Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 2 Number 1 (2007), pp. 123–133

http://www.ripublication.com/adsa.htm

Existence of Positive Radial Solutions for Elliptic Systems[1]

Shengli Xie
Department of Mathematics and Physics, Anhui Institute of Architecture and Industry, Hefei 230022, People's Republic of China
E-mail: xieshengli200@sina.com

Jiang Zhu[2]
Department of Mathematics, Xuzhou Normal University, Xuzhou 221116, People's Republic of China
E-mail: jzhuccy@yahoo.com.cn

Abstract

In this paper, we study the existence of positive radial solutions for the elliptic system by fixed point index theory.

AMS subject classification: 34B15, 45G15.
Keywords: Positive radial solution, fixed point index, cone.

1. Introduction and Preliminaries

There are many results on the study of positive radial solutions in the annulus for elliptic equations, see [1, 7, 9, 10] and references. However we are interested in problems of superlinearity and sublinearity for the elliptic system. In this paper, we study existence of positive radial solutions of the elliptic system

$$\begin{cases} \triangle u + h_1(r) f(u, v) = 0, \\ \triangle v + h_2(r) g(u) = 0, \ 0 < a < r < b \end{cases} \tag{1.1}$$

[1]This work is supported by Natural Science Foundation of the EDJP (05KGD110225), JSQLGC, National Natural Science Foundation 10671167, and EDAP2005KJ221, China.

[2]Corresponding author

Received April 28, 2006; Accepted January 28, 2007

with one of the following sets of boundary conditions

$$u = v = 0 \quad \text{on} \quad r = a, r = b, \tag{1.2a}$$

$$u = v = 0 \text{ on } r = a, \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0 \text{ on } r = b, \tag{1.2b}$$

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0 \text{ on } r = a, \quad u = v = 0 \text{ on } r = b, \tag{1.2c}$$

where $\{x \in \mathbb{R}^n : a < |x| < b\}$ is an annulus, $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $(n \geq 2)$, whereas $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+), g \in C(\mathbb{R}^+, \mathbb{R}^+), f(0,0) = g(0) = 0, h_i \in C((a,b), \mathbb{R}^+)$ $(i = 1, 2), \mathbb{R}^+ = [0, +\infty)$.

The existence of positive radial solutions for the elliptic system is studied in [5, 8]. The paper [8] only deals with the sublinear case

$$\lim_{|(x,y)| \to 0} \frac{f_i(x,y)}{|(x,y)|} = +\infty, \qquad \lim_{|(x,y)| \to \infty} \frac{f_i(x,y)}{|(x,y)|} = 0 \quad (i = 1, 2).$$

In [5], Jiang and Liu studied the case of sublinearity ($f_0 = 0$, $f_\infty = \infty$) or superlinearity ($f_0 = \infty$, $f_\infty = 0$), where

$$f_0 = \lim_{|u| \to 0} \frac{f(u)}{|u|^{p-1}}, \quad f_\infty = \lim_{|u| \to \infty} \frac{f(u)}{|u|^{p-1}} \quad (f \in \mathbb{R}^m, p > 1).$$

The purpose of this paper is to study the existence of positive radial solutions of the system (1.1)–(1.2a). Our particular interest is that $f(u, v)$ and $g(u)$ grow both superlinearly and sublinearly in u, v respectively. So our results are different from the ones of [1, 5, 7–11] and the conditions that we use are more general than the ones used in [1, 5, 7–11].

(1.1)–(1.2) is equivalent to the boundary value problems

$$\begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) + h_1(r) f(u(r), v(r)) = 0, \\ v''(r) + \dfrac{n-1}{r} v'(r) + h_2(r) g(u(r)) = 0, \ a < r < b, \end{cases} \tag{1.3}$$

$$u(a) = v(a) = u(b) = v(b) = 0, \tag{1.4a}$$

$$u(a) = v(a) = u'(b) = v'(b) = 0, \tag{1.4b}$$

$$u'(a) = v'(a) = u(b) = v(b) = 0. \tag{1.4c}$$

Let $s = -\int_r^b t^{1-n} dt, m = -\int_a^b t^{1-n} dt, w(s) = u(r(s)), z(s) = v(r(s))$. Then (1.3)–(1.4) can be rewritten as

$$\begin{cases} w''(s) + r(s)^{2(n-1)} h_1(r(s)) f(w(s), z(s)) = 0, \\ z''(s) + r(s)^{2(n-1)} h_2(r(s)) g(w(s)) = 0, \ m < s < 0, \end{cases}$$

$$w(m) = z(m) = w(0) = z(0) = 0,$$
$$w(m) = z(m) = w'(0) = z'(0) = 0,$$
$$w'(m) = z'(m) = w(0) = z(0) = 0.$$

Now, let $t = \dfrac{m-s}{m}$, $\varphi(t) = w(s)$ and $\psi(t) = z(s)$. Then (1.1)–(1.2) can also be written as

$$\begin{cases} \varphi''(t) + p_1(t) f(\varphi(t), \psi(t)) = 0, \\ \psi''(t) + p_2(t) g(\varphi(t)) = 0, \ 0 < t < 1, \end{cases} \tag{1.5}$$

$$\varphi(0) = \psi(0) = \varphi(1) = \psi(1) = 0, \tag{1.6a}$$

$$\varphi(0) = \psi(0) = \varphi'(1) = \psi'(1) = 0, \tag{1.6b}$$

$$\varphi'(0) = \psi'(0) = \varphi(1) = \psi(1) = 0, \tag{1.6c}$$

where $p_i(t) = m^2 r^{2(n-1)}(m(1-t)) h_i(r(m(1-t)))(i = 1, 2)$.

From now on, we concentrate on (1.5)–(1.6). Indeed, (1.1)–(1.2) has a positive radial solution for any annulus if we can prove that there exists a positive solution to BVP (1.5)–(1.6) for any $m \neq 0$ (cf. [8]).

For convenience of notation, we list the following assumptions:

(H_1) $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $h_i \in C((a, b), \mathbb{R}^+)$, $h_i(t) \not\equiv 0$ in any subinterval of (a, b), and

$$\int_a^b h_i(t) dt < +\infty \quad (i = 1, 2).$$

(H_2) There exists $\alpha \in (0, 1]$ such that

$$\liminf_{u \to +\infty} \frac{g(u)}{u^{\frac{1}{\alpha}}} = +\infty, \quad \liminf_{v \to +\infty} \frac{f(u, v)}{v^{\alpha}} > 0$$

uniformly with respect to $u \in \mathbb{R}^+$.

(H_3) There exists $\beta \in (0, +\infty)$ such that

$$\limsup_{u \to 0^+} \frac{g(u)}{u^{\frac{1}{\beta}}} = 0, \quad \limsup_{v \to 0^+} \frac{f(u, v)}{v^{\beta}} < +\infty$$

uniformly with respect to $u \in \mathbb{R}^+$.

(H_4) There exists $p \in (0, +\infty)$ such that

$$\limsup_{u \to +\infty} \frac{g(u)}{u^{\frac{1}{p}}} = 0, \quad \limsup_{v \to +\infty} \frac{f(u, v)}{v^{p}} < +\infty$$

uniformly with respect to $u \in \mathbb{R}^+$.

(H_5) There exists $q \in (0, 1]$ such that

$$\liminf_{u \to 0^+} \frac{g(u)}{u^{\frac{1}{q}}} = +\infty, \quad \liminf_{v \to 0^+} \frac{f(u, v)}{v^q} > 0$$

uniformly with respect to $u \in \mathbb{R}^+$.

(H_6) $f(u, v)$ and $g(u)$ are increasing in u and v and there exists $R > 0$ such that

$$\gamma_1 f(R, \gamma_2 g(R)) < R,$$

where $\gamma_i = \int_0^1 p_i(t)dt \quad (i = 1, 2)$.

The following examples to illustrate (H_2)–(H_5) are in order.

Example 1.1. Let $f(u, v) = (1+e^{v-u})v^2, g(u) = u^3, \alpha = \frac{1}{2}, \beta = 2$. Then (H_2)–(H_3) hold. Here $f(u, v)$ grows sublinearly in u and superlinearly in v respectively, whereas $g(u)$ grows superlinearly in u.

Example 1.2. Let $f(u, v) = (1 + e^{-(u+v)})v^{\frac{1}{2}}, g(u) = u^3, \alpha = \beta = \frac{1}{2}$. Then (H_2)–(H_3) hold, in which $f(u, v)$ grows sublinearly, whereas $g(u)$ grows superlinearly in u.

Example 1.3. Let $f(u, v) = (1 + e^{-(u+v)})v^{\frac{1}{2}}, g(u) = u^{\frac{3}{2}}$ or $g(u) = u^{\frac{1}{2}}, p = q = \frac{1}{2}$. Then (H_4) and (H_5) hold, $f(u, v)$ grows sublinearly, whereas $g(u)$ grows superlinearly or sublinearly.

Example 1.4. Let $f(u, v) = (1 + e^{u+v})v^{\frac{1}{2}}$ or $f(u, v) = (1 + e^u)v^{\frac{1}{2}}, g(u) = u^{\frac{1}{2}} + u^3, \alpha = q = \frac{1}{2}$. Then (H_2) and (H_5) hold, $f(u, v)$ and $g(u)$ are increasing in u and v. At $+\infty$, $f(u, v)$ grows superlinearly or $f(u, v)$ grows superlinearly and sublinearly in u and v respectively, whereas $g(u)$ grows superlinearly at $+\infty$.

By virtue of (H_1), we can define the integral operator $A : C[0, 1] \to C[0, 1]$ by

$$(A\varphi)(t) = \int_0^1 G_i(t, s)p_1(s)f(\varphi(s), (T\varphi)(s))ds, \tag{1.7}$$

where

$$(T\varphi)(t) = \int_0^1 G_i(t, s)p_2(s)g(\varphi(s))ds \quad (i = 1, 2, 3), \tag{1.8}$$

$$G_1(t, s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{1.9a}$$

$$G_2(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases} \tag{1.9b}$$

$$G_3(t,s) = \begin{cases} 1-s, & 0 \le t \le s \le 1, \\ 1-t, & 0 \le s \le t \le 1. \end{cases} \tag{1.9c}$$

Then the positive solutions of BVP (1.5)–(1.6) are equivalent to the positive fixed points of A.

Let $J = [0,1]$, $0 < c < d < 1$, $J_0 = [c,d]$, $\varepsilon_0 = c(1-d)$, $E = C[0,1]$, $\|u\| = \max_{t\in J} |u(t)|$ for $u \in E$,

$$K = \{u \in C[0,1] : u(t) \ge 0, u(t) \ge t(1-t)\|u\|, t \in J\}.$$

It is easy to show that $(E, \|\cdot\|)$ is a real Banach pace and K is a cone in E. From (1.9) we get that

$$G_1(t,s) \ge \varepsilon_0,\ G_2(t,s) \ge c,\ G_3(t,s) \ge 1-d, \qquad (t,s) \in J_0 \times J_0, \tag{1.10}$$

$$\begin{gathered} t(1-t)G_1(r,s) \le G_1(t,s) \le s(1-s), \\ tG_2(r,s) \le G_2(t,s) \le s, \\ (1-t)G_3(r,s) \le G_3(t,s) \le (1-s), \quad t,s,r \in J. \end{gathered} \tag{1.11}$$

Lemma 1.5. Let (H_1) hold. Then $A : K \to K$ is a completely continuous operator.

See [2, 6] for the proof of Lemma 1.5.

To prove our main results, we also need the following fixed point index theorems.

Let $(E, \|\cdot\|)$ be a real Banach space, P be a cone in E, and $B_\rho = \{u \in E : \|u\| < \rho\}$ $(\rho > 0)$ be the open ball of radius ρ. Let $A : \overline{B}_\rho \cap P \to P$ be a completely continuous operator, $i(A, B_\rho \cap P, P)$ denote the fixed point index of A on $B_\rho \cap P$. For the details of the fixed point index, one can refer to [4].

Lemma 1.6. [4] Assume that $A : \overline{B}_\rho \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that

$$u - Au \ne \lambda u_0 \text{ for all } \lambda \ge 0, u \in \partial B_\rho \cap P,$$

then $i(A, B_\rho \cap P, P) = 0$.

Lemma 1.7. [3] Assume that $A : \overline{B}_\rho \cap P \to P$ is a completely continuous operator and has no fixed point on $\partial B_\rho \cap P$.

(1) If $\|Au\| \le \|u\|$ for any $u \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 1$.

(2) If $\|Au\| \ge \|u\|$ for any $u \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 0$.

2. Main Results

Theorem 2.1. Let (H_1), (H_2) and (H_3) hold. Then (1.1)–(1.2a) has a positive radial solution for any annulus $a < r < b$.

Proof. Firstly, we consider (1.5)–(1.6a). By (H_2), there are $\nu > 0$ and sufficiently large $M > 0$ such that

$$f(\varphi, \psi) \geq \nu\psi^{\alpha} \text{ for all } \varphi \in \mathbb{R}^+,\ \psi > M, \tag{2.1}$$

$$g(\varphi) \geq C_0\varphi^{\frac{1}{\alpha}} \text{ for all } \varphi > M, \tag{2.2}$$

where $C_0 = \max\left\{\left(\frac{\nu\varepsilon_0^2}{2}\int_c^d p_2^{\alpha}(s)ds \max_{t\in J_0}\int_c^d G_1(t,s)p_1(s)ds\right)^{-\frac{1}{\alpha}}, \left(\gamma_2'\varepsilon_0^{\frac{1+\alpha}{\alpha}}\right)^{-1}\right\}$, where $\gamma_2' = \int_c^d p_2(r)dr$. Let $N = (M+1)\varepsilon_0^{-1}$, $\varphi_0(t) = \sin\pi t \in K \setminus \{\theta\}$. We claim that

$$\varphi - A\varphi \neq \lambda\varphi_0 \text{ for all } \lambda \geq 0,\ \varphi \in \partial B_N \cap K.$$

In fact, if there are $\lambda \geq 0$, $\varphi \in \partial B_N \cap K$ such that $\varphi - A\varphi = \lambda\varphi_0$, then

$$\varphi(t) \geq (A\varphi)(t) \geq \int_c^d G_1(t,s)p_1(s)f\left(\varphi(s), \int_0^1 G_1(s,r)p_2(r)g(\varphi(r))dr\right)ds,\ t \in J. \tag{2.3}$$

Owing to $\alpha \in (0, 1]$ and $\varphi(t) \geq \varepsilon_0\|\varphi\| = \varepsilon_0 N = M + 1 > M$, $\varphi(t) \in K$, $t \in J_0$, (2.2) implies that

$$\begin{aligned}\int_0^1 & G_1(s,r)p_2(r)g(\varphi(r))dr \geq \int_c^d G_1(s,r)p_2(r)g(\varphi(r))dr \\ &\geq C_0\int_c^d G_1(s,r)p_2(r)\varphi^{\frac{1}{\alpha}}(r)dr \geq C_0(\varepsilon_0 N)^{\frac{1}{\alpha}}\int_c^d G_1(s,r)p_2(r)dr \\ &\geq NC_0\gamma_2'\varepsilon_0^{\frac{1+\alpha}{\alpha}} \geq N,\ s \in J_0.\end{aligned} \tag{2.4}$$

By using $0 \leq G_1(t,s) \leq 1$, $\alpha \in (0, 1]$ and Jensen's inequality, it follows from (2.1)–

(2.4) that

$$\begin{aligned}
\varphi(t) &\geq \nu \int_c^d G_1(t,s)p_1(s)\left(\int_0^1 G_1(s,r)p_2(r)g(\varphi(r))dr\right)^{\alpha} ds \\
&\geq \nu \int_c^d G_1(t,s)p_1(s)\left(\int_c^d G_1(s,r)p_2(r)g(\varphi(r))dr\right)^{\alpha} ds \\
&\geq \nu \int_c^d G_1(t,s)p_1(s)\left(\int_c^d G_1^{\alpha}(s,r)p_2^{\alpha}(r)g^{\alpha}(\varphi(r))dr\right) ds \\
&\geq \nu \int_c^d G_1(t,s)p_1(s)\int_c^d G_1(s,r)p_2^{\alpha}(r)g^{\alpha}(\varphi(r))drds \\
&\geq \nu\varepsilon_0 C_0^{\alpha}\int_c^d G_1(t,s)p_1(s)ds\int_c^d p_2^{\alpha}(r)\varphi(r)dr \\
&\geq \nu\varepsilon_0^2 C_0^{\alpha}\int_c^d G_1(t,s)p_1(s)ds\int_c^d p_2^{\alpha}(r)dr\|\varphi\|,\ t \in J.
\end{aligned}$$

Thus

$$\|\varphi\| \geq C_0^{\alpha}\nu\varepsilon_0^2\int_c^d p_2^{\alpha}(r)dr\max_{t\in J_0}\int_c^d G_1(t,s)p_1(s)ds\|\varphi\| \geq 2\|\varphi\|.$$

This is a contradiction. By Lemma 1.6 we get

$$i(A, B_N \cap K, K) = 0. \tag{2.5}$$

On the other hand, according to the second limit of (H_3), there exists a sufficiently small $\rho_1 \in (0,1)$ such that

$$C_1 =: \sup\left\{\frac{f(\varphi,\psi)}{\psi^{\beta}} : \forall\,\varphi \in \mathbb{R}^+,\ \psi \in (0,\rho_1]\right\} < +\infty. \tag{2.6}$$

Let $\varepsilon_1 = \min\left\{\rho_1\gamma_2^{-1}, \gamma_2^{-1}\left(\frac{1}{2C_1\gamma_1}\right)^{\frac{1}{\beta}}\right\} > 0$. By the first limit of (H_3), there exists a sufficiently small $\rho_2 \in (0,1)$ such that

$$g(\varphi) \leq \varepsilon_1\varphi^{\frac{1}{\beta}},\ \ \varphi \in [0,\rho_2]. \tag{2.7}$$

Let $\rho = \min\{\rho_1,\rho_2\}$. (2.6) and (2.7) imply that

$$\begin{aligned}
\int_0^1 G_1(s,r)p_2(r)g(\varphi(r))dr &\leq \varepsilon_1\int_0^1 p_2(r)\varphi(r)^{\frac{1}{\beta}}dr \\
&\leq \rho_1\|\varphi\|^{\frac{1}{\beta}} = \rho_1^{1+\frac{1}{\beta}} < \rho_1,\ \ \varphi \in \overline{B}_{\rho}\cap K, s \in [0,1],
\end{aligned}$$

$$\begin{aligned}(A\varphi)(t) &\le C_1 \int_0^1 G_1(t,s)p_1(s)\left(\int_0^1 G_1(s,r)p_2(r)g(\varphi(r))dr\right)^{\beta} ds \\ &\le C_1\varepsilon_1^{\beta}\int_0^1 p_1(s)ds\left(\int_0^1 p_2(r)\varphi^{\frac{1}{\beta}}(r)dr\right)^{\beta} \\ &\le C_1\gamma_1\gamma_2^{\beta}\varepsilon_1^{\beta}\|\varphi\| \le \frac{1}{2}\|\varphi\|,\quad \varphi\in \overline{B}_\rho\cap K, t\in[0,1].\end{aligned}$$

Thus $\|A\varphi\| \le \frac{1}{2}\|\varphi\| < \|\varphi\|$ for any $\varphi \in \partial B_\rho \cap K$. Lemma 1.7 yields

$$i(A, B_\rho \cap K, K) = 1. \tag{2.8}$$

(2.5) together with (2.8) imply that

$$i(A,(B_N\setminus\overline{B}_\rho)\cap K, K) = i(A, B_N\cap K, K) - i(A, B_\rho\cap K, K) = -1.$$

So A has a fixed point $\varphi \in (B_N \setminus \overline{B}_\rho)\cap K$ and satisfies $0<\rho<\|\varphi\|\le N$. We know that $\varphi(t) > 0, t\in(0,1)$ by definition of K. This show that BVP (1.5)–(1.6a) has a positive solution $\varphi,\psi \in C^2(0,1)\cap C[0,1]$, and satisfies $\varphi(t)>0$, $\psi(t)>0$ for any $t\in(0,1)$. Similarly we can get the conclusions of (1.5)–(1.6b) and (1.5)–(1.6c). This completes the proof of Theorem 2.1. ∎

Theorem 2.2. Let (H_1), (H_4) and (H_5) hold. Then (1.1)–(1.2) has a positive radial solution for any annulus $a<r<b$.

Proof. First consider (1.5)–(1.6a). By (H_4), there exist $\delta>0$, $C_2>0$ and $C_3>0$ such that

$$f(\varphi,\psi) \le \delta\psi^p + C_2,\ g(\varphi) \le \left(\frac{\varphi}{2\delta\gamma_1\gamma_2^p}\right)^{\frac{1}{p}} + C_3,\quad \varphi,\psi\in\mathbb{R}^+. \tag{2.9}$$

(2.9) implies that

$$\begin{aligned}(A\varphi)(t) &\le \int_0^1 G_1(t,s)p_1(s)\left[\delta\left(\int_0^1 G_1(s,r)p_2(r)g(\varphi(r))dr\right)^p + C_2\right]ds \\ &\le \int_0^1 p_1(s)ds\left[\delta\left(\int_0^1 p_2(r)\left[\left(\frac{\varphi(r)}{2\delta\gamma_1\gamma_2^p}\right)^{\frac{1}{p}} + C_3\right]dr\right)^p + C_2\right] \\ &\le \delta\gamma_1\left[\left(\frac{\|\varphi\|}{2\delta\gamma_1}\right)^{\frac{1}{p}} + C_3\gamma_2\right]^p + \gamma_1 C_2.\end{aligned} \tag{2.10}$$

By means of simple calculation, we have

$$\lim_{u\to+\infty}\frac{\delta\gamma_1\left[\left(\frac{u}{2\delta\gamma_1}\right)^{\frac{1}{p}} + C_3\gamma_2\right]^p + \gamma_1 C_2}{u} = \frac{1}{2}.$$

Then there exists a sufficiently large $G > 0$ such that

$$\delta\gamma_1\left[\left(\frac{\|\varphi\|}{2\delta\gamma_1}\right)^{\frac{1}{p}} + C_3\gamma_2\right]^p + \gamma_1 C_2 < \frac{3}{4}\|\varphi\|, \quad \|\varphi\| > G.$$

From this and (2.10) we obtain that

$$\|A\varphi\| < \|\varphi\|, \quad \varphi \in \partial B_G \cap K.$$

This, along with Lemma 1.7, yields that

$$i(A, B_G \cap K, K) = 1. \tag{2.11}$$

In addition, by (H_5), there exist $\eta > 0$ and sufficiently small $\xi > 0$ such that

$$f(\varphi, \psi) \geq \eta\psi^q, \quad \varphi \in \mathbb{R}^+, 0 \leq \psi \leq \xi, \tag{2.12}$$

$$g^q(\varphi) \geq C_4\varphi, \quad 0 \leq \varphi \leq \xi, \tag{2.13}$$

where $C_4 = 2\left(\eta\varepsilon_0^2\int_c^d G_1\left(\frac{1}{2}, s\right)p_1(s)ds\int_c^d p_2^q(r)dr\right)^{-1}$. Since $g(0) = 0$, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$, there exists $\sigma \in (0, \min\{\xi, \gamma_2^{-1}\xi\})$ such that $g(\varphi) \leq \gamma_2^{-1}\xi$, for any $\varphi \in [0, \sigma]$. This implies that

$$\int_0^1 G_1(s, r)p_2(r)g(\varphi(r))dr \leq \xi, \quad \varphi \in \overline{B}_\sigma \cap K, s \in [0, 1]. \tag{2.14}$$

By using Jensen's inequality and $0 < q \leq 1$, from (2.12)–(2.14) we get that

$$\begin{aligned}
(A\varphi)\left(\frac{1}{2}\right) &\geq \eta\int_c^d G_1\left(\frac{1}{2}, s\right)p_1(s)\left(\int_0^1 G_1(s, r)p_2(r)g(\varphi(r))dr\right)^q ds \\
&\geq \eta\int_c^d G_1\left(\frac{1}{2}, s\right)p_1(s)\left(\int_c^d G_1(s, r)p_2(r)g(\varphi(r))dr\right)^q ds \\
&\geq \eta\int_c^d G_1\left(\frac{1}{2}, s\right)p_1(s)\left(\int_c^d G_1^q(s, r)p_2^q(r)g^q(\varphi(r)))dr\right) ds \\
&\geq \eta C_4\int_c^d G_1\left(\frac{1}{2}, s\right)p_1(s)\left(\int_c^d G_1(s, r)p_2^q(r)\varphi(r)dr\right) ds \\
&\geq \eta C_4\varepsilon_0^2\int_c^d G_1\left(\frac{1}{2}, s\right)p_1(s)ds\int_c^d p_2^q(r)dr\|\varphi\| \\
&= 2\|\varphi\|, \quad \varphi \in \overline{B}_\sigma \cap K,
\end{aligned}$$

thus

$$\|A\varphi\| > \|\varphi\| \text{ for all } \varphi \in \partial B_\sigma \cap K. \tag{2.15}$$

(2.15) together Lemma 1.7 yield

$$i(A, B_\sigma \cap K, K) = 0. \tag{2.16}$$

(2.11) and (2.16) imply that

$$i(A, (B_G \setminus \overline{B}_\sigma) \cap K, K) = i(A, B_G \cap K, K) - i(A, B_\sigma \cap K, K) = 1.$$

So A has a fixed point $\varphi \in (B_G \setminus \overline{B}_\sigma) \cap K$ and satisfies $0 < \sigma < \|\varphi\| < G$. This show that BVP (1.5)–(1.6a) has a positive solution $\varphi, \psi \in C^2(0, 1) \cap C[0, 1]$, and $\varphi(t) > 0, \psi(t) > 0$ for any $t \in (0, 1)$. Similarly we can get the conclusions of (1.5)–(1.6b) and (1.5)–(1.6c). This completes the proof of Theorem 2.2. ■

Theorem 2.3. Let (H_1), (H_2), (H_5) and (H_6) hold. Then (1.1)–(1.2) has two positive radial solutions for any annulus $a < r < b$.

Proof. We take $N > R > \sigma$ such that either (2.5) or (2.16) hold. (1.7), (1.8) and (H_6) indicate that

$$(A\varphi)(t) \le \int_0^1 p_1(s) f\left(\varphi(s), \int_0^1 p_2(r) g(\varphi(r) dr\right) ds \le \gamma_1 f(R, \gamma_2 g(R)) < R$$

for any $\varphi \in \partial B_R \cap K$, then $\|A\varphi\| < \|\varphi\|$ for any $\varphi \in \partial B_R \cap K$. Lemma 1.7 implies

$$i(A, B_R \cap K, K) = 1.$$

Consequently,

$$i(A, (B_N \setminus \overline{B}_R) \cap K, K) = i(A, B_N \cap K, K) - i(A, B_R \cap K, K) = -1,$$

$$i(A, (B_R \setminus \overline{B}_\sigma) \cap K, K) = i(A, B_R \cap K, K) - i(A, B_\sigma \cap K, K) = 1.$$

So A has two fixed points $\varphi_1 \in (B_R \setminus \overline{B}_\sigma) \cap K$ and $\varphi_2 \in (B_N \setminus \overline{B}_R) \cap K$ respectively, and $0 < \sigma < \|\varphi_1\| < R < \|\varphi_2\| \le N$. Then BVP (1.5)–(1.6) has two positive solution (φ_1, ψ_1), (φ_2, ψ_2), and satisfy $\varphi_i(t) > 0, \psi_i(t) > 0 (i = 1, 2)$ for any $t \in (0, 1)$. This completes the proof of Theorem 2.3. ■

Remark 2.4. From Examples 1.1–1.4 we know that all conclusions in this paper are different from the ones in [1, 5, 7–11] and the conditions that we use are more general than the ones in papers [1, 5, 7–11].

References

[1] C. Bandle, C. V. Coffman, and M. Marcus. Nonlinear elliptic problems in annular domains, *J. Differential Equations*, 69(3):322–345, 1987.

[2] Lynn H. Erbe and Ronald M. Mathsen. Positive solutions for singular nonlinear boundary value problems, *Nonlinear Anal.*, 46(7, Ser. A: Theory Methods):979–986, 2001.

[3] Da Jun Guo and V. Lakshmikantham. *Nonlinear problems in abstract cones*, volume 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press Inc., Boston, MA, 1988.

[4] Dajun Guo, Yeol Je Cho, and Jiang Zhu. *Partial ordering methods in nonlinear problems*, Nova Science Publishers Inc., Hauppauge, NY, 2004.

[5] Daqing Jiang and Huizhao Liu. On the existence of nonnegative radial solutions for p-Laplacian elliptic systems, *Ann. Polon. Math.*, 71(1):19–29, 1999.

[6] Kunquan Lan and Jeffrey R. L. Webb. Positive solutions of semilinear differential equations with singularities, *J. Differential Equations*, 148(2):407–421, 1998.

[7] Song-Sun Lin. On the existence of positive radial solutions for nonlinear elliptic equations in annular domains, *J. Differential Equations*, 81(2):221–233, 1989.

[8] Ruyun Ma. Existence of positive radial solutions for elliptic systems, *J. Math. Anal. Appl.*, 201(2):375–386, 1996.

[9] Haiyan Wang. On the existence of positive solutions for semilinear elliptic equations in the annulus, *J. Differential Equations*, 109(1):1–7, 1994.

[10] Haiyan Wang. On the structure of positive radial solutions for quasilinear equations in annular domains, *Adv. Differential Equations*, 8(1):111–128, 2003.

[11] Zhilin Yang. Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlinear Anal.*, 62(7):1251–1265, 2005.