Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 2 Number 2 (2007), pp. 135–140

http://www.ripublication.com/adsa.htm

Nodal Domains for the p-Laplacian

Aomar Anane, Omar Chakrone, Mohamed Filali, and Belhadj Karim
Département de Mathématiques et Informatique,
Faculté des sciences, Université Mohamed 1^{ier} *Oujda, Maroc*
E-mail: anane@sciences.univ-oujda.ac.ma,
chakrone@sciences.univ-oujda.ac.ma
filali@sciences.univ-oujda.ac.ma
karembelf@hotmail.com

Abstract

In this paper we consider the eigenvalue problem $-\triangle_p u = \lambda(m)|u|^{p-2}u$, $u \in W_0^{1,p}(\Omega)$ where $p > 1$, $\triangle_p$ is the p-Laplacian operator, $\lambda > 0$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ and m is a given positive function in $L^r(\Omega)$ (r depending on p and N). We prove that the second positive eigenvalue admits exactly two nodal domains.

AMS subject classification: 35J20, 35J70, 35P05, 35P30.
Keywords: Nodal domains, p-Laplacian, eigenvalues.

1. Introduction

Consider the problem

$$\begin{cases} -\triangle_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian, Ω is a bounded domain in $\mathbb{R}^N$, $N \in \mathbb{N}$ and λ is the eigenvalue parameter. We denote $M^+(\Omega)$= $\{m \in L^r(\Omega)$:

Received July 12, 2007; Accepted September 8, 2007

meas$\{x \in \Omega : m(x) > 0\} \neq 0\}$, with $r > \dfrac{N}{p}$ if $1 < p \leq N$ and $r = 1$ if $p > N$. We assume that m satisfies the hypothesis

(H) : $m(x) \geq 0$ a.e. $x \in \Omega$ and $m \in M^+(\Omega)$.

Let us start by considering the sequence $\mu_1 < \mu_2 \leq \mu_3 \cdots \to +\infty$ of all eigenvalues of $-\triangle$ on $H_0^1(\Omega)$, with $m \in L^\infty(\Omega)$, Ω being a bounded domain in $\mathbb{R}^N$, where each μ_k is repeated according to its multiplicity. A well-known theorem of Courant [4] states that if $u \in H_0^1(\Omega)$ is an eigenfunction associated to μ_k, then u admits at most k nodal domains. This theorem was partially extended to the p-Laplacian by Anane and Tsouli in [1]. Let us denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \to +\infty$ the sequence of eigenvalues of $-\triangle_p$ on $W_0^{1,p}(\Omega)$ obtained by the Ljusternik–Schnirelman method (see [7]). In the linear case $p = 2$ and $m \in L^\infty(\Omega)$, this sequence $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots$ yields all eigenvalues and coincides with the previous sequence $\mu_1 < \mu_2 \leq \mu_3 \ldots$ (see [3, page 23]). The result of [1] is the following. Let λ be an eigenvalue of $-\triangle_p$ on $W_0^{1,p}(\Omega)$ and suppose that for some k, $\lambda < \lambda_k$. Then the number of nodal domains of an eigenfunction associated to λ is strictly inferior to k. M. Cuesta et al. in [6] proved that in the nonlinear case and $m \equiv 1$, the number of nodal domains of an eigenfunction associated to λ_2, the second eigenvalue, is exactly 2. In this paper we prove that if m satisfies the hypothesis (H), then the number of nodal domains of an eigenfunction associated to $\underline{\lambda}_2(\Omega, m)$, the second eigenvalue for the problem (1.1), is exactly 2.

2. Preliminaries

Throughout this paper, Ω will be a bounded domain of $\mathbb{R}^N$ and we will always assume the hypothesis (H). $W_0^{1,p}(\Omega)$ will denote the usual Sobolev space with norm $||u||_{1,p} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$. We will write $||.||_p$ for the $L^p(\Omega)$ norm, p' denotes the Hölder conjugate exponent of p. We will write $Y = L^{r'p}(\Omega)$ if $1 < p \leq N$ and $Y = C(\Omega)$ if $p > N$. The infinity norm in the case $Y = C(\Omega)$ will be denoted by $||.||_Y$. $|\Omega|$ denotes the Lebesgue measure of Ω. We recall that a value $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if and only if there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \lambda \int_\Omega m|u|^{p-2}u\varphi dx$ for all $\varphi \in W_0^{1,p}(\Omega)$, and u is then called an eigenfunction associated to λ. Now let us formulate variational problem (1.1), for that we introduce the C^1 functionals Φ by $\Phi(u) = \int_\Omega |\nabla u|^p dx$ and $B : W_0^{1,p}(\Omega) \to \mathbb{R}$ by $B(u) = \int_\Omega m|u|^p dx$. $\widetilde{\Phi}$ will denote the restriction of Φ to $M = \{u \in W_0^{1,p}(\Omega) : B(u) = 1\}$. A first sequence of positive critical values of $\widetilde{\Phi}$ comes from the Ljusternik–Schnirelman critical point theory on C^1 manifolds proved in [8] that $\underline{\lambda}_n(\Omega, m) = \inf_{K\in\Gamma_n} \max_{u\in K} \Phi(u)$ is an eigenvalue. Moreover $\lim_{n\to+\infty} \underline{\lambda}_n(\Omega, m) = +\infty$. Here

$\Gamma_n = \{K \subset M : K$ compact, symmetric and $\gamma(K) \geq n\}$ and $\gamma(K)$ indicates the genus of K. Finally in [5], M. Cueta showed that $\underline{\lambda}_1(\Omega, m)$ is simple, isolated, and possesses the property of strict monotonicity respectively to the domain and the weight. This result has been shown in [2] in the case $m \in L^\infty(\Omega)$. Since $\underline{\lambda}_1(\Omega, m)$ is isolated in the spectrum and there exist eigenvalues different from $\underline{\lambda}_1(\Omega, m)$, it makes sense to define the second eigenvalue of (1.1) as $\lambda_2 := \min\{\lambda \in \mathbb{R} : \lambda$ eigenvalue and $\lambda > \underline{\lambda}_1(\Omega, m)\}$. This result is also proved in [1] in the case $m \in L^\infty(\Omega)$.

Theorem 2.1. [5] If m satisfies the hypothesis (H), then

$$\underline{\lambda}_2(\Omega, m) = \lambda_2 = \mu_2 = \inf_{h\in\mathcal{F}} \max_{u\in h([-1,1])} \int_\Omega |\nabla u|^p dx,$$

where $\mathcal{F} = \{\gamma \in C([-1,1], M) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}$ and $\varphi_1 \in M$ is the positive eigenfunction associated to $\underline{\lambda}_1(\Omega, m)$.

Proposition 2.2. Let Ω_1 be a proper open subset of a domain $\Omega_2 \subset \mathbb{R}^N$ such that $\text{meas}(\{x \in \Omega_1 : m(x) > 0\}) \neq 0$. Then $\underline{\lambda}_1(\Omega_2, m) < \underline{\lambda}_1(\Omega_1, m)$.

Proof. Consider $A = \left\{u \in W_0^{1,p}(\Omega_1) : \int_{\Omega_1} |\nabla u|^p = 1\right\}$. Now

$$\begin{aligned} \frac{1}{\underline{\lambda}_1(\Omega_1, m)} &= \sup_{u\in A} \int_{\Omega_1} m|u|^p dx = \int_{\Omega_1} m|\varphi_1|^p dx \\ &= \int_{\Omega_1} m|\tilde{\varphi}_1|^p dx < \int_{\Omega_2} m|\varphi_2|^p dx \\ &= \frac{1}{\underline{\lambda}_1(\Omega_2, m)}, \end{aligned}$$

where φ_1, φ_2 are respectively the eigenfunctions associated to $\underline{\lambda}_1(\Omega_1, m)$, $\underline{\lambda}_1(\Omega_2, m)$ and $\tilde{\varphi}_1 = \varphi_1$ on Ω_1, $\tilde{\varphi}_1 = 0$ on $\Omega_2 \setminus \Omega_1$. ■

3. Nodal Domains of the Second Eigenfunction

The main result in this section is the following theorem.

Theorem 3.1. An eigenfunction associated to $\underline{\lambda}_2(\Omega, m)$ admits exactly two nodal domains.

Consider $\Phi(u) = \int_\Omega |\nabla u|^p dx$ and $M = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega m|u|^p dx = 1\right\}$. We have $\underline{\lambda}_2(\Omega, m) = \inf_{K\in\Gamma_2} \max_{u\in K} \Phi(u)$, where $\Gamma_2 = \{K \subset M : K$ compact, symmetric and $\gamma(K) \geq 2\}$. Let φ_2 be the second eigenfunction associated to $\underline{\lambda}_2(\Omega, m)$. φ_2 must change sign and consequently admits at least one positive nodal domain Ω_1 and one negative nodal domain Ω_2. Let us assume by contradiction the existence of a third nodal domain with say $\varphi_2 > 0$ in Ω_3 (the argument would be similar if $\varphi_2 < 0$ in Ω_3). Thus we have the following lemma.

Lemma 3.2. There exists an open and connected set $\theta_2 \subset \Omega$ with $\Omega_2 \subsetneq \theta_2$ such that θ_2 is disjoint of Ω_1 or Ω_3.

Proof of Theorem 3.1. Let us admit Lemma 3.2 for a moment and show how to derive a contradiction. We will assume below that the lemma is satisfied with θ_2 disjoint of Ω_1 (the argument would be similar in the other case). We will show the existence of a function $v \in W_0^{1,p}(\Omega)$ which changes sign and satisfies

$$0 < \int_\Omega |\nabla v^+|^p dx < \underline{\lambda}_2(\Omega, m) \int_\Omega m(v^+)^p dx,$$

$$0 < \int_\Omega |\nabla v^-|^p dx < \underline{\lambda}_2(\Omega, m) \int_\Omega m(v^-)^p dx.$$

Since

$$\begin{cases} -\triangle_p(\varphi_2) = \underline{\lambda}_2(\Omega, m) m|\varphi_2|^{p-2}\varphi_2 & \text{in } \Omega, \\ \varphi_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where φ_2 is the second eigenfunction associated to the second eigenvalue $\underline{\lambda}_2(\Omega, m)$, we have

$$\begin{cases} -\triangle_p(\varphi_2) = \underline{\lambda}_2(\Omega, m) m|\varphi_2|^{p-2}\varphi_2 & \text{in } \Omega_1, \\ \varphi_2 = 0 & \text{on } \partial\Omega_1. \end{cases}$$

Thus $\underline{\lambda}_2(\Omega, m) = \underline{\lambda}_1(\Omega_1, m)$. Also, from Proposition 2.2 we conclude that $\underline{\lambda}_1(\theta_2, m) < \underline{\lambda}_1(\Omega_2, m)$, where $\underline{\lambda}_1(\theta, m)$ denotes the first eigenvalue of $-\triangle_p$ on $W_0^{1,p}(\theta)$. We then decrease θ_2 and increase Ω_1, so as to get two new open sets in Ω, $\tilde{\theta}_2$ and $\tilde{\Omega}_1$, with empty intersection such that $\underline{\lambda}_1(\tilde{\theta}_2, m) < \underline{\lambda}_2(\Omega, m)$ and $\underline{\lambda}_1(\tilde{\Omega}_1, m) < \underline{\lambda}_2(\Omega, m)$. On the other hand let u_1 be the first positive eigenfunction associated to $\underline{\lambda}_1(\tilde{\Omega}_1, m)$. Then we have

$$\begin{cases} -\triangle_p(u_1) = \underline{\lambda}_1(\tilde{\Omega}_1, m) m u_1^{p-1} & \text{in } \tilde{\Omega}_1, \\ u_1 = 0 & \text{on } \partial\tilde{\Omega}_1. \end{cases}$$

Let u_2 be the first positive eigenfunction associated to $\underline{\lambda}_1(\tilde{\theta}_2, m)$. Thus

$$\begin{cases} -\triangle_p(u_2) = \underline{\lambda}_1(\tilde{\theta}_2, m) m u_2^{p-1} & \text{in } \tilde{\theta}_2, \\ u_2 = 0 & \text{on } \partial\tilde{\theta}_2. \end{cases}$$

Put $v_1 = u_1/\tilde{\Omega}_1$, $v_1 = 0$ on $\Omega \setminus \tilde{\Omega}_1$ and $v_2 = u_2/\tilde{\theta}_2$, $v_2 = 0$ on $\Omega \setminus \tilde{\theta}_2$. Let $v = v_1 - v_2$. We have

$$\int_\Omega |\nabla v^+|^p dx = \int_\Omega |\nabla v_1|^p dx = \underline{\lambda}_1(\tilde{\Omega}_1, m) \int_\Omega m|v_1|^p dx < \underline{\lambda}_2(\Omega, m) \int_\Omega m|v_1|^p dx \tag{3.1}$$

and

$$\int_\Omega |\nabla v^-|^p dx = \int_\Omega |\nabla v_2|^p dx = \underline{\lambda}_1(\tilde{\theta}_2, m) \int_\Omega m|v_2|^p dx < \underline{\lambda}_2(\Omega, m) \int_\Omega m|v_2|^p dx, \tag{3.2}$$

where $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$. Consider the mapping

$$\phi : M \cap \langle v^+, v^- \rangle \to S \cap \langle v^+, v^- \rangle : u \mapsto \frac{u}{||u||_{1,p}}$$

which is odd homomorphic with

$$\phi^{-1} : S \cap \langle v^+, v^- \rangle \to M \cap \langle v^+, v^- \rangle : v \mapsto \frac{v}{(\int_\Omega m|v|^p dx)^{\frac{1}{p}}},$$

where $M = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega m|u|^p dx = 1\right\}$, $S = \{u \in W_0^{1,p}(\Omega) : ||u||_{1,p} = 1\}$ and $\langle v^+, v^- \rangle$ represents the space spanned by v^+ and v^-. Put $F_2 = M \cap \langle v^+, v^- \rangle$. Then we have $\gamma(F_2) = 2$, where $\gamma(F_2)$ indicates the genus of F_2. Now let $u \in F_2$. Then there exists $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $u = av^+ + bv^-$. Hence we have

$$0 < \int_\Omega |\nabla u|^p dx = |a|^p \int_\Omega |\nabla v^+|^p dx + |b|^p \int_\Omega |\nabla v^-|^p dx.$$

From (3.1) and (3.2), we conclude that

$$\int_\Omega |\nabla u|^p dx = |a|^p \underline{\lambda}_1(\tilde{\Omega}_1, m) \int_\Omega m|v_1|^p dx + |b|^p \underline{\lambda}_1(\tilde{\theta}_2, m) \int_\Omega m|v_2|^p dx.$$

Thus $\int_\Omega |\nabla u|^p dx < \underline{\lambda}_2(\Omega, m)$, which contradicts $\underline{\lambda}_2(\Omega, m) = \inf_{K \in \Gamma_2} \max_{u \in K} \Phi(u)$. ■

Proof of Lemma 3.2. The following proof adopts the scheme of M. Cuesta in [6]. Consider the two sets $\partial\Omega_2 \cap \Omega$ and $\partial\Omega_1 \cap \Omega$. We distinguish two cases (1) $\partial\Omega_2 \cap \Omega \not\subseteq \partial\Omega_1 \cap \Omega$ or (2) $\partial\Omega_2 \cap \Omega \subset \partial\Omega_1 \cap \Omega$. In case (1), there exists $x \in \partial\Omega_2 \cap \Omega$ such that x does not belong to $\partial\Omega_1$. Thus for some $\varepsilon > 0$, $B(x, \varepsilon) \subset \Omega$ and $B(x, \varepsilon) \cap \Omega_1 = \emptyset$. The set $\theta_2 = \Omega_2 \cup B(x, \varepsilon)$ is then disjoint of Ω_1 and yields the conclusion of the lemma. Let us now deal with case (2). The function φ_2 on Ω_2 is C^1, negative, and satisfies there $-\triangle_p \varphi_2 \leq 0$ in the weak sense. Let $z \in \partial\Omega_2 \cap \Omega$ satisfy the interior ball condition with respect to Ω_2. Since φ_2 is C^1 in a neighbourhood of z, we deduce from the Hopf maximum principle that $\frac{\partial \varphi_2}{\partial n(z)} > 0$, where n is the exterior normal direction to the interior ball at z. Thus at least one partial derive of φ_2 at z is nonzero. So there exists $1 \leq j \leq N$ such that $\frac{\partial \varphi_2}{\partial x_j} \neq 0$. Now consider the C^1 mapping $\Psi : \Omega \to \mathbb{R}^N : (x_1, \ldots, x_N) \mapsto (y_1, \ldots, y_N)$ defined by $y_i = x_i - z_i$ for all $1 \leq i \leq N$ and $i \neq j$, $y_j = \varphi_2(x_1, x_2, \ldots, x_N)$, by the inverse mapping theorem there is an open neighbourhood U of z which is diffeomorphic through Ψ to $V := \{y \in \mathbb{R}^N : |y| < \varepsilon\}$ for some $\varepsilon > 0$. Since $\varphi_2(\Psi^{-1}(y)) = y_j$, we have $\varphi_2 = 0$ on $\Psi^{-1}(V^0)$, $\varphi_2 > 0$ on $\Psi^{-1}(V^+)$ and $\varphi_2 < 0$ on $\Psi^{-1}(V^-)$, where $V^0 = \{y \in V : y_j = 0\}$, $V^+ = \{y \in V : y_j > 0\}$, $V^- = \{y \in V : y_j < 0\}$. Moreover $U = \Psi^{-1}(V^0) \cup \Psi^{-1}(V^+) \cup \Psi^{-1}(V^-)$. We have $z \in \partial\Omega_1 \cap \Omega$, $\Psi^{-1}(V^+)$ is open and connected, and Ω_1 is a positive nodal domain. Consequently $\Psi^{-1}(V^+) \subset \Omega_1$. Similarly we have $\Psi^{-1}(V^-) \subset \Omega_2$. Thus z does not belong to $\partial\Omega_3$. So there exists $\varepsilon > 0$ such that $B(z, \varepsilon) \cap \Omega_3 = \emptyset$, in which case we put $\theta_2 = B(z, \varepsilon) \cup \Omega_2$. ■

Corollary 3.3. Let Ω_1 and Ω_2 be the nodal domains of the second eigenfunction. Then we have $|\Omega| \geq |\Omega_1| + |\Omega_2| \geq 2(C\underline{\lambda}_2(\Omega, m)||m||_r)^{-\gamma}$, where $\gamma = \dfrac{rN}{rp - N}$ and C is some constant depending only on N and p if $p \neq N$ and on N and r' if $p = N$.

Proof. This follows from Theorem 3.1 and using [5, Theorem 3.2]. ■

References

[1] A. Anane and N. Tsouli. On a nonresonance condition between the first and the second eigenvalues for the p-Laplacian, *Int. J. Math. Math. Sci.*, 26(10):625–634, 2001.

[2] Aomar Anane. Simplicité et isolation de la première valeur propre du p-laplacien avec poids, *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987.

[3] Omar Chakrone. *Thèse de Doctorat*, PhD thesis, Université Mohamed Premier, Oujda, Maroc, 1998.

[4] R. Courant and D. Hilbert. *Methoden der Mathematischen Physik. Vols. I, II*, Interscience Publishers, Inc., N.Y., 1943.

[5] Mabel Cuesta. Eigenvalue problems for the p-Laplacian with indefinite weights, *Electron. J. Differential Equations*, pages No. 33, 9 pp. (electronic), 2001.

[6] Mabel Cuesta, Djairo G. De Figueiredo, and Jean-Pierre Gossez. A nodal domain property for the p-Laplacian, *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8):669–673, 2000.

[7] L. Ljusternik and L. Schnirelmann. *Méthodes topologiques dans les problèmes variationelles*, Hermann and Cie, Paris, 1934.

[8] Andrzej Szulkin. Ljusternik–Schnirelmann theory on C^1-manifolds, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(2):119–139, 1988.