Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 2 Number 1 (2007), pp. 107–121

http://www.ripublication.com/adsa.htm

Boundedness and Exponential Asymptotic Stability in Dynamical Systems with Applications to Nonlinear Differential Equations with Unbounded Terms

Youssef N. Raffoul
Department of Mathematics, University of Dayton,
Dayton, OH 45469-2316, USA
E-mail: youssef.raffoul@notes.udayton.edu

Abstract

Nonnegative definite Lyapunov functions are employed to obtain sufficient conditions that guarantee boundedness of solutions of a nonlinear differential system. Also, sufficient conditions will be given to insure that the zero solution is exponentially and asymptotically stable. Our theorems will make a significant contribution to the theory of differential equations when dealing with equations that might contain unbounded terms. The theory is illustrated with several examples.

AMS subject classification: 34C11, 34C35, 34K15.
Keywords: Nonlinear differential system, boundedness, exponential stability, Lyapunov functions.

1. Introduction

For motivational purpose, we consider the scalar linear initial value problem

$$\begin{aligned} \dot{x} &= -a(t)x(t) + g(t), \quad t \geq 0, \\ x(t_0) &= x_0, \quad t_0 \geq 0, \end{aligned} \tag{1.1}$$

where a and g are continuous in t with $a(t) \geq 0, t \geq 0$. By the variational of parameters formula

$$x(t) = x_0 e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^t g(u)\, e^{-\int_u^t a(s)ds}\, du. \tag{1.2}$$

Received February 3, 2007; Accepted April 28, 2007

Thus, if

$$\Big|\int_{t_0}^{t} g(u)\, e^{-\int_u^t a(s)ds}\, du\Big| \le M, \;\; t \ge t_0 \ge 0$$

for some positive constant M, then expression (1.2) implies that all solutions of Eqn. (1.1) are uniformly bounded. Then, it is reasonable to ask if the same can be done for nonlinear scalar or vector equations of the form

$$\begin{aligned} \dot{x} &= -a(t)x(t) + g(t,x), \quad t \ge 0, \\ x(t_0) &= x_0, \quad t_0 \ge 0, \end{aligned} \tag{1.3}$$

where $a(t)$ is continuous in t with $a(t) \ge 0$, and $g(t,x)$ is continuous in t and x, $t \ge t_0 \ge 0$. The answer is of course yes. Indeed, one gets the variational of parameters formula

$$x(t) = x_0 e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^{t} g(u, x(u))\, e^{-\int_u^t a(s)ds}\, du. \tag{1.4}$$

But expression (1.4) hardly gives any information about the boundedness of solutions of Eq. (1.3) unless we assume that

$$|g(t,x)| \le |\lambda(t)|, \;\; t \ge 0, \tag{1.5}$$

where $\lambda(t)$ is continuous. Moreover, by making the assumption that

$$\int_{t_0}^{t} |\lambda(u)|\, e^{-\int_u^t a(s)ds}\, du \le M$$

for some positive constant M, one concludes that all solutions of Eqn. (1.3) are uniformly bounded. For more on this discussion, we refer the reader to [10,12]. Condition (1.5) is very restrictive since the function $g(t)$ is bounded above by a function of t. In this paper, we are interested in dynamical systems that are similar to Eqn. (1.3) with the function $g(t,x)$ satisfying the less restrictive growth condition

$$|g(t,x)| \le |\lambda(t)||x(t)|^n + h(t), \;\; t \ge 0, \tag{1.6}$$

where $\lambda(t), h(t)$ are continuous, unbounded and n is a positive rational number. To better illustrate our point, we consider Eqn. (1.3) along with (1.5) with $n = 1$. Now, we consider the Lyapunov function $V(x) = x^2$. Then along solutions of (1.3) we have

$$\begin{aligned} V'(x) &\le -2a(t)x^2(t) + 2|\lambda(t)|x^2(t) + x^2(t) + h^2(t) \\ &= (-2a(t) + 2|\lambda(t)| + 1)x^2(t) + h^2(t), \end{aligned} \tag{1.7}$$

where $h(t)$ may be unbounded. If $a(t) > \dfrac{1}{2} + |\lambda(t)|$ and if we let $\alpha(t) = 2a(t) - 2|\lambda(t)| - 1$, then inequality (1.7) is equivalent to

$$V'(x) \le -\alpha(t)V(x) + h^2(t). \tag{1.8}$$

From (1.8), we easily obtain the variation of parameters inequality

$$|x(t)| \leq |x_0| e^{-\int_{t_0}^{t} \alpha(s)ds} + \int_{t_0}^{t} h^2(u)\, e^{-\int_{u}^{t} \alpha(s)ds}\, du. \tag{1.9}$$

Thus, the solutions are uniformly bounded provided that

$$\Big| \int_{t_0}^{t} h^2(u)\, e^{-\int_{u}^{t} \alpha(s)ds}\, du \Big| \leq M$$

for some positive constant M. We note that (1.9) is a major improvement over (1.4) and consequently (1.5). Inequality (1.9) was easily obtained due to inequality (1.8). The question is, what if instead of (1.8), we have the differential inequality

$$V'(x) \leq -\alpha(t) W(x) + h^2(t),\; W(x) \neq V(x) \text{ for } x \neq 0. \tag{1.10}$$

Here $\alpha : [0, \infty) \to [0, \infty)$ is continuous and $W : [0, \infty) \to [0, \infty)$ is continuous with $W(0) = 0$, $W(r)$ is strictly increasing, and $W(r) \to \infty$ as $r \to \infty$. Such a function is called a wedge.

In the case that $h^2(t)$ of (1.10) is bounded by some positive constant, then solutions of (1.3) are bounded. For a reference, we ask the reader to see [1, page 9–13], [4–6,8,12,13] and the references therein. In the last five years, fixed point theory is being used to overcome some of the difficulties that one may encounter using Lyapunov functions or functionals. One of the major hurdles when using the Lyapunov function method is the fact that $a(t)$ may not be big enough in the negative way so that a condition similar to (1.8) or (1.10) is satisfied. As we shall see in Section 4, our results will offer a better alternative than the alternative of fixed point theory. Thus, this paper will offer new and powerful theorems that will significantly advance the theory of existence of solutions in differential equations and their exponential decay to zero in the case where the differential equation in consideration has unbounded terms including unbounded forcing terms. Our motivation is to use the negative magnitude of $\alpha(t)$ to offset the unboundedness of the term $h(t)$, and then arrive at new variational of parameters inequalities that will serve as upper bounds on all the solutions of (1.3).

For illustration purpose, we write (1.3) as

$$\begin{aligned} \dot{x} &= -a(t)x(t) + g(t, x) + f(t), \quad t \geq 0, \\ x(t_0) &= x_0, \quad t_0 \geq 0 \end{aligned} \tag{1.11}$$

where a, f and g are continuous with $a(t) \geq 0, t \geq 0$ and $f(t)$ is unbounded. By displaying a suitable Lyapunov function, one may arrive at the inequality

$$V'(x) \leq -\alpha(t) W(x) + M, \tag{1.12}$$

where M is a constant. But, this will put so much weight on the size of $-a(t)$, which makes it impossible for $-\alpha(t) \leq 0$ to hold for all $t \geq 0$. As we shall show in Example 3.1, the condition $-\alpha(t) \leq 0$ may not hold for all $t \geq 0$, for the right choice of $f(t)$ and $a(t)$ despite the fact that $a(t)$ might be unbounded.

Currently, the author is extending the content of this paper to functional differential equations with bounded or unbounded delays.

2. Boundedness and Exponential Asymptotic Stability

In this section we use nonnegative Lyapunov type functions and establish sufficient conditions to obtain boundedness results on all solutions x of the dynamical system

$$\dot{x} = f(t, x), \quad t \geq 0, \tag{2.1}$$

subject to the initial conditions

$$x(t_0) = x_0, \quad t_0 \geq 0, \ \ x_0 \in \mathbb{R}^n, \tag{2.2}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear continuous function in t and x, where $t \in \mathbb{R}^+$. Here $\mathbb{R}^n$ is the n-dimensional Euclidean vector space, $\mathbb{R}^+$ is the set of all nonnegative real numbers, $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$. Also, in the case $f(t, 0) = 0$, we obtain conditions that insure the zero solution of (2.1) and (2.2) is exponentially asymptotically stable. For more on boundedness and stability, we refer the interested reader to [3, 7, 9, 10, 14]. In the spirit of the work in [11, 12], in this investigation, we establish sufficient conditions that yield all solutions of (2.1) and (2.2) are uniformly bounded. We achieve this by assuming the existence of a Lyapunov function that is bounded below and above and that its derivative along the trajectories of (2.1) to be bounded by a negative definite function, plus a positive continuous function that might be unbounded. From this point forward, if a function is written without its argument, then the argument is assumed to be t.

Definition 2.1. We say that solutions of system (2.1) are bounded, if any solution $x(t, t_0, x_0)$ of (2.1) and (2.2) satisfies

$$||x(t, t_0, x_0)|| \leq C\Big(||x_0||, t_0\Big) \quad \text{for all } t \geq t_0,$$

where $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant that depends on t_0 and x_0. We say that solutions of system (2.1) are uniformly bounded if C is independent of t_0.

Definition 2.2. We say $V : \mathbb{R}^n \to \mathbb{R}^+$ is a **"type I" Lyapunov function** on $\mathbb{R}^n$ provided

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \ldots + V_n(x_n),$$

where each $V_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable and $V_i(0) = 0$. If x is any solution of system (2.1) and (2.2), then for a continuously differentiable function

$$V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+,$$

we define the derivative V' of V by

$$V'(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} f_i(t, x).$$

In [12], the author proved the following theorem.

Theorem 2.3. [7] Let D be a set in $\mathbb{R}^n$. Suppose there exists a type I Lyapunov function $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$ that satisfies

$$\lambda_1 ||x||^p \le V(t,x) \le \lambda_2 ||x||^q \tag{2.3}$$

and

$$V'(t,x) \le -\lambda_3 ||x||^r + L \tag{2.4}$$

for some positive constants $\lambda_1, \lambda_2, \lambda_3, p, q, r$ and L. Moreover, if for some constant $\gamma \ge 0$ the inequality

$$V(t,x) - MV^{r/q}(t,x) \le \gamma$$

holds for $M = \lambda_3/\lambda_2^{r/q}$, then all solutions of (2.1) and (2.2) that stay in D satisfy

$$||x|| \le \left\{\frac{1}{\lambda_1}\right\}^{1/p} \left[\lambda_2 ||x_0||^q + \frac{M\gamma + L}{M}\right]^{\frac{1}{p}}$$

for all $t \ge t_0$. In this paper we are interested in proving similar theorems, where the constant L in (2.4) is replaced by a continuous function $\beta(t)$, where $\beta(t)$ might be unbounded. Thus, we have the following theorem.

Theorem 2.4. Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$:

$$W(\|x\|) \le V(x) \le \phi(\|x\|), \tag{2.5}$$

$$V'(t,x) \le -\alpha(t)\psi(\|x\|) + \beta(t), \tag{2.6}$$

$$V(x) - \psi(\phi^{-1}(V(x))) \le \gamma, \tag{2.7}$$

where W, ϕ, ψ are continuous functions such that $\phi, \psi, W : [0, \infty) \to [0, \infty)$, $\alpha, \beta : [0, \infty) \to [0, \infty)$ and continuous in t, ψ, ϕ and W are strictly increasing, γ is nonnegative constant. Then all solutions of (2.1) and (2.2) that stay in D satisfy

$$||x(t)|| \le W^{-1}\Big\{V(t_0, x_0)e^{-\int_{t_0}^t \alpha(s)ds} + \int_{t_0}^t \Big(\gamma\alpha(u) + \beta(u)\Big)\, e^{-\int_u^t \alpha(s)ds}\, du\Big\}$$

for all $t \ge t_0$.

Proof. Let x be a solution to (2.1) and (2.2) that stays in D for all $t \ge t_0 \ge 0$. Consider

$$\begin{aligned}
\frac{d}{dt}\Big(e^{\int_{t_0}^t \alpha(s)ds} V(t,x)\Big) &= \Big[V'(t,x) + \alpha(t)V(t,x)\Big] e^{\int_{t_0}^t \alpha(s)ds} \\
&\le \Big(- \alpha(t)\psi(\|x(t)\|) + \beta(t) + \alpha(t)V(t,x\Big) e^{\int_{t_0}^t \alpha(s)ds}, \quad \text{by (2.6)} \\
&\le \Big(- \alpha(t)\psi(\phi^{-1}(V(t,x))) + \beta(t) + \alpha(t)V(t,x)\Big) e^{\int_{t_0}^t \alpha(s)ds} \\
&= \Big[\alpha(t)\Big(V(t,x) - \psi(\phi^{-1}(V(t,x)))\Big) + \beta(t)\Big] e^{\int_{t_0}^t \alpha(s)ds}, \quad \text{by (2.5)} \\
&\le \Big(\gamma\alpha(t) + \beta(t)\Big) e^{\int_{t_0}^t \alpha(s)ds}, \quad \text{by (2.7)}.
\end{aligned}$$

Integrating both sides from t_0 to t with $x_0 = x(t_0)$, we obtain, for $t \in [t_0, \infty)$,

$$V(t,x)e^{\int_{t_0}^{t}\alpha(s)ds} \leq V(t_0,x_0) + \int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\, e^{\int_{t_0}^{u}\alpha(s)ds}\, du.$$

It follows that

$$V(t,x) \leq V(t_0,x_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + \int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\, e^{-\int_{u}^{t}\alpha(s)ds}\, du$$

for all $t \in [t_0, \infty)$. Thus by (2.5), we arrive at the formula

$$||x(t)|| \leq W^{-1}\Big\{\Big(V(t_0,x_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + \int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\, e^{-\int_{u}^{t}\alpha(s)ds}\, du\Big\}$$

for all $t \geq t_0$. This concludes the proof. ■

We now provide a special case of Theorem 2.4 for certain functions ϕ and ψ.

Theorem 2.5. Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$:

$$\|x\|^p \leq V(t,x) \leq \|x\|^q, \tag{2.8}$$

$$V'(t,x) \leq -\alpha(t)\|x\|^r + \beta(t), \tag{2.9}$$

$$V(t,x) - V^{r/q}(t,x) \leq \gamma, \tag{2.10}$$

where $\alpha(t)$ and $\beta(t)$ are nonnegative and continuous functions, p, q, r are positive constants, γ is nonnegative constant. Then all solutions of (2.1) and (2.2) that stay in D satisfy

$$||x(t)|| \leq \Big\{V(t_0,x_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + \int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\, e^{-\int_{u}^{t}\alpha(s)ds}\, du\Big\}^{1/p} \tag{2.11}$$

for all $t \geq t_0$.

Proof. Let x be a solution to (2.1) and (2.2) that stays in D for all $t \geq t_0 \geq 0$. Consider

$$\begin{aligned}\frac{d}{dt}\Big(e^{\int_{t_0}^{t}\alpha(s)ds}V(t,x)\Big) &= \Big[V'(t,x)+\alpha(t)V(t,x)\Big]e^{\int_{t_0}^{t}\alpha(s)ds}\\ &\leq \Big(-\alpha(t)\|x\|^r + \beta(t) + \alpha(t)V(t,x)\Big)e^{\int_{t_0}^{t}\alpha(s)ds}.\end{aligned}$$

From condition (2.5) we have $||x||^q \geq V(t,x)$, and consequently $-||x||^r \leq -V^{r/q}(t,x)$. Thus, by using (2.7) we get

$$\begin{aligned}\frac{d}{dt}\Big(e^{\int_{t_0}^{t}\alpha(s)ds}V(t,x)\Big) &\leq \Big(\alpha(t)(V(t,x)-V^{r/q}(t,x)) + \beta(t)\Big)e^{\int_{t_0}^{t}\alpha(s)ds}\\ &\leq \Big(\alpha(t)\gamma + \beta(t)\Big)e^{\int_{t_0}^{t}\alpha(s)ds}.\end{aligned}$$

By integrating both sides from t_0 to t, the rest of the proof follows along the lines of the proof of Theorem 2.4. This concludes the proof. ∎

The next theorem is an immediate consequence of Theorem 2.5 and hence we omit its proof.

Theorem 2.6. Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$:

$$\|x\|^p \leq V(t, x), \tag{2.12}$$

$$V'(t, x) \leq -\alpha(t)V^q(t, x) + \beta(t), \tag{2.13}$$

$$V(t, x) - V^q(t, x) \leq \gamma, \tag{2.14}$$

where $\alpha(t)$ and $\beta(t)$ are nonnegative and continuous functions, p, q are positive constants, γ is nonnegative constant. Then all solutions of (2.1)–(2.2) that stay in D satisfy

$$||x(t)|| \leq \left\{V(t_0, x_0)e^{-\int_{t_0}^{t}\alpha(s)ds} + \int_{t_0}^{t}\Big(\gamma\alpha(u) + \beta(u)\Big)\, e^{-\int_{u}^{t}\alpha(s)ds}\, du\right\}^{1/p} \tag{2.15}$$

for all $t \geq t_0$. From formula (2.11) we deduce a wealth of information regarding the qualitative behavior of all solutions of (2.1) and (2.2). But first, we make the following definition.

Definition 2.7. Suppose $f(t, 0) = 0$. We say the zero solution of (2.1), (2.2), $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, is α-exponentially asymptotically stable if there exists a positive continuous function $\alpha(t)$ such that $\displaystyle\int_{t_0}^{t}\alpha(s)ds \to \infty$ as $t \to \infty$ and constants d and $C \in \mathbb{R}^+$ such that for any solution $x(t, t_0, x_0)$ of (2.1), (2.2)

$$\|x(t, t_0, x_0)\| \leq C\Big(||x_0||, t_0\Big)\left(e^{-\int_{t_0}^{t}\alpha(s)ds}\right)^d \quad \text{for all } t \in [t_0, \infty).$$

The zero solution of (2.1) is said to be α-uniformly exponentially asymptotically stable if C is independent of t_0.

Corollary 2.8. Assume either of the hypothesis of Theorem 2.5 or Theorem 2.6 hold.

i) If

$$\int_{t_0}^{t}\Big(\gamma\alpha(u) + \beta(u)\Big)\, e^{-\int_{u}^{t}\alpha(s)ds}\, du \leq M, \quad t \geq t_0 \geq 0 \tag{2.16}$$

for some positive constant M, then all solutions of (2.1) and (2.2) are uniformly bounded.

ii) If

$$f(t,0)=0,$$

$$\int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\,e^{\int_{t_0}^{u}\alpha(s)ds}\,du \le M \tag{2.17}$$

for some positive constant M, and

$$\int_{t_0}^{t}\alpha(s)ds \to \infty \;\text{ as }\; t\to\infty \;\text{ for all }\; t\ge t_0, \tag{2.18}$$

then the zero solution of (2.1) is α-exponentially asymptotically with $d = 1/p$.

Proof. Let x be a solution to (2.1), (2.2) that stays in D for all $t \ge t_0 \ge 0$. Hence the proof of *i)* is an immediate consequence of inequality (2.16). For the proof of *ii)*, we consider the inequality from the proof of Theorem 2.4

$$V(t,x)e^{\int_{t_0}^{t}\alpha(s)ds} \;\le\; V(t_0,x_0)+\int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\,e^{\int_{t_0}^{u}\alpha(s)ds}\,du$$

for all $t \in [t_0,\infty)$. This yields

$$V(t,x) \;\le\; \Big[V(t_0,x_0)+\int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\,e^{\int_{t_0}^{u}\alpha(s)ds}\,du\Big]e^{-\int_{t_0}^{t}\alpha(s)ds}$$

for all $t \in [t_0,\infty)$. Using (2.5) we have

$$||x(t)|| \le \Big[V(t_0,x_0)+\int_{t_0}^{t}\Big(\gamma\alpha(u)+\beta(u)\Big)\,e^{\int_{t_0}^{u}\alpha(s)ds}\,du\Big]^{1/p}e^{-\frac{1}{p}\int_{t_0}^{t}\alpha(s)ds}.$$

This concludes the proof. ■

Remark 2.9. If $f(t,0)\ne 0$ and (2.17) and (2.18) hold, then all solutions of (2.1) decay exponentially to zero.

3. Examples

In this section we give three examples as application to our theorems, where at times, we consider $\beta(t)$ to be unbounded.

Example 3.1. For $a(t)\ge 0$, we consider the scalar semi-linear differential equation

$$x' = -\Big(a(t)+\frac{7}{6}\Big)x + b(t)x^{1/3} + h(t), \;\; x(t_0)=x_0, \;\; t\ge t_0\ge 0. \tag{3.1}$$

Let $V(t,x) = x^2$. Then along solutions of (3.1) we have

$$
\begin{aligned}
V'(t,x) &= 2xx' \\
&= -2\left(a(t) + \frac{7}{6}\right)x^2 + 2b(t)x^{4/3} + 2xh(t) \\
&\le -2\left(a(t) + \frac{7}{6}\right)x^2 + 2|b(t)|x^{4/3} + x^2 + h^2(t). \qquad (3.2)
\end{aligned}
$$

To further simplify (3.2), we make use of Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$
wz \le \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } \frac{1}{e} + \frac{1}{f} = 1.
$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$
\begin{aligned}
2|b(t)|x^{4/3} &\le 2\Big[\frac{1}{3}|b(t)|^3 + \frac{(x^{4/3})^{3/2}}{3/2}\Big] \\
&= \frac{4}{3}x^2 + \frac{2}{3}|b(t)|^3.
\end{aligned}
$$

By substituting the above inequality into (3.2), we arrive at

$$
\begin{aligned}
V'(t,x) &\le -2a(t)x^2 + \frac{2}{3}|b(t)|^3 + h^2(t) \\
&= -\alpha(t)x^2 + \beta(t),
\end{aligned}
$$

where $\alpha(t) = 2a(t)$ and $\beta(t) = \frac{2}{3}|b(t)|^3 + h^2(t)$. One can easily check that conditions (2.8)–(2.10) of Theorem 2.5 are satisfied with $p = q = r = 2$ and $\gamma = 0$. Let $a(t) = t/2$, $b(t) = t^{1/3}$ and $h(t) = t^{1/2}$. Then condition (2.16) implies that

$$
\int_{t_0}^{t} \Big(\gamma\alpha(u) + \beta(u)\Big)\, e^{-\int_u^t \alpha(s)ds}\, du = \frac{5}{3}\int_{t_0}^{t} u\, e^{-\int_u^t s ds}\, du \le \frac{5}{3},
$$

$t \ge t_0 \ge 0$. Hence, condition (2.16) is satisfied, which implies that all solutions of

$$
x' = -\left(\frac{t}{2} + \frac{7}{6}\right)x + t^{1/3}x^{1/3} + t^{1/2}, \quad x(t_0) = x_0, \quad t \ge t_0 \ge 0 \qquad (3.3)
$$

are uniformly bounded. On the other hand, if we let $a(t) = \frac{1}{2}$, $b(t) = e^{-\frac{k}{3}t}$ and $h(t) = 0$, $t \ge t_0 \ge 0$ and $k > 1$, then we have $\alpha(t) = 1$ and $\beta(t) = \frac{2}{3}e^{-kt}$. One can easily see that conditions (2.17) and (2.18) are satisfied, and hence the zero solution of

$$
x' = -\frac{5}{3}x + e^{-\frac{k}{3}t}x^{1/3}, \quad x(t_0) = x_0, \quad t \ge t_0 \ge 0 \qquad (3.4)
$$

is 1-exponentially asymptotically stable.

Next, for the sake of comparison, we will use the same Lyapunov function and the same values for $a(t), b(t)$ and $h(t)$, as in Example 3.1, and try to get a differential inequality similar to (1.8) where the condition $-\alpha(t) \leq 0$ will not hold for all $t \geq 0$.

Suppose $|b(t)| \leq p_1(t)q_1$ and $|h(t)| \leq p_2(t)q_2$ for some positive unbounded functions $p_1(t)$, $p_2(t)$ and positive constants q_1 and q_2. Let $V(t,x)$ be as in Example 3.1. Then along the solutions of (3.1) we have

$$\begin{aligned} V'(t,x) &\leq -2\left(a(t)+\frac{7}{6}\right)x^2 + 2p_1(t)q_1x^{4/3} + 2p_2(t)q_2|x(t)| \\ &\leq -2\left(a(t)+\frac{7}{6}\right)x^2 + 2p_1(t)q_1x^{4/3} + p_2^2(t)x^2(t) + q_2^2. \end{aligned}$$

Also, by making use of Young's inequality, we have

$$2p_1(t)q_1x^{4/3} \leq \frac{4}{3}p_1^{3/2}(t)x^2 + \frac{2}{3}q_1^3.$$

With this in mind, we have

$$\begin{aligned} V'(t,x) &\leq \left[-2a(t) - \frac{7}{3} + \frac{4}{3}p_1^{3/2}(t) + p_2^2(t)\right]x^2(t) + \frac{2}{3}q_1^3 + q_2^2 \\ &\leq -\alpha(t)x^2(t) + M, \end{aligned}$$

where $\alpha(t) = 2a(t) + \frac{7}{3} - \frac{4}{3}p_1^{3/2}(t) - p_2^2(t)$, and $M = \frac{2}{3}q_1^3 + q_2^2$ is a positive constant. Let $a(t) = t/2$, $b(t) = t^{1/3}$ and $h(t) = t^{1/2}$. Then

$$\alpha(t) = 2a(t) + \frac{7}{3} - \frac{4}{3}p_1^{3/2}(t) - p_2^2(t) = \frac{7}{3} - \frac{4}{3}t^{1/2} < 0$$

for $t > \frac{\sqrt{7}}{2}$. That is, $-\alpha(t) > 0$ for $t > \frac{\sqrt{7}}{2}$.

Thus, condition (1.12) cannot hold. We conclude that the current available literature which makes use of conditions similar to (1.8) cannot be applied to our example. Yet, according to our theorems, the solutions are uniformly bounded. ■

In Example 3.1, condition (2.10) did not come into play, which was due to the fact that $r = q = 2$. In the next example, we consider a nonlinear system in which condition (2.3) naturally comes into play.

Example 3.2. Let $D = \{x \in \mathbb{R} : ||x|| \geq 1\}$. For $a(t) \geq \frac{5}{12}$ and continuous $h(t)$, consider the nonlinear differential equation

$$x' = -a(t)x^3 + b(t)x^{1/3} + h(t), \quad t \geq 0, \quad x(0) = 1.$$

Consider the Lyapunov functional $V(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^+$ such that $V(t,x) = x^2$. Then along solutions of the differential equation we have

$$\begin{aligned} V' &= 2xx' \\ &= -2a(t)x^4 + 2b(t)x^{4/3} + 2xh(t) \\ &\leq -2a(t)x^4 + 2|b(t)|x^{4/3} + 2|x||h(t)|. \end{aligned} \tag{3.5}$$

Using Young's inequality with $e = 3$ and $f = 3/2$, we get

$$|x|^{4/3}|b(t)| \leq \frac{x^4}{3} + \frac{2}{3}|b(t)|^{3/2}.$$

By a similar argument we have

$$2|x||h(t)| \leq \frac{x^4}{2} + \frac{3|h|^{4/3}}{2}.$$

Hence

$$\begin{aligned} V'(t,x) &\leq \left(-2a(t) + \frac{5}{6}\right)x^4 + \frac{2}{3}|b(t)|^{3/2} + \frac{3|h|^{4/3}}{2} \\ &= -\alpha(t)x^4 + \beta(t), \end{aligned}$$

where $\alpha(t) = 2a(t) - \frac{5}{6}$ and $\beta(t) = \frac{2}{3}|b(t)|^{3/2} + \frac{3|h(t)|^{4/3}}{2}$. Hence, we have $p = q = 2$ and $r = 4$. Thus, for $x \in D$

$$V(t,x) - V^{r/q}(t,x) = x^2(1 - x^2) \leq 0.$$

Thus, condition (2.3) is satisfied for $\gamma = 0$. Thus, by inequality (2.11) we have

$$\begin{aligned} ||x(t)|| \leq \Big\{ & e^{-\int_{t_0}^t (2a(s)-\frac{5}{6})ds} \\ & + \int_{t_0}^t \left(\frac{2}{3}|b(u)|^{3/2} + \frac{3|h(u)|^{4/3}}{2}\right) e^{-\int_u^t (2a(s)-\frac{5}{6})ds}\, du \Big\}^{1/p} \end{aligned}$$

for all $t \geq t_0$.

We see that if the right side of the above inequality is uniformly bounded, then all solutions that are in D are bounded. As a matter of fact, we have that every solution x with $x(t) \in D$ satisfies

$$\begin{aligned} 1 &\leq |x(t)| \\ &\leq \left\{ e^{-\int_{t_0}^t (2a(s)-\frac{5}{6})ds} + \int_{t_0}^t \left(\frac{2}{3}|b(u)|^{3/2} + \frac{3|h(u)|^{4/3}}{2}\right) e^{-\int_u^t (2a(s)-\frac{5}{6})ds}\, du \right\}^{1/p}, \end{aligned} \tag{3.6}$$

for all $t \geq t_0$.

Let $a(t) = \dfrac{t}{2} + \dfrac{5}{6}$, $b(t) = t^{2/3}$ and $h(t) = t^{3/4}$. Then inequality (3.5) implies that all solutions of the nonlinear differential equation

$$x' = -\left(\frac{t}{2} + \frac{5}{6}\right) x + t^{2/3} x^{1/3} + t^{3/4}, \quad x(t_0) = 1, \quad t \ge t_0 \ge 0$$

satisfy

$$1 \le |x(t)| \le \left\{1 + \frac{13}{6}\right\}^{1/2} \qquad \text{for all } t \ge t_0.$$

As an application of Theorem 2.6, we furnish the following example.

Example 3.3. Let $D = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \ge 2\}$. For $a(t) > 0$, consider the following two dimensional system

$$\begin{aligned} y_1' &= y_2 - a(t) y_1 |y_1| + \frac{y_1 h_1(t)}{1 + y_1^2} \\ y_2' &= -y_1 - a(t) y_2 |y_2| + \frac{y_2 h_2(t)}{1 + y_2^2} \\ y_1(t_0) &= a_1, \quad y_2(t_0) = b_1 \ \text{ for } \ t_0 \ge 0 \end{aligned}$$

such that $a_1^2 + b_1^2 = 2$. Let us take $V(t, y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$. Then

$$\begin{aligned} V'(t, y_1, y_2) &= -a(t) y_1^2 |y_1| - a(t) y_2^2 |y_2| + \frac{y_1^2 h_1(t)}{1 + y_1^2} + \frac{y_2^2 h_2(t)}{1 + y_2^2} \\ &\le -a(t) \left(|y_1|^3 + |y_2|^3\right) + (|h_1(t)| + |h_2(t)|) \\ &= -2a(t) \left[\frac{|y_1|^3}{2} + \frac{|y_2|^3}{2}\right] + (|h_1(t)| + |h_2(t)|) \\ &= -2a(t) \left[\frac{\left(|y_1|^2\right)^{3/2}}{2} + \frac{\left(|y_2|^2\right)^{3/2}}{2}\right] + (|h_1(t)| + |h_2(t)|) \\ &\le -2a(t) \left(|y_1|^2 + |y_2|^2\right)^{3/2} 2^{-3/2} + (|h_1(t)| + |h_2(t)|) \\ &= -2a(t) V^{3/2}(t, y_1, y_2) + \beta(t), \end{aligned}$$

where we have used the inequality $\left(\dfrac{a+b}{2}\right)^l \le \dfrac{a^l}{2} + \dfrac{b^l}{2}$, $a, b > 0$, $l > 1$. Thus, we have that $p = 1$ and $q = 3/2$. To verify (2.19), we note that for $y = (y_1, y_2) \in D$, we have

$$V(y) - V^q(y) = \frac{y_1^2 + y_2^2}{2}\Big(1 - \frac{(y_1^2 + y_2^2)^{1/2}}{\sqrt{2}}\Big) \le 0.$$

Hence, for $\gamma = 0$, $\alpha = 2a(t)$ and $\beta(t) = (|h_1(t)| + |h_2(t)|$, all the conditions of Theorem 2.6 are satisfied. We conclude that all solutions of the above two dimensional system that are in D satisfy,

$$2 \leq |y(t)| \leq \left\{2e^{-\int_{t_0}^{t}(2a(s))ds} + \int_{t_0}^{t} \Big(|h_1(u)| + |h_2(u)|\Big) \, e^{-\int_u^t 2a(s)ds} \, du\right\}^{1/p}$$

for all $t \geq t_0$.

4. Comparison

Recently, fixed point theory has been revived to alleviate some of the restrictions on the coefficient $\alpha(t)$, or $a(t)$ in (1.11). When using Lyapunov function of functionals method, the condition that $a(t) \geq 0$ for all $t \geq t_0$ is a pointwise condition. In several papers, Burton used fixed point theory to study the stability of the trivial solution and boundedness of solutions of functional differential equations. The purpose was to relax the pointwise condition on $a(t)$ and replace it with a weaker condition which is the average $\int_{t_0}^{t} a(s)ds \geq 0$. In particular, Burton and Furumochi, [2], used the contraction mapping principle and showed that under certain conditions, all the solutions of the delay differential equation

$$\dot{x} = -a(t)x(t) + b(t)g(x(t-h)) + f(t), \quad t \geq 0, \tag{4.1}$$

are bounded. They make the remark that regular Lyapunov functional arguments will not work unless the functions $a(t)$, $b(t)$, and $f(t)$ are bounded. For the sake of comparison, we consider the nonlinear differential equation

$$\begin{aligned} \dot{x} = -a(t)x(t) + b(t)g(x(t)) + f(t), \quad t \geq 0, \\ x(t_0) = x_0, \quad t_0 \geq 0, \end{aligned} \tag{4.2}$$

where all functions are continuous on their respective domains. We adjust [2, Theorem 9.1], which is about the solutions of (4.1) so that it can be applied to (4.2) and then compare it with our results. First we make the following assumptions. Suppose there are positive constants M, L, K, μ with

$$\int_0^t e^{-\int_s^t a(u)du}|f(s)|ds \leq M, \tag{4.3}$$

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \leq L, \tag{4.4}$$

$$|g(x) - g(y)| \leq \mu|x - y|, \tag{4.5}$$

$$\mu LK + M + 1 < K, \tag{4.6}$$

and for each $t_1 > 0$ and $\varepsilon > 0$ there exists $t_2 > t_1$ such that $t > t_2$ implies

$$e^{-\int_0^t a(u)du} < \varepsilon \text{ and } e^{-\int_{t_1}^t a(u)du} < \varepsilon. \tag{4.7}$$

Theorem 4.1. If (4.3)–(4.7) hold, then all solutions of (4.2) are bounded for large t.

The proof of Theorem 4.1 follows along the lines of the proof of [2, Theorem 9.1] and hence we omit it. Instead, we make the following observations.

1) In relation to equation (3.3) we have $g(x) = x^{\frac{1}{3}}$ and $f(t) = t^{\frac{1}{2}}$. One can easily see that our $g(x)$ cannot satisfy condition (4.5). Actually, $g(x) = x^{\frac{1}{3}}$ is not even differentiable at $x = 0$.

2) Condition (4.6) implies that $\mu L < 1$ which may restrict the type of functions g that can be considered.

3) Our method will not work unless we ask that $a(t) \geq 0$ for all $t \geq t_0$, a condition that Burton replaces with an averaging one $\displaystyle\int_{t_0}^t a(u)du$.

In conclusion, if we consider equations that are similar to (1.11), then our methods in this paper will handle cases when the function f is unbounded provided that the coefficient $a(t)$ is large enough in the negative way and when g is not Lipschitz. Thus, we have shown that our results improve the existing results in the literature when using the traditional way by constructing a suitable Lyapunov function or functional.

References

[1] T. A. Burton. *Stability and periodic solutions of ordinary and functional-differential equations*, volume 178 of *Mathematics in Science and Engineering*, Academic Press Inc., Orlando, FL, 1985.

[2] T. A. Burton and Tetsuo Furumochi. Fixed points and problems in stability theory for ordinary and functional differential equations, *Dynam. Systems Appl.*, 10(1):89–116, 2001.

[3] D. N. Cheban. Uniform exponential stability of linear periodic systems in a Banach space, *Electron. J. Differential Equations*, pages No. 3, 12 pp. (electronic), 2001.

[4] R. D. Driver. *Ordinary and delay differential equations*. Springer-Verlag, New York, 1977. Applied Mathematical Sciences, Vol. 20.

[5] Rodney D. Driver. Existence and stability of solutions of a delay-differential system, *Arch. Rational Mech. Anal.*, 10:401–426, 1962.

[6] John V. Erhart. Lyapunov theory and perturbations of differential equations, *SIAM J. Math. Anal.*, 4:417–432, 1973.

[7] Jack Hale. *Theory of functional differential equations*. Springer-Verlag, New York, second edition, 1977, Applied Mathematical Sciences, Vol. 3.

[8] Philip Hartman. *Ordinary differential equations*, John Wiley & Sons Inc., New York, 1964.

[9] Shigeo Katō. Existence, uniqueness, and continuous dependence of solutions of delay-differential equations with infinite delays in a Banach space, *J. Math. Anal. Appl.*, 195(1):82–91, 1995.

[10] N. M. Linh and V. N. Phat. Exponential stability of nonlinear time-varying differential equations and applications, *Electron. J. Differential Equations*, pages No. 34, 13 pp. (electronic), 2001.

[11] Allan C. Peterson and Youssef N. Raffoul. Exponential stability of dynamic equations on time scales, *Adv. Difference Equ.*, (2):133–144, 2005.

[12] Youssef N. Raffoul. Boundedness in nonlinear differential equations, *Nonlinear Stud.*, 10(4):343–350, 2003.

[13] T. Yoshizawa. *Stability theory and the existence of periodic solutions and almost periodic solutions*, Springer-Verlag, New York, 1975, Applied Mathematical Sciences, Vol. 14.

[14] Taro Yoshizawa. *Stability theory by Liapunov's second method*, Publications of the Mathematical Society of Japan, No. 9. The Mathematical Society of Japan, Tokyo, 1966.