}$.

Proof. Suppose that $\hat{A} \in \Delta(A)$ is (g, U)-invariant. Then by virtue of (3.2) along with the formula (3.4), we have

$$\mathcal{B} = \sqrt{1-\kappa^2}\,U(I + \kappa\hat{A})^{-1}|\mathfrak{N}.$$

From this formula, we easily deduce that

$$\mathcal{B} = \mathcal{B}_{\mu}\left[I_{\mathfrak{N}} + \Delta\mathcal{E}\mathcal{N}_{\mu}\left(-\frac{1}{\kappa}\right)\right]^{-1} = \mathcal{B}_{\mu}\left[I_{\mathfrak{N}} + \Delta\mathcal{E}T_{\mu}\right]^{-1}.$$

Therefore, formula (3.19) takes the form

$$\begin{aligned}\mathcal{N}(g(z)) \;=\; & \left[\mathcal{B}^{*}\mathcal{N}(z)\mathcal{B} + T\right] \times \\ & \times\left\{\left[\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E} + \Delta\mathcal{E}\mathcal{B}_{\mu}^{*} - \Delta\mathcal{E}T_{\mu}\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E}\right]\mathcal{N}(z)\mathcal{B} + I_{\mathfrak{N}}\right\}^{-1}.\end{aligned}$$

Since the operator $\hat{A}$ is assumed to be invariant, (3.6) is fulfilled. It means that the "denominator" in the last expression is equal to $I_{\mathfrak{N}}$ identically. Since $\mathcal{B}$ and $\mathcal{N}(z)$ are invertible operators, we deduce that

$$\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E} + \Delta\mathcal{E}\mathcal{B}_{\mu}^{*} - \Delta\mathcal{E}T_{\mu}\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E} = 0.$$

Suppose now that $\hat{A} \in \Delta(A)$ and suppose that $\mathcal{E}$ satisfies (3.20). Then the "denominator" in (3.19) does not depend on z, and (3.19) takes the form

$$\mathcal{N}(g(z)) = \mathcal{B}^{*}\mathcal{N}(z)\mathcal{B}_{\mu}\left[I_{\mathfrak{N}} + \Delta\mathcal{E}T_{\mu}\right]^{-1} + \mathcal{N}\left(-\frac{1}{\kappa}\right).$$

The same condition (3.20) makes the "denominator" of (3.18) independent of z, and (3.18) takes the form

$$\mathcal{N}(g(z)) = \left[I_{\mathfrak{N}} + T_{\mu}\Delta\mathcal{E}\right]^{-1}\mathcal{B}_{\mu}^{*}\mathcal{N}(z)\mathcal{B} + \mathcal{N}\left(-\frac{1}{\kappa}\right).$$

The equality $\mathcal{B}^{*} = \mathcal{B}_{\mu}^{*} - T_{\mu}\mathcal{B}_{\mu}^{-1}\Delta\mathcal{E} = \left[I_{\mathfrak{N}} + T_{\mu}\Delta\mathcal{E}\right]^{-1}\mathcal{B}_{\mu}^{*}$ follows from (3.20). Therefore the value of $\mathcal{N}(g(z))$ is expressed through the value of $\mathcal{N}(z)$ according to the formula (3.6). From Theorem 3.6 it follows that $\hat{A}$ is (g, U)-invariant.

Remark 3.12. From our consideration it follows that (3.20) can be written in the form

$$\mathcal{B}_\mu^{-1}\Delta\mathcal{E} + \Delta\mathcal{E}\mathcal{B}^* = 0.$$

In particular, for $\dim\mathfrak{N} = 1$, and $\hat{A}_\mu \neq \hat{A}_M$ (that is, $\mathcal{E}_M - \mathcal{E}_\mu \neq 0$) we obtain

$$\mathcal{B}_\mu^{-1} + \mathcal{B}_M^* = 0. \tag{3.21}$$

Taking into account Theorem 3.7 and Corollary 3.10, one deduces that the following three conditions are equivalent:

1. $\hat{A}_\mu \neq \hat{A}_M$;
2. $|\mathcal{B}_M| < 1$;
3. $|\mathcal{B}_\mu| > 1$.

As before the inequalities above depend on the assumption that $\kappa > 0$. For $\kappa < 0$ we would have opposite inequality signs.

4. Case $\dim\mathfrak{N} = 1$

Let A be a (g, U)-invariant nondensely defined Hermitian contraction with $\dim\mathfrak{N} = 1$ and such that $\hat{A} \in \Delta(A)$ is also (g, U)-invariant. Therefore $\hat{A}$ is either $\hat{A}_\mu$ or $\hat{A}_M$. Then equation (3.7) takes the form

$$\mathcal{N}(g(z)) = |B|^2\mathcal{N}(z) + T, \qquad z \notin (-1, 1) \tag{4.1}$$

where $T \in \mathbb{R}$, $T > 0$. As was pointed out above (Corollary 3.10) $|B| \neq 1$ if and only if $\hat{A}_\mu \neq \hat{A}_M$. If this condition is fulfilled, then from Theorem 3.7 it follows that $|B_\mu| > 1$, while $|B_M| < 1$.

Assume now that there is a continuous group of unitary operators U_t, $-\infty < t < \infty$ such that for every $t \in \mathbb{R}$ the operator A is (g_t, U_t)-invariant, where g_t is a transformation of the form (3.1) and κ depends on t, namely $\kappa(t) = \tanh t$.

From $\mathcal{N}(g_{t_1}(g_{t_2}(z))) = \mathcal{N}(g_{t_1+t_2}(z))$ and (4.1) we have $|B_{t_1+t_2}|^2 = |B_{t_1}|^2|B_{t_2}|^2$. Therefore, there exists $p \in \mathbb{R}$ such that

$$|B_t|^2 = e^{pt}.$$

Theorem 4.1. Let A be a nondensely defined Hermitian contraction with $\dim\mathfrak{N} = 1$ which is (g_t, U_t)-invariant for all $t \in \mathbb{R}$.

1. If $\hat{A}_\mu = \hat{A}_M$, then

$$\mathcal{N}_\mu(z) = \mathcal{N}_M(z) = \frac{1}{2}\log\frac{z-1}{z+1}, \quad z \notin (-1, 1); \tag{4.2}$$

2. If $\hat{A}_\mu \neq \hat{A}_M$, then there exists a real number ν, $0 < \nu < 1$, such that

$$\mathcal{N}_\mu(z) = \frac{1}{2\nu}\left[\left(\frac{z-1}{z+1}\right)^\nu - 1\right], \quad z \notin (-1, 1), \tag{4.3}$$

and

$$\mathcal{N}_M(z) = -\frac{1}{2\nu}\left[\left(\frac{z-1}{z+1}\right)^{-\nu} - 1\right], \quad z \notin (-1, 1). \tag{4.4}$$

Remark 4.2. The functions on the right-hand sides of equations (4.2)–(4.4) are holomorphic and single-valued in the whole complex plane with cut along the real axis from -1 to 1. It is also clear that these functions belong to the Herglotz–Nevanlinna class.

Proof of Theorem 4.1. Let the operator A satisfy the condition of the theorem. Then equation (4.1) takes the form

$$\mathcal{N}(g_t(z)) = e^{pt}\mathcal{N}(z) + T_t, \tag{4.5}$$

where $-\infty < t < \infty$, and p is a real number. From Corollary 3.10, it follows that conditions $p = 0$ (that is, $|\mathcal{B}| = 1$) and $\hat{A}_\mu = \hat{A}_M$ are equivalent. From Theorem 3.7 and Remark 3.12, it follows that if $\hat{A}_\mu \neq \hat{A}_M$, then $p > 0$ (that is, $|\mathcal{B}_\mu| > 1$ for $t > 0$) if $\mathcal{N} = \mathcal{N}_\mu$, and $p < 0$ ($|\mathcal{B}_M| < 1$ for $t > 0$) if $\mathcal{N} = \mathcal{N}_M$.

We differentiate both sides of (4.5) at first with respect to z, then with respect to t, and put $t = 0$. The result is

$$\frac{d}{dz}\left[(z^2-1)\mathcal{N}'(z)\right] = p\mathcal{N}'(z).$$

Taking into account that $\mathcal{N}(z) \to 0$ as $z \to \infty$, from the last equation we obtain that $\mathcal{N}(z)$ is of the form

$$\mathcal{N}(z) = C_1 \log\frac{z-1}{z+1}, \quad p = 0, \quad z \notin (-1, 1),$$

or

$$\mathcal{N}(z) = C_2\left[\left(\frac{z-1}{z+1}\right)^\nu - 1\right] \quad \nu = p/2 \neq 0, \quad z \notin (-1, 1). \tag{4.6}$$

The constants C_1 and C_2 are obtained from the condition $\lim_{z\to\infty}(z\mathcal{N}(z)) = -1$. Thus $C_1 = 1/2$ and $C_2 = 1/(2\nu)$. The condition $0 < |\nu| < 1$ follows from the fact that $\mathcal{N}(z)$ has nonnegative imaginary part in the upper half-plane. For a fixed value of t, from (4.6) we obtain

$$|\mathcal{B}|^2 = \left(\frac{1+\kappa}{1-k}\right)^\nu = e^{2\nu t}.$$

Since $|B_\mu| > 1$ for $\hat{A}_\mu \neq \hat{A}_M$ and $0 < \kappa < 1$ (Theorem 3.7, Remark 3.12), we conclude that in this case $\nu > 0$ for $\mathcal{N}_\mu$. Finally, the statement that for $\mathcal{N}_\mu$ and $\mathcal{N}_M$ the values of the parameter ν are of opposite signs follows from (3.21). This completes the proof. ■

Remark 4.3. From Theorem 3.6 and Corollary 3.10, it follows that the converse of Theorem 4.1 is also valid. Namely, if A is a nondensely defined Hermitian contraction with $\dim\mathfrak{N} = 1$, and if $\hat{A} \in \Delta(A)$ such that the $\mathfrak{N}$-resolvent of $\hat{A}$ is given by (4.2), or (4.3), or (4.4), then the pair $(A, \hat{A})$ is (g_t, U_t)-invariant for all $t \in \mathbb{R}$. Moreover, $\hat{A} = \hat{A}_\mu = \hat{A}_M$ if the $\mathfrak{N}$-resolvent of $\hat{A}$ is given by (4.2), $\hat{A} = \hat{A}_\mu$ if the $\mathfrak{N}$-resolvent of $\hat{A}$ is given by (4.3), $\hat{A} = \hat{A}_M$ if the $\mathfrak{N}$-resolvent of $\hat{A}$ is given by (4.4).

It is easily seen that the function $\mathcal{N}(z)$ defined by (4.2) is representable in the form (2.4) with

$$d\sigma(\lambda) = d\lambda/2, \quad -1 < \lambda < 1,$$

and the functions defined by (4.3) and (4.4) are also representable in such form with

$$d\sigma_\mu(\lambda) = \frac{\sin(\pi\nu)}{2\pi\nu}\left(\frac{1-\lambda}{1+\lambda}\right)^\nu d\lambda, \quad -1 < \lambda < 1,$$

and

$$d\sigma_M(\lambda) = \frac{\sin(\pi\nu)}{2\pi\nu}\left(\frac{1-\lambda}{1+\lambda}\right)^{-\nu} d\lambda, \quad -1 < \lambda < 1,$$

respectively.

Recall that the operator $\hat{A}$ (respectively $\hat{A}_\mu$, $\hat{A}_M$) is an operator of multiplication by λ in the space $L^2((-1, 1), d\sigma)$ (respectively, $L^2((-1, 1), d\sigma_\mu)$, $L^2((-1, 1), d\sigma_M)$). Direct calculations show now that the operator $\mathcal{E}$ in the representation (2.1) is equal to zero, if the $\mathfrak{N}$-resolvent of $\hat{A}$ is given by formula (4.2), $\mathcal{E} = -\nu$, if the $\mathfrak{N}$-resolvent is given by (4.3), and $\mathcal{E} = \nu$, if the $\mathfrak{N}$-resolvent is given by (4.4).

5. Example

Let $\mathfrak{H} = L^2(0, \infty)$ and let the operator $\mathcal{H}_0$ be defined by the differential expression

$$\mathcal{H}_0 f = -\frac{d^2 f}{dx^2} + \frac{\nu^2 - 1/4}{x^2} f \tag{5.1}$$

for smooth functions f which have a compact support within $(0, \infty)$. It is well known that for $\nu \geq 1$ the operator $\mathcal{H}_0$ is essentially self-adjoint, and for $0 \leq \nu < 1$ operator $\mathcal{H}$, the closure of $\mathcal{H}_0$, is a prime positive symmetric operator with index of defect $(1, 1)$ (see [22, 24]). In what follows we consider the case $0 \leq \nu < 1$.

According to M. G. Kreĭn [18], with any positive densely defined closed symmetric operator $\mathcal{H}$ on a Hilbert space $\mathfrak{H}$ one can associate a nondensely defined Hermitian contraction A in the following way: The domain $\mathfrak{D}$ of A is the set of all vectors $h \in \mathfrak{H}$ representable in the form

$$h = f + \mathcal{H} f, \quad f \in \mathfrak{D}(\mathcal{H}), \tag{5.2}$$

and

$$Ah = f - \mathcal{H}f, \tag{5.3}$$

where $\mathfrak{D}(\mathcal{H})$ is the domain of $\mathcal{H}$. The expression for A can be written in the form

$$A = (I - \mathcal{H})(I + \mathcal{H})^{-1}. \tag{5.4}$$

The operator $\mathcal{H}$ is recovered from A by the formula

$$\mathcal{H} = (I - A)(I + A)^{-1}.$$

Because $\mathcal{H}$ is not self-adjoint and closed, set $\mathfrak{D} = \overline{\mathfrak{D}} \neq \mathfrak{H}$. The dimension of its orthogonal complement $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}$ is equal to the defect number of $\mathcal{H}$.

Any element $\hat{A} \in \Delta(A)$ defines a positive self-adjoint extension H of $\mathcal{H}$ according to the formula

$$H = (I - \hat{A})(I + \hat{A})^{-1}. \tag{5.5}$$

The extreme extensions $\hat{A}_\mu$ and $\hat{A}_M$ correspond to the Friedrichs extension H_F and the Kreĭn extension H_K respectively.

Define a unitary operator U_t, $t \in \mathbb{R}$, on $\mathfrak{H}$ by the formula

$$(U_t f)(x) = e^{-t/2} f(e^{-t}x), \quad x > 0.$$

It is easily seen that the operators U_t, $t \in \mathbb{R}$, form a strongly continuous group. The operator $\mathcal{H}_0$ satisfies the relation

$$U_t \mathcal{H}_0 U_t^* = e^{2t} \mathcal{H}_0$$

and, therefore, the operator $\mathcal{H}$ satisfies the same relation. This property follows from the following general statement.

Lemma 5.1. Let $\mathcal{H}_0$ be a symmetric (not necessarily closed) operator on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\mathcal{H}_0)$ and let $\mathcal{H}$ be the closure of $\mathcal{H}_0$. Suppose that a unitary operator U and the operator $\mathcal{H}_0$ satisfy the following conditions:

$$U\mathfrak{D}(\mathcal{H}_0) = \mathfrak{D}(\mathcal{H}_0); \tag{5.6}$$

$$U\mathcal{H}_0 = a\mathcal{H}_0 U, \quad a \in \mathbb{R}, \quad a \neq 0. \tag{5.7}$$

Then the operator $\mathcal{H}$ satisfies the same conditions.

Proof. Suppose that $f \in \mathfrak{D}(\mathcal{H})$. This means (see, for example, [1]) that there exists a sequence $\{f_n\}_{n=1}^{\infty}$, such that $f_n \in \mathfrak{D}(\mathcal{H}_0)$, $f_n \to f$, and $\mathcal{H} f_n (= \mathcal{H}_0 f_n) \to \mathcal{H} f$.

From conditions (5.6) and (5.7) it follows that $Uf_n \in \mathfrak{D}(\mathcal{H})$ and $Uf = \lim\limits_{n\to\infty} Uf_n$ (since U is a continuous operator). Also

$$a^{-1} U\mathcal{H} f = a^{-1} \lim_{n\to\infty} U\mathcal{H} f_n = \lim_{n\to\infty} \mathcal{H} U f_n.$$

Therefore, $\lim \mathcal{H}Uf_n$ exists and $U\mathcal{H}f = a\mathcal{H}Uf$. In order to complete the proof, one observes that (5.7) can be written in the form $aU^*\mathcal{H}_0 = \mathcal{H}_0U^*$ and repeats the same arguments. ■

Lemma 5.2. Let $\mathcal{H}$ be a densely defined positive symmetric operator on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\mathcal{H})$. Let U be a unitary operator on $\mathfrak{H}$ such that conditions (5.6) and (5.7) hold with $a > 0$, $a \neq 1$. Let A be a nondensely defined Hermitian contraction defined by formulas (5.2) and (5.3). Then A is a (g, U)-invariant with

$$g : z \to g(z) = \frac{z-\kappa}{1-\kappa z}, \quad \kappa = \frac{a-1}{a+1}.$$

Proof. From formulas (5.2) and (5.3) it follows that for $h \in \mathfrak{D}$ and $f \in \mathfrak{D}(\mathcal{H})$

$$f = \frac{1}{2}(h + Ah), \quad \mathcal{H}f = \frac{1}{2}(h - Ah).$$

Now, for a vector h given by (5.2), one gets $Uh = Uf + U\mathcal{H}f = Uf + a\mathcal{H}Uf$. Since $Uf \in \mathfrak{D}(\mathcal{H})$, there exists $h' \in \mathfrak{D}$ such that $Uf = (h'+Ah')/2$ and $\mathcal{H}Uf = (h'-Ah')/2$. Therefore $Uh = (h'+Ah')/2+a(h'-Ah')/2 = (h_1-\kappa Ah_1)$, where $h_1 = (1+a)h'/2$, and $\kappa = (a-1)/(a+1)$. That is $U\mathfrak{D} \subset (I-\kappa A)\mathfrak{D}$.

Conversely, if $g = h - \kappa Ah$, for $h \in \mathfrak{D}$, then $g = (f + \mathcal{H}f) - \kappa(f - \mathcal{H}f)$ with $f \in \mathfrak{D}(\mathcal{H})$. Therefore

$$\begin{aligned} U^*g &= U^*f + U^*\mathcal{H}f - \kappa U^*f + \kappa U^*\mathcal{H}f = (1-\kappa)U^*f + \frac{1+\kappa}{a}\mathcal{H}U^*f \\ &= \frac{2}{1+a}U^*f + \frac{2}{1+a}\mathcal{H}U^*f \in \mathfrak{D}. \end{aligned}$$

Thus $U\mathfrak{D} = (I-\kappa A)\mathfrak{D}$ with κ as in the statement. Similar calculations show that $UAh = (Ah_1 - \kappa h_1)$. This completes the proof. ■

The nondensely defined Hermitian contraction A obtained according to (5.4) with $\mathcal{H}_0$ given by (5.1) and its closure $\mathcal{H}$ is defined by

$$\begin{aligned} (Ah)(x) = -h(x) + 2\sqrt{x}K_\nu(x) \int_0^x \sqrt{\tau}h(\tau)I_\nu(\tau)d\tau \\ -2\sqrt{x}I_\nu(x) \int_0^x \sqrt{\tau}h(\tau)K_\tau d\tau, \end{aligned} \tag{5.8}$$

with the domain

$$\mathfrak{D} = \left\{ h \in L^2(0,\infty) : \int_0^\infty h(\tau)\sqrt{\tau}K_\nu(\tau)d\tau = 0 \right\}, \tag{5.9}$$

where K_ν and I_ν are modified Bessel functions. Formula (5.9) follows from (5.2) and the fact that the orthogonal complement to the domain $\mathfrak{D}$ is the set of those functions $\psi \in L^2(0,\infty)$ that satisfy the differential equation

$$-\frac{d^2\psi}{dx^2} + \frac{\nu^2 - 1/4}{x^2}\psi = -\psi.$$

Theorem 5.3. Let A be a nondensely defined Hermitian contraction defined by (5.2) and (5.9), and let $0 < \nu < 1$. Then $\hat{A}_\mu \neq \hat{A}_M$, and the $\mathcal{N}$-resolvents $\mathcal{N}_\mu(z)$ and $\mathcal{N}_M(z)$ of the operators $\hat{A}_\mu$ and $\hat{A}_M$ are given by formulas (4.3) and (4.4).

Proof. It is possible to perform further calculations directly using formulas (5.2) and (5.9). We use another way based on the notion of the Weyl–Titchmarsh function of a symmetric operator and its self-adjoint extension. Corresponding theorems are formulated in the appendix.

For $0 < \nu < 1$ the domain operator H_F, the Friedrichs extension of the operator $\mathcal{H}$, defined above, was described in many articles (see, for example, [16] and references therein). The domain of this extension consists of all functions $f \in L^2(0, \infty)$ which satisfy the following conditions:

1. f is absolutely continuous;
2. f' is absolutely continuous and $f' \in L^2(0, \infty)$;
3. $-f'' + \dfrac{\nu^2 - 1/4}{x^2} f$ is in $L^2(0, \infty)$;
4. $f(x)/x \in L^2(0, \infty)$;
5. $\lim\limits_{x \to 0^+} f(x) = 0$.

In order to calculate the Weyl–Titchmarsh function $M_F(z)$ of this extension (we use notation $M_F(z)$ for $M_{\mathcal{H},H_F}(z)$), it is necessary to calculate $R_F(z)\varphi_i$, where φ_i is the defect vector of $\mathcal{H}$, $\varphi_i \in \mathfrak{N}_i$, and $R_F(z)$ is the resolvent of the operator H_F. In the process of calculations, we use the formulas for integrals of products of Bessel functions [5, Formulas (9) and (10) from Section 7.14], power series representation of Bessel functions [5, Section 7.2] and formulas for asymptotic behavior of Bessel functions (especially, [5, Formulas (1) and (2) from Section 7.13]). It is also assumed that for a complex number z, its argument satisfies $0 \leq \arg z < 2\pi$.

The defect vector φ_i is the solution of equation

$$-\varphi_i''(x) + \frac{\nu^2 - 1/4}{x^2} \varphi_i = i\varphi_i$$

which belongs to $L^2(0, \infty)$. It is given by

$$\varphi_i(x) = \sqrt{x} H_\nu^{(1)}(\sqrt{i}x),$$

where $H_\nu^{(1)}(\zeta)$ is the Hankel function of the first kind of order ν. In particular,

$$\|\varphi_i\|^2 = \frac{1}{\pi \cos(\pi\nu/2)}.$$

If $R_F(z)\varphi_i = f$, then the function f is in the domain of the operator H_F and satisfies the differential equation

$$-f''(x) + \frac{\nu^2 - 1/4}{x^2} f(x) - zf(x) = \varphi_i(x).$$

The function f from $L^2(0, \infty)$ which satisfies the last equation is given by the formula

$$f(x) = \frac{A\sqrt{x}H_\nu^{(1)}(\sqrt{z}x) - \sqrt{x}H_\nu^{(1)}(e^{i\pi/4}x)}{z - i},$$

where A is a constant which depends upon an extension. For the Friedrichs extension H_F and for $\nu > 0$, the conditions above give $A = z^{\nu/2}e^{-i\pi\nu/4}$ and

$$(R_F(z)\varphi_i)(x) = e^{-i\pi\nu/4}\frac{z^{\nu/2}\sqrt{x}H_\nu^{(1)}(\sqrt{z}x) - e^{i\pi\nu/4}\sqrt{x}H_\nu^{(1)}(e^{i\pi/4}x)}{z - i}.$$

According to (6.1), $M_F(z) = z + (1 + z^2)(R_F(z)\varphi_i, \varphi_i)/\|\varphi_i\|^2$. Now straightforward calculations that involve the above mentioned formulas for integrals of products of Bessel functions result in

$$M_F(z) = \cot\frac{\pi\nu}{2} - z^\nu \frac{e^{-i\pi\nu}}{\sin(\pi\nu/2)}. \tag{5.10}$$

Now direct verification shows that the function from the right-hand side of (5.10) can be written in form (6.2) with measure $d\sigma_F(\lambda)$ defined as

$$d\sigma_F(\lambda) = \frac{2}{\pi}\cos\frac{\pi\nu}{2}\frac{\lambda^\nu}{1 + \lambda^2}d\lambda.$$

Therefore, according to Theorem 6.2, the Friedrichs extension H_F of the operator $\mathcal{H}$ is unitarily equivalent to the operator of multiplication by independent variable in the space $L^2((0, \infty), d\sigma_F(\lambda))$: $(H_F g)(\lambda) = \lambda g(\lambda)$. In such representations, the defect subspace $\mathfrak{N}_z = [(\mathcal{H} - \bar{z}I)\mathfrak{D}(\mathcal{H})]^\perp$ is given by $\mathfrak{N}_z = l.h.\{(\lambda - i)/(\lambda - z)\xi : \xi \in \mathbb{C}\}$. In particular, the unit vector φ, $\|\varphi\|^2 = 1$ that generates the subspace $\mathfrak{N}_{-1} = \mathfrak{N}$ is given by

$$\varphi = \sqrt{\frac{1}{\nu}\sin\left(\frac{\pi\nu}{2}\right)}\frac{\lambda - i}{\lambda + 1}, \quad 0 < \nu < 1.$$

Therefore

$$d(E_F(\lambda)\varphi, \varphi) = \frac{1}{\pi\nu}\sin(\pi\nu)\frac{\lambda^\nu}{(1 + \lambda)^2},$$

where $E_F(\lambda)$ is the resolution of identity of the operator H_F.

Since according to (5.5) the $\mathfrak{N}$-resolvent $\mathcal{N}_\mu(z)$ of the operator $\hat{A}_\mu$, the minimal self-adjoint contractive extension of operator (5.2), can be written as

$$((I - H_F)(I + H_F)^{-1} - zI)^{-1}\varphi, \varphi) = \frac{\sin\pi\nu}{\pi\nu}\int_0^\infty \left(\frac{1 - \lambda}{1 + \lambda} - z\right)^{-1}\frac{\lambda^\nu}{1 + \lambda^2}d\lambda,$$

the previous formula after substituting λ by $(1-\lambda)/(1+\lambda)$ yields

$$\mathcal{N}_\mu(z) = \frac{sin(\pi\nu)}{2\pi\nu}\int_{-1}^{1}\frac{1}{\lambda - z}\left(\frac{1-\lambda}{1+\lambda}\right)^{\nu} d\lambda,$$

that is, coincides with (4.3). Consequently, the $\mathfrak{N}$-resolvent $\mathcal{N}_M(z)$ of the largest self-adjoint contractive extension $\hat{A}_M$ of operator (5.2) is given by (4.4). This completes the proof. ■

For $\nu = 0$, calculations similar to those from [6] were performed directly using formulas (5.2) and (5.9). We are looking for a value of $\mathcal{E}$ from the block representation (2.1) of $\hat{A}$ such that $\hat{A}U^*(I - \kappa\hat{A}) = U^*(\hat{A} - \kappa I)$. Such calculations result in $\mathcal{E}_\mu = \mathcal{E}_M = 0$, and $\mathcal{N}_\mu(z) = \mathcal{N}_M(z)$ is given by formula (4.2).

Now Theorem 2.4, Theorem 4.1, and Remark 4.3 give the following statement.

Theorem 5.4. Let A be a nondensely defined Hermitian contractive operator with domain $\mathfrak{D}$ which is (g_t, U_t)-invariant for all $t \in \mathbb{R}$. Suppose that $\dim\mathfrak{D}^\perp = 1$. Then the operator A is unitarily equivalent to the operator on $L^2(0,\infty)$ defined by (5.2) for some ν, $0 \le \nu < 1$. Moreover, $\nu = 0$ if and only if the operator A has only one self-adjoint contractive extension.

6. Appendix

In this appendix, the notion of the Weyl–Titchmarsh function is defined and some important properties of this object are formulated. This information, in particular, provides a functional model for a prime symmetric operator and its self-adjoint extension. For detailed developments of the theory of the Weyl–Titchmarsh function and some applications, we refer readers to the articles [7, 9–11, 13–15, 20] and references therein.

Let $\mathcal{H}$ be a densely defined prime symmetric operator on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(\mathcal{H})$. We assume that the index of defect of $\mathcal{H}$ is (m, m) $(m < \infty)$, and denote by H a self-adjoint extension of $\mathcal{H}$.

In what follows we use the notations $\mathfrak{M}_z = (\mathcal{H} - zI)\mathfrak{D}(\mathcal{H})$, $\mathfrak{N}_z = \mathfrak{M}_{\bar{z}}^{\perp}$. Therefore, $\mathfrak{N}_z$ is the eigenspace of $\mathcal{H}^*$ which corresponds to the eigenvalue z.

The Weyl–Titchmarsh function $M_{\mathcal{H},H}(z)$ of the pair $(\mathcal{H}, H)$ is an operator-valued function whose values are operators on m-dimensional space $\mathfrak{N}_i$. The function $M_{\mathcal{H},H}(z)$ is defined on the resolvent set $\rho(H)$ of the operator H by

$$M_{\mathcal{H},H}(z) = P(zH + I)(H - zI)^{-1}|_{\mathfrak{N}_i}, \tag{6.1}$$

where P is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_i$.

From the spectral representation of H, one obtains that $M_{\mathcal{H},H}(z)$ can be written as

$$M_{\mathcal{H},H}(z) = \int_{\mathbb{R}} \frac{\lambda z + 1}{\lambda - z} d\sigma(\lambda). \tag{6.2}$$

The values of the nondecreasing function $\sigma(\lambda)$ are operators on $\mathfrak{N}_i$, where

$$\sigma(\lambda) = PE(\lambda)|_{\mathfrak{N}_i},$$

where $E(\lambda)$ is the resolution of identity associated with H. We normalize $E(\lambda)$ by $E(\lambda) = (E(\lambda+0) + E(\lambda-0))/2$. It is evident that $M_{\mathcal{H},H}(z)$ is analytic on $\rho(H)$, particularly, for $\Im z \neq 0$, and from (6.2) it follows that $\Im M_{\mathcal{H},H}(z) \geq 0$ for $z \in \mathbb{C}_+$. Therefore, $M_{\mathcal{H},H}(z)$ belongs to the Herglotz–Nevanlinna class.

The function σ has the following properties:

$$\int_{\mathbb{R}} d\sigma(\lambda) = I_{\mathfrak{N}_i}; \tag{6.3}$$

$$\int_{\mathbb{R}} (1+\lambda^2)(d\sigma(\lambda)h, h) = \infty \quad \text{for all} \quad h \in \mathfrak{N}(i), \tag{6.4}$$

and $\sigma(\lambda) = (\sigma(\lambda+0)+\sigma(\lambda-0))/2$. Condition (6.3) is obvious, condition (6.4) follows from the fact, that according to von Neumann's formulas, for vector $h \in \mathfrak{N}_i$, $h \notin \mathfrak{D}(H)$. Condition (6.3) provides a normalization condition for the Weyl–Titchmarsh function: $M_{\mathcal{H},H}(i) = iI_{\mathfrak{N}_i}$. From condition (6.4), it follows that points of growth of σ form a noncompact set.

Some important properties of the Weyl–Titchmarsh function of the pair $(\mathcal{H}, H)$ are summarized in the following statements.

Theorem 6.1. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be prime symmetric operators with equal defect numbers in Hilbert spaces $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$ respectively, and H and $\tilde{H}$ be their self-adjoint extensions. Suppose that there is a unitary operator $W : \mathfrak{H} \to \tilde{\mathfrak{H}}$ such that $W\mathcal{H} = \tilde{\mathcal{H}}W$ and $WH = \tilde{H}W$. Then there is a unitary operator $W_0 : \mathfrak{N}_i \to \tilde{\mathfrak{N}}_i$ such that $W_0 M_{\mathcal{H},H}(z) = M_{\tilde{\mathcal{H}},\tilde{H}}(z)W_0$.

Therefore, Theorem 6.1 can be reformulated as follows:
If pairs $(\mathcal{H}, H)$ and $(\tilde{\mathcal{H}}, \tilde{H})$ are unitarily equivalent, then there are bases with respect to which matrices of their Weyl–Titchmarsh functions are equal.

The next theorem is the statement about realization. It provides the functional model of the pair with prescribed Weyl–Titchmarsh function.

Theorem 6.2. Let F be a function whose values are linear operators on m-dimensional space $\mathfrak{N}$, and which admits the integral representation

$$F(z) = \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z} d\sigma(\lambda),$$

where $\sigma(\lambda)$ is a nondecreasing function with values on the set of linear operators on $\mathfrak{N}$, and which satisfies (6.3) and (6.4) (with $\mathfrak{N}$ instead of $\mathfrak{N}_i$). Then there exist a Hilbert space $\tilde{\mathfrak{H}}$, a prime symmetric operator $\tilde{\mathcal{H}}$ with defect index (m, m), and its self-adjoint extension $\tilde{H}$ in $\tilde{\mathfrak{H}}$, such that $F(z) = M_{\tilde{\mathcal{H}},\tilde{H}}(z)$. If $(\hat{\mathfrak{H}}, \hat{\mathcal{H}}, \hat{H})$ is another realization of F, then there is a unitary operator $\Psi : \tilde{\mathfrak{H}} \mapsto \hat{\mathfrak{H}}$ such that $\Psi\tilde{\mathcal{H}} = \hat{\mathcal{H}}\Psi$, and $\Psi\tilde{H} = \hat{H}\Psi$.

The Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$ is the set of functions f defined on the real line $\mathbb{R}$ with values in $\mathfrak{N}$ such that $\int_{\mathbb{R}} (d\sigma(\lambda) f(\lambda), f(\lambda))_{\mathfrak{N}} < \infty$. The operator $\tilde{H}$ is defined by

$$\mathfrak{D}(\tilde{H}) = \left\{ f \in \tilde{\mathfrak{H}} : \int_{\mathbb{R}} (1+\lambda^2)(d\sigma(\lambda) f(\lambda), f(\lambda))_{\mathfrak{N}} < \infty \right\}$$

and

$$(\tilde{H} f)(\lambda) = \lambda f(\lambda).$$

The operator $\tilde{\mathcal{H}}$ has domain

$$\mathfrak{D}(\tilde{\mathcal{H}}) = \left\{ f \in \mathfrak{D}(H) : \int_{\mathbb{R}} (\lambda + i) d\sigma(\lambda) f(\lambda) = 0 \right\},$$

and

$$(\tilde{\mathcal{H}} f)(\lambda) = \lambda f(\lambda).$$

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