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Coupled Maps: Local Bifurcations of Fixed Points Initiate Global Phase Portrait Changes

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Abstract

We present two map examples such that bifurcations of their fixed point which is embedded in a topologically transitive invariant chaotic set can generate global map phase portrait changes. To be more precise, we consider two coupled map families such that the family maps all have the same fixed point which is nested within the same topologically transitive invariant set which is nested in turn within the same invariant subspace. We prove in such a case that these point bifurcations which are transversal to the invariant subspace generate two periodic of period 2 points in a neighbourhood of the given point and besides can simultaneously give rise to orbits that are homoclinic to the periodic points. These orbits appear suddenly and consist of points of transversal intersections of stable manifolds and unstable ones built up at the periodic points. Therefore, at a moment immediately just after the bifurcation, a countable set of periodic points and, moreover, a whole large invariant topologically transitive set appear in a neighbourhood of the invariant set. Thus, in the case under study, a local bifurcation of fixed point initiates a global one of phase portrait of map.

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1. Introduction and Main Results

Although there are very many results concerning dynamical system properties, one can determine very seldom *a priori* whether a system has a certain property or not. A bright example is orbits homoclinic (biasymptotic) to fixed and periodic points. It is well known how much of dynamical system phase pattern complexity arises in a vicinity of these orbits. However we know very little on causes and conditions favourable for an appearance of such orbits. The present paper is devoted to the mechanics of appearance of new homoclinic orbits at maps used to study a synchronization phenomenon. These are so-called coupled maps. We shall examine further the coupled maps which have a fixed point embedded into the map "diagonal". The latter is their invariant subspace. For such kind of maps, we shall show that these point bifurcations which are transversal to the "diagonal" can initiate a new transversal homoclinic orbit appearance and, as a corollary, generate global changes of the phase map portrait. Thus bifurcations of the fixed point embedded in the invariant subspace (to be more precise, in the invariant topologically transitive set of the given subspace) can differ very strongly from those for isolated fixed points.

So, let $\hat{z} \equiv (\hat{x}, \hat{y}) \in \mathbb{R}^2$. Consider

$$\hat{F} : \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} (1-\varepsilon)\Phi_a(\hat{x}) + \varepsilon\Phi_a(\hat{y}) \\ (1-\varepsilon)\Phi_a(\hat{y}) + \varepsilon\Phi_a(\hat{x}) \end{pmatrix}. \tag{1.1}$$

Here $\Phi_a(t) = at(1-t)$ and $a \in (2, 4)$, $\varepsilon \in (0, 1/2)$ are parameters. Usually ε is called the coupling parameter. This map family is known as one of coupled logistic maps and is used very often as a sample for researching of coupled map properties. It is easy to show that there are parameter values such that the map $\hat{F}$ has a nontrivial topologically transitive attractor on the "diagonal" $\hat{x} = \hat{y}$. Indeed, denote by $\mho \subset (0,4)$ a set of values of a such that $\hat{\Phi}_a : t \mapsto \hat{\Phi}_a(t)$ has the aforesaid attractor provided $a \in \mho$. As is well known [6], $\mho \neq \emptyset$. Moreover, $\mathrm{mes}(\mho) > 0$, where $\mathrm{mes}(\cdot)$ is the Lebesgue measure. Therefore, if $a \in \mho$ and the "diagonal" is an asymptotically stable subspace, then $\hat{F}$ has the aforementioned attractor.

Further we shall deal with the following map family of $\mathbb{R}^2$:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -(1-2\varepsilon)xy \\ a(a-2)/2 - (x^2+y^2)/2 \end{pmatrix}, \tag{1.2}$$

where $z \equiv (x, y) \in \mathbb{R}^2$, and a, ε are the same as in (1.1). The family (1.2) can be obtained from (1.1) by a change of variables (see [4]). At that the "diagonal" $\hat{x} = \hat{y}$ transforms to the y-axis.

Denote $O = (0, 0)$, $\Lambda = [F^2(O), F(O)]$, $P_+ = (0, a-2)$ and let U_{P_+}, U_Λ be neighbourhoods of the point P_+ and the set Λ respectively.

Theorem 1.1. There exists $A \in \left(1 + \sqrt{4 + 2\sqrt{2}}, 4\right)$ so that, for any $a \in (A, 4)$ and arbitrary U_{P_+}, U_Λ, there is $\varepsilon_a \in \left(0, \dfrac{a-3}{2(a-2)}\right)$ such that if $\varepsilon \in \left(\varepsilon_a, \dfrac{a-3}{2(a-2)}\right)$, then the following is satisfied:

1) F has a couple of periodic of period 2 points belonging to $U_{P_+} \setminus \{(x, y) : x = 0\}$;

2) F has a nontrivial, invariant, topologically transitive set belonging to $U_\Lambda \setminus \{(x, y) : x = 0\}$, and periodic points of F are dense on it.

Corollary 1.2. There exists $A \in \left(1 + \sqrt{4 + 2\sqrt{2}}, 4\right)$ so that, for any $a \in (A, 4) \cap \mho$ and arbitrary U_{P_+}, U_Λ, there is $\varepsilon_a \in \left(0, \dfrac{a-3}{2(a-2)}\right)$ such that if $\varepsilon \in \left(\varepsilon_a, \dfrac{a-3}{2(a-2)}\right)$, then the following is satisfied:

1) Λ is a simply connected topologically transitive attractor for the restriction $F|_{\{(x,y):x=0\}}$;

2) F has a couple of periodic of period 2 points belonging to $U_{P_+} \setminus \{(x, y) : x = 0\}$;

3) F has a nontrivial, invariant, topologically transitive set belonging to $U_\Lambda \setminus \{(x, y) : x = 0\}$, and periodic points of F are dense on it.

To show that the same results take place for diffeomorphisms also, let us consider the following maps of $\mathbb{R}^4$:

$$\Upsilon_0 : \begin{pmatrix} x \\ z \\ y \\ t \end{pmatrix} \longmapsto \begin{pmatrix} -(1-2\varepsilon)xy \\ x \\ \{a(a-2) - (x^2 + y^2)\}/2 \\ y \end{pmatrix},$$

$$\Upsilon_b : \begin{pmatrix} x \\ z \\ y \\ t \end{pmatrix} \longmapsto \begin{pmatrix} -(1-2\varepsilon)xy + (1-2\eta)bz \\ x \\ \{a(a-2) - (x^2 + y^2)\}/2 + bt \\ y \end{pmatrix}.$$

Here $0 < |b| < 1$, $\eta \neq 1/2$ and a, ε are the same as above. Let $O = (0, 0, 0, 0)$, $P_+(\Upsilon_b)$ be the fixed point of Υ_b such that $\lim_{b \to 0} P_+(\Upsilon_b) = (0, 0, a-2, a-2) \equiv P_+(\Upsilon_0)$. Denote by Λ_b, $U_{P_+(\Upsilon_b)}$, $U_{\Lambda_b(\Upsilon_b)}$ a closure of the unstable manifold of Υ_b at $P_+(\Upsilon_b)$, a neighbourhood of $P_+(\Upsilon_b)$ and a neighbourhood of $\Lambda_b(\Upsilon_b)$, respectively.

Theorem 1.3. There exist $A \in \left(1 + \sqrt{4 + 2\sqrt{2}}, 4\right)$ and $B > 0$ so that, for any $a \in (A, 4)$, $|b| \in (0, B)$ and arbitrary $U_{P_+(\Upsilon_b)}$, $U_{\Lambda_b(\Upsilon_b)}$, there is $\varepsilon_a \in \left(0, \dfrac{a-3}{2(a-2)}\right)$ such that if $\varepsilon \in \left(\varepsilon_a, \dfrac{a-3}{2(a-2)}\right)$, then the following is satisfied:

1) Υ_b has a couple of periodic of period 2 points belonging to $U_{P_+(\Upsilon_b)} \setminus \{(x, z, y, t) : x = z = 0\}$;

2) Υ_b has a nontrivial, invariant, topologically transitive set belonging to $U_{\Lambda_b(\Upsilon_b)} \setminus \{(x, z, y, t) : x = z = 0\}$, and periodic points of Υ_b are dense on it.

Considering Υ_b one can easily see that the plane $\{(x, z, y, t) : x = z = 0\}$ is invariant with respect to Υ_b and the restriction of Υ_b to this plane $\Upsilon_b|_{\{(x,z,y,t):x=z=0\}}$ is (within the accuracy of change of variables and parameters) the Hénon diffeomorphism.

2. Proof of Theorem 1.1

An idea is to show that, at the moment of bifurcation of fixed point embedded in an invariant subspace, the following takes place: 1) two "saddle" hyperbolic periodic of period 2 points appear outside this subspace in a vicinity of the given fixed point; 2) there are nontrivial points of transversal intersection of the stable manifolds and unstable ones of F which are built up at the periodic points. The latter implies an existence of transversal orbits homoclinic to these points. As is well known [5, 9] an arbitrary small neighbourhood of such kind of orbit contains a nontrivial, invariant, topologically transitive set. Periodic points of F are everywhere dense on this set. Since the stable manifolds of periodic points (due to their definition) are located outside the invariant subspace, the aforesaid homoclinic orbits as well as the invariant sets associated with ones are located outside this subspace too. The phase map portrait bifurcation enlarges upon a whole neighbourhood of the chaotic topologically transitive set because the above mentioned unstable manifolds expand along the whole chaotic set at once as soon as these manifolds appear.

From now on we assume that $a > 3$. Designate a closure by $\mathrm{Cl}(\cdot)$, an interior by $\mathrm{Int}(\cdot)$, a neighbourhood by $U(\cdot)$, an ε-neighbourhood by $U_\varepsilon(\cdot)$ and an origin of coordinates by $O = (0, 0)$. Let $P_+ = (0, a-2)$ be the fixed point of F, $\lambda_1(P_+) = 2 - a$, $\lambda_2(P_+) = (2-a)(1-2\varepsilon)$ be eigenvalues of DF at P_+, $S_\pm = \Big(\pm\sqrt{a(a-2) - (3-4\varepsilon)(1-2\varepsilon)^{-2}}, 1/(1-2\varepsilon)\Big)$ be two periodic of period 2 points of F and $\mu_\imath(S_\pm)$, $\imath \in \{1, 2\}$ be eigenvalues of DF^2 at $S_\pm$. One can easily see that $S_\pm$ exists for any $a > 3$ provided $\varepsilon > 0$ such that $1/(1-2\varepsilon) < a - 2$. If $1/(1-2\varepsilon) \to a - 2$, then $S_\pm \to P_+$ and coincides with the latter provided $\varepsilon = \varepsilon_0$, where $\varepsilon_0 \equiv \dfrac{a-3}{2(a-2)}$.

Lemma 2.1. There exists $\tilde{\varepsilon} \in (0, \varepsilon_0)$ such that $0 < \mu_1(S_\pm) < 1 < \mu_2(S_\pm)$ for all $\varepsilon \in (\tilde{\varepsilon}, \varepsilon_0)$.

Proof. It is easy to see that $\mu_\imath(S_\pm)$, $\imath \in \{1, 2\}$ are solutions of the equation

$$\begin{aligned} 0 = &\left[\mu + a(a-2)(1-2\varepsilon) - 3 - (1-2\varepsilon)^{-1}\right] \\ &\times \left[\mu + a(a-2)(1-2\varepsilon) - 2 - (1-2\varepsilon)^{-1} - (1-2\varepsilon)^{-2}\right] \end{aligned}$$

$$+\,4\varepsilon^2/(1-2\varepsilon)\left[a(a-2)-2/(1-2\varepsilon)-(1-2\varepsilon)^{-2}\right].$$

$\mu_\imath(\varepsilon) \to \mu_\imath(\varepsilon_0)$ as $\varepsilon \nearrow \varepsilon_0$. Here $\mu_1(\varepsilon_0) = 1$, $\mu_2(\varepsilon_0) = (a-2)^2 > 1$. It is clear too that $\mu_2(\varepsilon) > 1$ for $\varepsilon < \varepsilon_0$ close enough to ε_0. Let us show that $\mu_1(\varepsilon) < 1$ for the same ε. Choose ε so that $\gamma \equiv a - 2 - 1/(1-2\varepsilon) > 0$ is small enough. Accurate to $\mathcal{O}(\gamma^2)$, eigenvalues of DF^2 at $S_\pm$ coincide with solutions of the equation

$$\begin{aligned}0 = \mu^2 - \left[1 + (a-2)^2 - 2\gamma(a^2-2a+2)/(a-2)\right]\mu \\ + (a-2)^2 - \gamma(3a^2 - 10a - 8)/(a-2).\end{aligned} \quad (2.1)$$

Solving (2.1) one can find accurate to $o(\gamma)$ that

$$\begin{aligned}\mu_1(\varepsilon) &= \frac{1+(a-2)^2}{2} - \frac{a^2-2a+2}{a-2} \\ &\quad - \sqrt{\frac{[1-(a-2)^2]^2}{4} - \frac{\gamma[(a-2)^2-1](a^2-2a+1)}{a-2}} \\ &= 1 - \frac{\gamma}{a-2}.\end{aligned}$$

Hence $0 < \mu_1(\varepsilon) < 1$ for $\varepsilon_0 - \varepsilon > 0$ small enough. Thus $S_\pm$ are the hyperbolic "saddles". ∎

Denote by $W^s_{\text{loc}}(S_\pm)$, $W^u_{\text{loc}}(S_\pm)$ locally stable and unstable manifolds of F at $S_\pm$. Due to analyticity of F, these manifolds are analytic [1]. Consider $W^u_{\text{loc}}(S_+)$, $W^u_{\text{loc}}(S_-)$. It is easy to see that $W^u_{\text{loc}}(S_+)$, $W^u_{\text{loc}}(S_-)$ are curves symmetric with respect to the y-axis. Let $x = \omega(y)$ be a function whose graph coincides with $W^u_{\text{loc}}(S_+)$. The analyticity of $W^u_{\text{loc}}(S_+)$ implies the one of $x = \omega(y)$. Substituting coordinates of S_+ one can find that $\omega\left(1/(1-2\varepsilon)\right) = \sqrt{a(a-2) - (3-4\varepsilon)(1-2\varepsilon)^{-2}}$. Let us designate the graph of $x = \omega(y)$ by graph(ω) and $\text{Graph}(\omega) = \bigcup_{\jmath=0}^{\infty} F^{2\jmath}(\text{graph}(\omega))$. Since $F\left(W^u(S_\pm)\right) = W^u(S_\mp)$, we have $F : \begin{pmatrix}\omega(y)\\ y\end{pmatrix} \longmapsto \begin{pmatrix}-\omega(\bar y)\\ \bar y\end{pmatrix}$, where

$$\bar y = \frac{a(a-2) - (\omega(y))^2 - y^2}{2}, \quad \omega(\bar y) = (1-2\varepsilon)y\omega(y).$$

Thus the equation

$$\omega\left(\frac{a(a-2)}{2} - \frac{[\omega(y)]^2 + y^2}{2}\right) = (1-2\varepsilon)y\omega(y) \quad (2.2)$$

holds. The latter means an invariance of Graph(ω) with respect to the map

$$\bar F : \begin{pmatrix}x\\ y\end{pmatrix} \longmapsto \begin{pmatrix}(1-2\varepsilon)xy\\ a(a-2)/2 - (x^2+y^2)/2\end{pmatrix}.$$

Denoting

$$Y = \frac{a(a-2)}{2} - \frac{[\omega(y)]^2 + y^2}{2}, \quad \delta = \sqrt{a(a-2) - (3-4\varepsilon)(1-2\varepsilon)^{-2}},$$

we obtain

$$\omega\left(\frac{1}{1-2\varepsilon}\right) = \delta, \quad \omega(Y) = (1-2\varepsilon)y\omega(y). \tag{2.3}$$

Differentiating the latter we find

$$\omega'(Y) = -(1-2\varepsilon)\frac{y\omega'(y) + \omega(y)}{\omega(y)\omega'(y) + y}. \tag{2.4}$$

Changing y, $\omega(y)$ here to $1/(1-2\varepsilon)$, δ respectively and taking into account that $Y = 1/(1-2\varepsilon)$ when $y = 1/(1-2\varepsilon)$, we obtain an equation with respect to $\omega'(1/[1-2\varepsilon])$. Solving it we find

$$\omega'(1/[1-2\varepsilon]) = -\frac{(1-\varepsilon)(1-2\varepsilon)^{-1}}{\delta} \pm \sqrt{(1-\varepsilon)^2\delta^{-2}(1-2\varepsilon)^{-2} - (1-2\varepsilon)}.$$

Since the sign "+" corresponds to a direction tangent to $W^u_{\mathrm{loc}}(S_+)$ at S_+, accurate to $\mathcal{O}(\delta^3)$, we have

$$\omega'\left(\frac{1}{1-2\varepsilon}\right) = -\delta\frac{(1-2\varepsilon)^2}{2(1-\varepsilon)}.$$

Formulas for $\omega''(1/[1-2\varepsilon])$, $\omega'''(1/[1-2\varepsilon])$ can be found in the same way. Keeping in mind that $1/(1-2\varepsilon_0) = a-2$, let us take limits for $\omega'(1/[1-2\varepsilon])$, $\omega''(1/[1-2\varepsilon])$, $\omega'''(1/[1-2\varepsilon])$ as $\varepsilon \nearrow \varepsilon_0$. We find $\lim\limits_{\varepsilon\nearrow\varepsilon_0}\omega'\left(\frac{1}{1-2\varepsilon}\right) = \omega'(a-2) < 0$, $\lim\limits_{\varepsilon\nearrow\varepsilon_0}\omega''\left(\frac{1}{1-2\varepsilon}\right) = \omega''(a-2) < 0$, $\lim\limits_{\varepsilon\nearrow\varepsilon_0}\omega'''\left(\frac{1}{1-2\varepsilon}\right) = \omega'''(a-2) < 0$. Therefore, for ε close enough to ε_0, there is a neighbourhood of $y = 1/(1-2\varepsilon)$ within which $x = \omega(y)$ is a convex function. That is, for the same ε, there exists a neighbourhood of S_+ in which $W^u_{\mathrm{loc}}(S_+)$ is a convex curve. Here and further we call a curve convex (concave) when it is a graph of a convex (concave) function.

Denote $\kappa = \omega'(1/[1-2\varepsilon])$. In view of (2.3) and (2.4), $\delta\kappa^2 + 2\kappa(1-\varepsilon)/(1-2\varepsilon) = -(1-2\varepsilon)\delta$. Since $x = \omega(y)$ is a function convex in a vicinity of $y = 1/(1-2\varepsilon)$, there exist $\tilde{y}_- < 0 < \tilde{y}_+$ such that graph(ω) is located under and to the left of $x = \delta + \kappa\,[y - 1/(1-2\varepsilon)]$ provided $y \in (1/[1-2\varepsilon]+\tilde{y}_-, 1/[1-2\varepsilon]+\tilde{y}_+)$. We show now that the whole graph graph(ω) is located under and to the left of $x = \delta + \kappa\,[y - 1/(1-2\varepsilon)]$ too. Let us make a change of variables $\tilde{x} = x - \delta$, $\tilde{y} = y - 1/(1-2\varepsilon)$. Taking into account that $a(a-2) - (1-2\varepsilon)^{-2} - \delta^2 = 2/(1-2\varepsilon)$, one can easily observe that the map (1.2) takes the following form in coordinates $\tilde{x}$, $\tilde{y}$:

$$\tilde{F} : \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} \longmapsto \begin{pmatrix}\tilde{x} + (1-2\varepsilon)(\delta + \tilde{x})\tilde{y}\\ -\delta\tilde{x} - \tilde{y}/(1-2\varepsilon) - (\tilde{x}^2 + \tilde{y}^2)/2\end{pmatrix}. \tag{2.5}$$

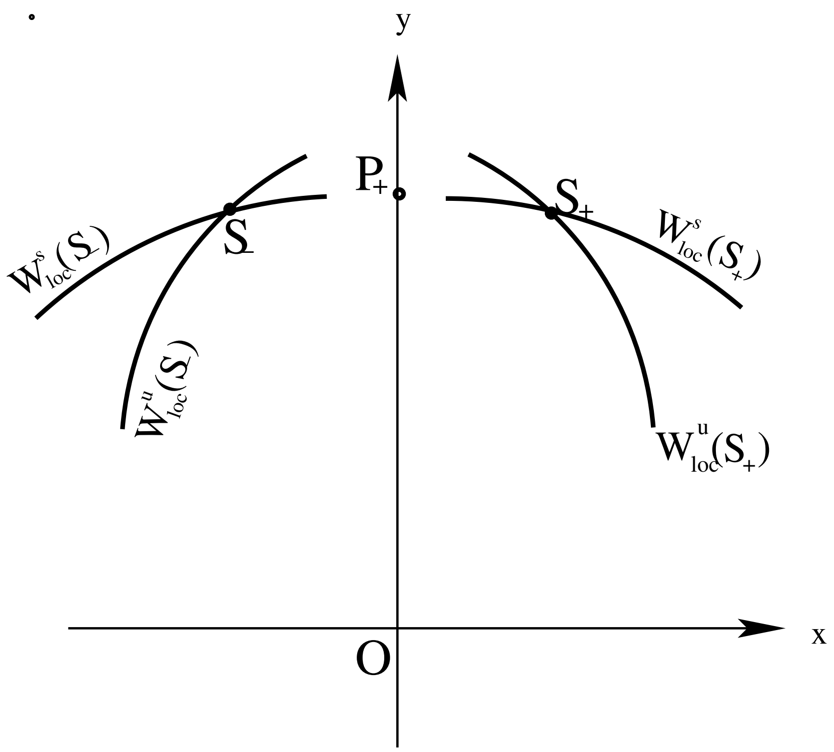

Figure 1.

In doing so the tangent equation simplifies to $\tilde{x} = \kappa\tilde{y}$. Consider $\tilde{F}((\tilde{x} = \kappa\tilde{y}))$. This is a curve $\tilde{x} = q(\tilde{y})$, where

$$\tilde{x} = \kappa t + (1-2\varepsilon)(\delta + \kappa t)t,\ \ \tilde{y} = -\left[1/(1-2\varepsilon) + \kappa\delta\right]t - (1+\kappa^2)t^2/2.$$

We prove that the curve locates under and to the left of $\tilde{x} = \kappa\tilde{y}$. Computing its derivative we find $q'(\tilde{y}) = -\dfrac{\kappa + (1-2\varepsilon)\delta + 2(1-2\varepsilon)\kappa t}{1/(1-2\varepsilon) + \kappa\delta + (1+\kappa^2)t}$. Let us show that $q'(\tilde{y}) > \kappa$ provided $\tilde{y} < 0$ and $q'(\tilde{y}) < \kappa$ when $\tilde{y} > 0$. In view of (2.5) it is clear that

$$\kappa\left[1 + 1/(1-2\varepsilon)(3-4\varepsilon+\kappa^2)t + \kappa\delta\right] < -(1-2\varepsilon)\delta, \qquad \text{for} \qquad t > 0, \tag{2.6}$$

$$\kappa\left[1 + 1/(1-2\varepsilon)(3-4\varepsilon+\kappa^2)t + \kappa\delta\right] > -(1-2\varepsilon)\delta, \qquad \text{for} \qquad t < 0. \tag{2.7}$$

Besides, (2.6) implies $-\kappa - (1-2\varepsilon)\delta - 2(1-2\varepsilon)\kappa t > \kappa/(1-2\varepsilon) + \kappa^2\delta + \kappa(1+\kappa^2)t$. Since $\delta \to 0$, $\kappa \to 0$ as $\varepsilon \nearrow \varepsilon_0$, there exists $\tilde{\varepsilon} < \varepsilon_0$ such that $1/(1-2\varepsilon) > \delta|\kappa|$ when $\varepsilon \in (\tilde{\varepsilon}, \varepsilon_0)$ and $a \in (1+\sqrt{5}, 4)$. It implies in turn that if a and ε are the same as above, then the inequality $1/(1-2\varepsilon) + \kappa\delta + (1+\kappa^2)t > 0$ is satisfied for any $t > 0$. Keeping in mind that $\tilde{y} < 0$ for $t > 0$, we see that $q'(\tilde{y}) > \kappa$ for $\tilde{y} < 0$. On the other hand, in the same way (2.7) implies $-\kappa - (1-2\varepsilon)\delta - 2(1-2\varepsilon)\kappa t < \kappa/(1-2\varepsilon) + \kappa^2\delta + \kappa(1+\kappa^2)t$.

If $-\dfrac{[1/(1-2\varepsilon)+\kappa\delta]}{1+\kappa^2} < t < 0$, then $q'(\tilde{y}) < \kappa$ and simultaneously $\tilde{y} > 0$. However if $t < -\dfrac{[1/(1-2\varepsilon)+\kappa\delta]}{1+\kappa^2}$, then $q'(\tilde{y}) > \kappa$ and simultaneously $\tilde{y} < 0$. Hence the graph $\tilde{x} = q(\tilde{y})$ is indeed located under and to the left of $\tilde{x} = \kappa\tilde{y}$.

Let us study $\tilde{x} = q(\tilde{y})$ in detail. One can easily see that its graph consists of two arcs which merge in one curve at the point $(\tilde{x}_0, \tilde{y}_0)$, where $\tilde{x}_0 = \tilde{x}(t_0)$, $\tilde{y}_0 = \tilde{y}(t_0)$, $t_0 = -\dfrac{[1/(1-2\varepsilon)+\kappa\delta]}{1+\kappa^2}$. One of the arcs is convex and another is concave. Indeed, computing $q''(\tilde{y})$, we find $q''(\tilde{y}) = -\dfrac{(1-\kappa^2)[(1-2\varepsilon)\delta-\kappa]}{\left[1/(1-2\varepsilon)+\kappa\delta+(1+\kappa^2)t\right]^3}$. Keeping this in mind, one can easily infer that $q''(\tilde{y}) < 0$ for $t > t_0$ and $q''(\tilde{y}) > 0$ for $t < t_0$. Then $\lim\limits_{\tilde{y}\to\pm\infty} q'(\tilde{y}) = -2(1-2\varepsilon)\kappa/(1+\kappa^2)$ exists.

Let us locate graph(ω) when $0 \le y \le \sqrt{a(a-2)}$. Consider the quadrangle Δ bounded by segments of the lines $x = 0$, $y = 0$, $y = \sqrt{a(a-2)}$ and $x = \delta + \kappa\,[y - 1/(1-2\varepsilon)]$. Computing $\bar{F}(\Delta)$, we find that $\bar{F}(\Delta)$ is a "curvilinear" quadrangle bounded by arcs of the curves $y = 0$, $y = a(a-2)/2 - x/(1-2\varepsilon)$, $x^2 = -2a(a-2)(1-2\varepsilon)^2 y$ and $\tilde{x} = q(\tilde{y})$. Taking into account mutual positions of $\tilde{x} = \kappa\tilde{y}$ and $\tilde{x} = q(\tilde{y})$, we see that $\bar{F}(\Delta) \cap \{(x, y) : 0 \le y \le \sqrt{a(a-2)}\} \subset \Delta$. Denote $\Delta_n = \bar{F}(\Delta_{n-1}) \cap \{(x, y) : 0 \le y \le \sqrt{a(a-2)}\}$, $n = 1, 2, \ldots$, $\Delta_0 = \Delta$ and $\Delta_* = \bigcap\limits_{n=0}^{\infty} \Delta_n$. By construction, $\{\Delta_n\}_{n=0}^{\infty}$ is a monotone decreasing sequence of nested "curvilinear" quadrangles and $\bar{F}(\Delta_*) \cap \{(x, y) : 0 \le y \le \sqrt{a(a-2)}\} = \Delta_*$. Due to the invariance of Δ_* with respect to $\bar{F}$, it is clear that graph$(\omega) \subset \Delta_*$. This implies

$$\max_{0\le y\le\sqrt{a(a-2)}} \omega(y) < \delta - \kappa/(1-2\varepsilon) \approx \delta\left[1 + 0.5(1-2\varepsilon)^2/(1-\varepsilon)\right] < 2\delta.$$

Let us consider the fraction of boundary of Δ_* which is a limit of sequence of images of the line $x = \delta + \kappa\,[y - 1/(1-2\varepsilon)]$. We denote it by L_ω and show that L_ω is a smooth function graph.

Let $\Xi > 0$ be a constant, $\rho = \sqrt{a(a-2)}$, $I \equiv [0.99\sqrt{\rho^2 - 2\rho}, \rho]$ be a line segment, $x = \chi(y)$ be a C^2-smooth function such that $0 \le \chi(y) < 2\delta < \sqrt[4]{\delta} < 0.4\sqrt{\rho^2 - 2\rho}$, $|\chi'(y)| < \Xi$, $|\chi''(y)| < 100\Xi$ on I. Set $x = \chi_F(y)$ by the formulas $x = (1-2\varepsilon)t\chi(t)$, $y = \left[a(a-2) - t^2 - \chi^2(t)\right]/2$.

Lemma 2.2. Let $a > 1 + \sqrt{4 + 2\sqrt{2}}$ and $x = \chi(y)$ as above. There exists $\varepsilon_F < \varepsilon_0$ so close to ε_0 that a domain of the function $x = \chi_F(y)$ includes I and $|\chi_F'(y)| < \Xi$, $|\chi_F''(y)| < 100\Xi$ for $\varepsilon \in (\varepsilon_F, \varepsilon_0)$.

Proof. Computations give $y(0.99\sqrt{\rho^2 - 2\rho}) > \rho$, $y(\rho) < 0$. Thus, the domain of

$x = \chi_F(y)$ includes I. Computing $|\chi_F'(y)|$, we find

$$|\chi_F'(y)| < (1-2\varepsilon)\left|\frac{t\chi'(t)+\chi(t)}{\chi(t)\chi'(t)+t}\right| < (1-2\varepsilon)\cdot\left(\left|\frac{t\chi'(t)}{\chi(t)\chi'(t)+t}\right|+\left|\frac{\chi(t)}{\chi(t)\chi'(t)+t}\right|\right),$$

where $t \in [0.99\sqrt{\rho^2-2\rho}, \rho]$. If $a > 1+\sqrt{5}$, then $1/(a-2) < (1+\sqrt{5})/4 < 0.9$. So, $1-2\varepsilon < 0.9$ for ε sufficiently close to ε_0. Fix ε_F so that $\delta\Xi < 0.001\sqrt{\rho^2-2\rho}$, $1-2\varepsilon < 0.9$, $\delta^2 < \Xi$ for all $\varepsilon \in (\varepsilon_F, \varepsilon_0)$. In doing so we get $|\chi_F'(y)| < \Xi$ for any $y \in I$. As for $|\chi_F''(y)|$ we can estimate it by

$$\begin{aligned}|\chi_F''(y)| &= (1-2\varepsilon)\left|\frac{[t\chi'(t)-\chi(t)][1-\chi'^2(t)]+\chi''(t)[t^2-\chi^2(t)]}{[\chi(t)\chi'(t)+t]^3}\right|\\ &< (1-2\varepsilon)\left[\frac{|t\chi'(t)-\chi(t)|}{|\chi(t)\chi'(t)+t|^3}+\frac{t^2|\chi''(t)|}{|\chi(t)\chi'(t)+t|^3}\right].\end{aligned}$$

Keeping in mind that $\sqrt{\rho^2-2\rho} = \sqrt{a(a-2)-2\sqrt{a(a-2)}} > 1$ for $a > 1+\sqrt{4+2\sqrt{2}}$, we find $|\chi''(t)| < 100\Xi$. ■

Let us choose $\Xi = 10^{-6}$. As is shown above, values of the x-coordinate of points belonging to images of the line $x = \delta + \kappa\,[y - 1/(1-2\varepsilon)]$ are less than 2δ. Since $|\kappa| < \Xi^2$ for ε_F sufficiently close to ε_0, it means due to Lemma 2.2 that the absolute values of derivatives of these images are uniformly bounded on I and do not exceed 10^{-6}. By construction, the images constitute a monotone sequence. Therefore the sequence limit is a one-valued C^1-smooth function. Obviously its domain includes I and the absolute value of its derivative is less than 10^{-6}. But this is a fraction of L_ω only. However if we consider the $\bar{F}$-image of the given fraction, one can prove then the same for whole L_ω. Let us show now that $L_\omega = \text{graph}(\omega)$. Indeed, by definition, the arc L_ω is a graph of the one-valued C^1-smooth function having uniformly bounded derivative. It is invariant with respect to $\bar{F}$ and includes S_+. However there is a unique curve which is situated in a vicinity of S_+, belongs to Δ_*, is invariant with respect to $\bar{F}$ and has the absolute value of the derivative less than 10^{-6}. It is $W^u_{\text{loc}}(S_+)$. Hence $W^u_{\text{loc}}(S_+) \subset L_\omega$. The contrary is true: $W^u(S_+) \supset L_\omega$. Actually if not, then $\text{Cl}(W^u(S_+)) \setminus W^u(S_+)$ is a couple of periodic of period 2 points which belong to L_ω by construction. Straightforward computations show that these points exist when $\varepsilon \in (0, \hat{\varepsilon})$, where $\hat{\varepsilon} = \left(1-\sqrt{3/[a(a-2)]}\right)/2$, and disappear or, to be more precise, merge with S_+ at $\varepsilon = \hat{\varepsilon}$. Moreover, the same computations show that if $\varepsilon \in (\hat{\varepsilon}, \varepsilon_0)$, then, excluding $S_\pm$, F has no other periodic points in $\mathbb{R}^2 \setminus \{(x, y) : x = 0\}$. Therefore $L_\omega = \text{graph}(\omega)$.

Remark 2.3. The same arguments show $\text{Graph}(\omega) \equiv W^u(S_+)$ provided $\varepsilon \in (\hat{\varepsilon}, \varepsilon_0)$.

Keeping in mind that $\text{Graph}(\omega)$ consists of the $\bar{F}$-images of $\text{graph}(\omega)$, and using (2.3), one can estimate the absolute x-coordinate value of points of $\text{Graph}(\omega)$ by $|\omega(y)| <$

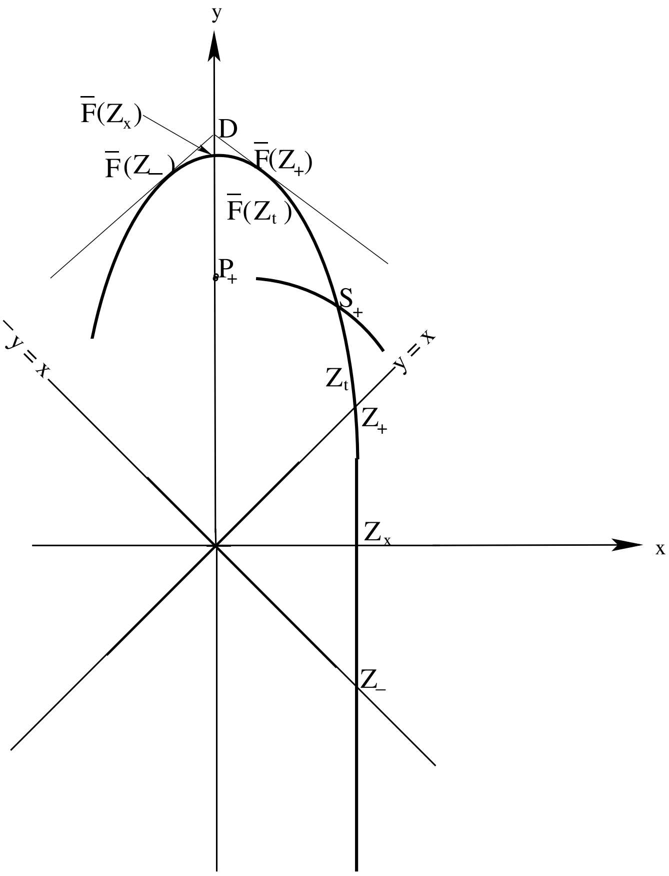

Figure 2.

$2\delta\,[a(a-2)/2]^n$. Here n is the number of iterations of graph(ω). We shall see later that it is enough to consider not more than 5 iterations of L_ω. As for values of derivatives of Graph(ω), those can be estimated in an appropriate way on suitable subsets of Graph(ω). To determine these subsets, one should use a critical set of F [2, 3]. In the case under study, the critical set is $K = \{(x, y) : x = y\} \cup \{(x, y) : x = -y\}$.

Consider $\bar{F}^2(L_\omega)$. Let $t > 0$ be small enough. Denote $Z_+ \equiv (X_+, Y_+) = \bar{F}^2(L_\omega) \cap \{(x, y) : x > 0,\ y > 0\}$, $Z_- \equiv (X_-, Y_-) = \bar{F}^2(L_\omega) \cap \{(x, y) : x > 0,\ y < 0\}$, $Z_t = L_\omega \cap \{(x, y) : y = Y_+ + t\}$ and by $L_{\omega t}$ the arc of Graph(ω) stretching of $\bar{F}^2(Z_t)$ to $\bar{F}(Z_t)$.

Lemma 2.4. Let $a \in (1 + \sqrt{5}, 4)$ and $t > 0$ small enough. There exists $\varepsilon_F > 0$ such that $\max\limits_{(x,y)\in L_{\omega t}} \omega(y) < t^2 \ll t$, $\max\limits_{(x,y)\in L_{\omega t}} |\omega'(y)| < t^2 \ll t$ provided $\varepsilon \in (\varepsilon_F, \varepsilon_0)$.

Proof. Choose $\breve{\varepsilon} > 0$ so close to ε_0 that $\delta < t^4$, $\omega(y) < 0.01\delta$ for all $(x, y) \in L_\omega$, $\varepsilon \in (\breve{\varepsilon}, \varepsilon_0)$. In this case $\omega(y) < \delta/3$ for all $(x, y) \in L_{\omega t}$. Fix $\breve{\varepsilon} > 0$ so that $100\Xi < t^4 \ll t$, $|\omega'(y)| < \Xi$ for all $(x, y) \in L_\omega$, $\varepsilon \in (\breve{\varepsilon}, \varepsilon_0)$ and consider $\bar{F}(Z_t)$. Using (2.4) we find accurate to $\mathcal{O}(\delta^2)$ that

$$\omega'\left(a[a-2]/2 - [\omega(Y_+ + t)]^2/2 - [Y_+ + t]^2/2\right) = -(1-2\varepsilon)\left[\omega'(Y_+ + t) + \frac{\omega(Y_+ + t)}{Y_+ + t}\right].$$

Taking into account $\omega(Y_+ + t) < 0.01t^3$, $|\omega'(Y_+ + t)| < 0.01t^2$, we can see that

$$|1-2\varepsilon|\left|\omega'(Y_+ + t) + \frac{\omega(Y_+ + t)}{Y_+ + t}\right| < \frac{t}{30}$$

of course, provided $\varepsilon \in (\varepsilon_F, \varepsilon_0)$, where $\varepsilon_F = \min(\breve{\varepsilon}, \breve{\varepsilon})$. Computing and estimating $|\omega'(y)|$ on $\bar{F}^2(Z_t)$, we find that it is much less than t. It is clear that analogous reasons can be used for an arbitrary point of $(x, y) \in L_\omega$ with $y > Y_+ + t$. ■

Remark 2.5. To prove that the other fractions of Graph(ω) possess similar properties, one should make arguments similar to those aforesaid. For example, let us consider an arc of Graph(ω) between points $\bar{Z}_t = \bar{F}^2(L_\omega) \cap \{(x, y) : y = Y_- - t\}$ and $\bar{F}^2(Z_t)$. If we try to apply the reasoning presented above to this arc, we can see then that these arguments all remain valid. A crucial circumstance determining the possibility to use the aforesaid arguments to any arc is that the arc points as well as its images all do not belong to the critical set K. Moreover, the arc closure should be separated from the set $\bar{K} \equiv \{(x, y) : -\Xi \le x \le 0, x \le y \le -x\} \cup \{(x, y) : 0 \le x \le \Xi, -x \le y \le x\}$ as well as from its images.

We now discuss the behaviour of $\bar{F}^3(L_\omega)$ inside $\bar{F}(\bar{K})$. Analyzing formulas (2.3) and (2.4), we can see the following. First, $\bar{F}(Z_x) \in \{(x, y) : x = 0\}$, where $Z_x \equiv L_\omega \cap \{(x, y) : y = 0\}$. Second, since $|\omega(y)\omega'(y)|$ is $\mathcal{O}(\delta^2)$ and $Y_\pm = \pm\delta$, there exists $\breve{y} \in (-\delta, \delta)$ such that $\omega(\breve{y})\omega'(\breve{y}) + \breve{y} = 0$. The latter means in turn that a tangent to Graph(ω) is parallel to the x-axis. Third, $\bar{F}(Z_+)$ and $\bar{F}(Z_-)$ are situated on different sides of the y-axis, i.e., $\bar{F}(Z_+) \in \{(x, y) : x > 0\}$, $\bar{F}(Z_-) \in \{(x, y) : x < 0\}$.

We have studied properties of $\bar{F}^3(L_\omega)$ on $\bar{F}^2(L_\omega)$ and within $\bar{F}(\bar{K})$. It is clear that the same approach can be used to research a behaviour of the "remainder" $\bar{F}^3(L_\omega) \setminus \left(\bar{F}^2(L_\omega) \cup \bar{F}(\bar{K})\right)$. Outside a neighbourhood of $\bar{F}(\bar{K}) \cup \bar{F}^2(\bar{K})$, the given "remainder" has the same properties as $L_{\omega t}$. However, inside $\bar{F}^2(\bar{K})$, its behaviour is similar to that of $\bar{F}^3(L_{\omega t})$ inside $\bar{F}(\bar{K})$.

An analogous analysis can be made with respect to properties of $\bar{F}^4(L_\omega)$, $\bar{F}^5(L_\omega)$ and $W^u(S_+)$ in whole. The only thing that should be kept in mind is the following. When a bit of the arc $\bar{F}^3(L_\omega)$ falls inside $\bar{F}(\bar{K})$, this gives rise to a necessity to study its image properties inside $\bar{F}^2(\bar{K})$. Similarly the most essential and principal piece of researching of behaviour of $\bar{F}^4(L_\omega)$ consists in a study of properties of all fractions of $\bar{F}^4(\bar{K}) \setminus \bar{F}^3(\bar{K})$ which have a nonempty intersection with $\bar{F}^2(\bar{K})$. In order to do this, it is necessary to locate where the given intersection image is. In order to locate the latter, it is necessary in turn to consider $\bar{F}^3(\bar{K})$, and so on. However if we analyze reasons which guarantee a validity of the arguments presented above in detail, it becomes clear then that it is enough to study the behaviour of $\bar{F}^2(L_{\omega t})$ only. Among the reasons which should be taken into account, the principal ones are following: 1) a point $\mathcal{P}_a \stackrel{\text{def}}{=} \big(0, a(a-2)[4+2a-a^2]/8\big)$ is a fixed one of F and $\bar{F}$ at $a=4$; 2) there is a continuous dependence of $\bar{F}^j(\bar{K})$, $j=1,2,\dots$ on parameters; 3) for any $j=1,2,\dots,$ $\bar{F}^{j+1}(\bar{K})$ tends to $\mathcal{P}_a$ as $\varepsilon \nearrow \varepsilon_0$, $a \nearrow 4$ in the sense that $\lim\limits_{a\nearrow 4}\lim\limits_{\varepsilon\nearrow\varepsilon_0} \operatorname{dist}(\bar{F}^{j+1}(\bar{K}), \mathcal{P}_a) = 0$. Here $\operatorname{dist}(\cdot,\cdot)$ is the distance.

Consider the local stable manifolds of F at $S_\pm$ now. As well as the unstable manifolds, $W^s_{\text{loc}}(S_+)$, $W^s_{\text{loc}}(S_-)$ are curves symmetric with respect to the y-axis. Let $y=\Omega(x)$ be the function whose graph coincides with $W^s_{\text{loc}}(S_+)$. Since $W^s_{\text{loc}}(S_+)$ is an analytic manifold, $y=\Omega(x)$ is an analytic function. Taking into account coordinates of S_+, we find $\Omega(\delta)=1/(1-2\varepsilon)$. Denote the graph of $y=\Omega(x)$ by graph(Ω). Because $F(W^s(S_\pm)) = W^s(S_\mp)$, we infer that $F : \begin{pmatrix} x \\ \Omega(x)\end{pmatrix} \mapsto \begin{pmatrix} \breve{x} \\ \Omega(\breve{x})\end{pmatrix}$, where $\breve{x} = -(1-2\varepsilon)x\Omega(x)$, $\Omega(\breve{x}) = \big[a(a-2)-(\Omega(x))^2-x^2\big]/2$. Keeping in mind that $\Omega(\breve{x}) = \Omega(-\breve{x})$, we obtain the functional equation $a(a-2)/2 - \{[\Omega(x)]^2+x^2\}/2 = \Omega\left((1-2\varepsilon)x\Omega(x)\right)$. Therefore graph$(\Omega)$ is invariant with respect to $\bar{F}$. Setting $X=(1-2\varepsilon)x\Omega(x)$, we arrive at the system

$$\Omega(\delta) = \frac{1}{1-2\varepsilon}, \qquad \Omega(X) = \frac{a(a-2)}{2} - \frac{[\Omega(x)]^2+x^2}{2}. \tag{2.8}$$

Differentiation of (2.8) gives $-\big[\Omega(x)\Omega'(x)+x\big] = (1-2\varepsilon)[x\Omega'(x)+\Omega(x)]\Omega'(X)$. Changing x, $\Omega(x)$ to δ, $\dfrac{1}{1-2\varepsilon}$ respectively and taking into account that $X=\delta$ when $x=\delta$, we obtain an equation with respect to $\Omega'(\delta)$. Its solution is $\Omega'(\delta) = -\dfrac{1-\varepsilon}{\delta(1-2\varepsilon)^2} \pm \sqrt{\left[\dfrac{1-\varepsilon}{\delta(1-2\varepsilon)^2}\right]^2 - \dfrac{1}{1-2\varepsilon}}$. The sign "+" corresponds to the direction tangent to $W^s_{\text{loc}}(S_+)$ at S_+. Therefore, accurate to $\mathcal{O}(\delta^3)$,

$$\Omega'(\delta) = -\frac{1-\varepsilon}{\delta(1-2\varepsilon)^2} + \sqrt{\left[\frac{1-\varepsilon}{\delta(1-2\varepsilon)^2}\right]^2 - \frac{1}{1-2\varepsilon}} = -\frac{(1-2\varepsilon)\delta}{2(1-\varepsilon)}.$$

Formulas of $\Omega''(\delta)$, $\Omega'''(\delta)$ can be found in a similar way. Taking limit as $\varepsilon \nearrow \varepsilon_0$ in these formulas and keeping in mind that $1/(1-2\varepsilon_0) = a-2$, one can easily check that $\lim\limits_{\varepsilon\nearrow\varepsilon_0} \Omega'(\delta) < 0$, $\lim\limits_{\varepsilon\nearrow\varepsilon_0} \Omega''(\delta) < 0$, $\lim\limits_{\varepsilon\nearrow\varepsilon_0} \Omega'''(\delta) < 0$. Thus, for ε close enough to ε_0, there is a neighbourhood of $x=\delta$ within which $y=\Omega(x)$ is a convex function.

Therefore, for the same ε, there exists a neighbourhood of S_+ such that $W^s_{\mathrm{loc}}(S_+)$ is a convex curve inside this neighbourhood.

Consider a curvilinear quadrangle $AH\bar{F}(H)\bar{F}(A) \subset \{(x, y) : x \geq 0, y \geq 0\}$ which is constituted in such a manner. AH, $\bar{F}(A)\bar{F}(H)$ are arcs of curves $(1-2\varepsilon)^2x^2y^2 + \left[a(a-2) - x^2 - y^2\right]^2/4 = a(a-2)$, $x^2 + y^2 = a(a-2)$ which are adjacent to A and $\bar{F}(A)$, respectively. H, $\bar{F}(H)$ are points of intersection of these arcs with L_ω. In doing so, $H\bar{F}(H)$ is a fraction of L_ω between H and $\bar{F}(H)$, and $A\bar{F}(A)$ is a segment of the y-axis. Obviously, there exist a and ε such that $\bar{F}$ is a diffeomorphism of $AH\bar{F}(H)\bar{F}(A)$ on $\bar{F}(A)\bar{F}(H)\bar{F}^2(H)\bar{F}^2(A)$. Indeed, since $\bar{F}(O) = (0, a(a-2)/2)$, $\bar{F}(P_+) = P_+$ and $a - 2 < \sqrt{a(a-2)} < a(a-2)/2$ provided $a > 1 + \sqrt{5}$, there exists a pre-image of $(0, \sqrt{a(a-2)}) \equiv \bar{F}(A)$ inside the interval (O, P_+). This is A. Let $\hat{a} \in (1 + \sqrt{5}, 4)$, $\eta > 0$ such that $U_\eta(O) \not\ni A$ for all $a \in (\hat{a}, 4)$. Since L_ω converges to a segment of the y-axis uniformly as $\varepsilon \nearrow \varepsilon_0$, there is $\hat{\varepsilon} > 0$ so that $\bar{F}(A)\bar{F}(H) \subset \bar{F}(U_\eta(A))$ for $\varepsilon \in (\hat{\varepsilon}, \varepsilon_0)$. Therefore $AH \subset \{(x, y) : x \geq 0, y > 0\}$. Observing that $\bar{F}(AH) = \bar{F}(A)\bar{F}(H)$, $H\bar{F}(H) \subset \bar{F}(H\bar{F}(H))$, $A\bar{F}(A) \subset \bar{F}(A\bar{F}(A))$, $\bar{F}^2(A)\bar{F}^2(H) \subset \{(x, y) : x \geq 0, y = 0\}$ and taking into account that a restriction of $\bar{F}$ to $\{(x, y) : x \geq 0, y \geq 0\}$ is a diffeomorphism, one can easily find that $\bar{F}$ is a diffeomorphism of $AH\bar{F}(H)\bar{F}(A)$ to $\bar{F}(A)\bar{F}(H)\bar{F}^2(H)\bar{F}^2(A)$.

Lemma 2.6. Let $a > 1 + \sqrt{5}$. Then there exists $\breve{\varepsilon} < \varepsilon_0$ such that $y > x$ for all points $(x, y) \in AH$ provided that $\varepsilon \in (\breve{\varepsilon}, \varepsilon_0)$.

Proof. Fix $\zeta = 0.1\sqrt{a(a-2) - 2\sqrt{a(a-2)}}$. Then $y > x$ for any $(x, y) \in U_\zeta(A)$, where $A = (0, \sqrt{a(a-2) - 2\sqrt{a(a-2)}})$. Choosing $\breve{\varepsilon}$ so close to ε_0 that $\delta \leq \zeta$ for all $\varepsilon \in (\breve{\varepsilon}, \varepsilon_0)$ concludes the proof. ■

Denote the map inverse of $\bar{F}$ on $AH\bar{F}(H)\bar{F}(A)$ by $\bar{F}^{-1}$. Taking into account that $y \geq x \geq 0$ for all $(x, y) \in AH\bar{F}(H)\bar{F}(A)$, we find

$$\bar{F}^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \dfrac{x\sqrt{2}}{1-2\varepsilon}\left[\sqrt{\dfrac{a(a-2)}{2} - y + \dfrac{x}{1-2\varepsilon}} + \sqrt{\dfrac{a(a-2)}{2} - y - \dfrac{x}{1-2\varepsilon}}\right]^{-1} \\ \dfrac{\sqrt{2}}{2}\left[\sqrt{\dfrac{a(a-2)}{2} - y + \dfrac{x}{1-2\varepsilon}} + \sqrt{\dfrac{a(a-2)}{2} - y - \dfrac{x}{1-2\varepsilon}}\right] \end{pmatrix}.$$

Consider $\tilde{W}^s_+ \equiv \bigcup\limits_{j=0}^{\infty} \bar{F}^j \left(W^s_{\mathrm{loc}}(S_+) \cap AH\bar{F}(H)\bar{F}(A)\right)$. Since $\bar{F}^3(AH) \subset \{(x, y) : x = 0\}$ while $\tilde{W}^s_+ \cap \{(x, y) : x = 0\} = \emptyset$, it is clear that $\tilde{W}^s_+$ is situated between AH and $\bar{F}(A)\bar{F}(H)$.

Let us show that $\mathrm{Cl}(\tilde{W}^s_+)$ leans against $\{(x, y) : x = 0\}$. The map $\bar{F}^{-1}$ induces a smooth function map in such a manner. Let $0 < \sigma \ll \sqrt[3]{\sigma} < 10^{-3}$ be small enough. Denote by $\mathcal{S}_\psi$ the space of C^2-smooth functions $y = \psi(x)$ such that

$|\psi(x) - 1/(1-2\varepsilon)| < \sigma$, $|\psi'(x)| < 0.01$, $|\psi''(x)| < 5$ on $\breve{I} \equiv [0,\sigma]$. Define $\Phi : \mathcal{S}_\psi \longmapsto \mathcal{S}_\psi$ by means of the formula $\Phi\left(y = \psi(x)\right) = \left(y_F = \psi_F(x_F)\right)$, where

$$y_F = \frac{\sqrt{2}}{2}\left[\sqrt{\frac{a(a-2)}{2} - \psi(x) + \frac{x}{1-2\varepsilon}} + \sqrt{\frac{a(a-2)}{2} - \psi(x) - \frac{x}{1-2\varepsilon}}\right],$$
$$x_F = \frac{\sqrt{2}}{2}\left[\sqrt{\frac{a(a-2)}{2} - \psi(x) + \frac{x}{1-2\varepsilon}} - \sqrt{\frac{a(a-2)}{2} - \psi(x) - \frac{x}{1-2\varepsilon}}\right]. \tag{2.9}$$

Lemma 2.7. Let $a > 1 + \sqrt{4 + 2\sqrt{2}}$ and $y = \psi(x)$ as above. There exists $\varepsilon_F < \varepsilon_0$ so that if $\varepsilon \in (\varepsilon_F, \varepsilon_0)$, then $\left|\psi_F(x_F) - 1/(1-2\varepsilon)\right| < \sigma$, $\left|\psi'_F(x_F)\right| < 0.01$, $\left|\psi''_F(x_F)\right| < 5$ for all $x \in \breve{I}$.

Proof. Denote $\psi(x) = a - 2 + r(x)$. Substituting this expression in the formulas (2.9) instead of $\psi(x)$, we obtain

$$y_F = \frac{\sqrt{2}}{2}\left[\sqrt{\frac{(a-2)^2}{2} - r(x) + \frac{x}{1-2\varepsilon}} + \sqrt{\frac{(a-2)^2}{2} - r(x) - \frac{x}{1-2\varepsilon}}\right],$$
$$x_F = \frac{\sqrt{2}}{2}\left[\sqrt{\frac{(a-2)^2}{2} - r(x) + \frac{x}{1-2\varepsilon}} - \sqrt{\frac{(a-2)^2}{2} - r(x) - \frac{x}{1-2\varepsilon}}\right].$$

Taking into account that $\sqrt{b \pm c} = \sqrt{b}\left[1 \pm c/(2b) + \mathcal{O}\left(c^2(2b)^{-2}\right)\right]$, $|r(x)| < \sigma$, $|x/(1-2\varepsilon)| < 2\sigma$, we find $y_F = a - 2 + r(x)/(a-2) + \mathcal{O}\left(\sigma^2\right)$, $x_F = \dfrac{x}{(a-2)(1-2\varepsilon)} + \mathcal{O}\left(\sigma^2\right)$ for ε close enough to ε_0. Since $1 - 2\varepsilon_0 = 1/(a-2)$, the latter implies: $\jmath$) that $|y_F - (a-2)| < \sigma\sqrt{2}/2$ when $x \in \breve{I}$ and $\jmath\jmath$) that the largest value of $x_F \approx \sigma$ when $x = \sigma$. As for derivatives, those can be found by differentiating of the identities $x = (1-2\varepsilon)x_F\psi_F(x_F)$, $\psi(x) = \left[a(a-2) - x_F^2 - \psi_F^2(x_F)\right]/2$. We finally find

$$\psi'_F(x_F) = -\frac{(1-2\varepsilon)\psi_F(x_F)\psi'(x) + x_F}{\psi_F(x_F) + (1-2\varepsilon)x_F\psi'(x)},$$

$$\psi''_F(x_F) = \frac{\left[1 - \psi'^2_F(x_F)\right]\left[x_F\psi'_F(x_F) - \psi_F(x_F)\right] - (1-2\varepsilon)^2\left[x_F\psi'_F(x_F) + \psi_F(x_F)\right]^3\psi''(x)}{\psi_F^2(x_F) - x_F^2}.$$

Because $x_F < 2\sigma \ll \psi_F(x_F) \approx a - 2$, we obtain that $|\psi'_F(x_F)| = |(1-2\varepsilon)\psi'(x) - \mathcal{O}(\sigma)| < 0.01$ when ε is close enough to ε_0. Since $\psi'^2_F(x_F) < 0.0001$, $|x_F\psi'_F(x_F)| < 0.02$, $\psi_F(x_F) = a - 2 + \mathcal{O}(\sigma)$, we find further that

$$\begin{aligned}|\psi''_F(x_F)| &= \left|\frac{1}{\psi_F(x_F)} + (1-2\varepsilon)^2\psi_F(x_F)\psi''(x) - \mathcal{O}(\sigma)\right| \\ &= \left|1/(a-2) + (1-2\varepsilon)^2\psi''(x)/(a-2) + \mathcal{O}(\sigma)\right|.\end{aligned}$$

Choosing ε_F so that $1 - 2\varepsilon = 1/(a-2) + \mathcal{O}(\sigma^2)$ for any $\varepsilon \in (\varepsilon_F, \varepsilon_0)$, we infer that $|\psi_F''(x_F)| < 5$ for all $a > 1 + \sqrt{4 + 2\sqrt{2}}$ and $\varepsilon \in (\varepsilon_F, \varepsilon_0)$. ■

Given $a > 1 + \sqrt{4 + 2\sqrt{2}}$, let us consider the function $y = \sqrt{a(a-2) - x^2}$ on $[0, \tilde{\delta}]$, where $\tilde{\delta} < \sigma$. Its graph coincides with the arc $\bar{F}(A)\bar{F}(H)$. Denote this function by $y = \psi_0(x)$. The functions $y = \psi_j(x)$, $j = 1, 2, \ldots$ are defined by means of their graphs in such a manner: The arc $\bar{F}^{1-j}(AH)$ is the graph of $y = \psi_j(x)$. It is easy to see that $W^u(P_+) \supset \bar{F}(A)\bar{F}^2(A)$ provided $a > 1 + \sqrt{5}$. On the other hand, $W^u(S_+) \supset L_{\omega t}$ for $\varepsilon \in (\hat{\varepsilon}, \varepsilon_0)$. Because of this, $\bar{F}^{-j}(A) \to P_+$, $\bar{F}^{-j}(H) \to S_+$ as $j \to \infty$. Hence there is J that $\bar{F}^{-j}(A) \in U_\sigma(P_+)$ for all $j \geq J$. Due to the orthogonality of the coordinate axes as well as that the matrix of DF is diagonal at points belonging to the y-axis, pre-images of $\bar{F}^{-j}$ $(\{(x, y) : y = 0\})$ are orthogonal to the y-axis at their intersection points with the y-axis. Since $\{(x, y) : x^2 + y^2 = a(a-2)\} = \bar{F}^{-1}$ $(\{(x, y) : y = 0\})$, pre-images of $y = \sqrt{a(a-2) - x^2}$ have the same property. Due to their smoothness and smooth dependence on parameters, it is clear that the following statement holds: There is $\check{\varepsilon} < \varepsilon_0$ such that, for any $\varepsilon \in (\check{\varepsilon}, \varepsilon_0)$, there is a sub-arc of $\bar{F}^{-J+1}\left(\{(x, y) : y = \sqrt{a(a-2) - x^2}, 0 \leq x \leq \sigma\}\right)$ which belongs to $\tilde{U} = \{(x, y) : 1/(1-2\varepsilon_0) - \sigma < y < 1/(1-2\varepsilon_0) + \sigma, 0 \leq x \leq \tilde{\delta}\}$ and stretches throughout the whole $\tilde{U}$ from one end to another. Due to Lemma 2.4, we know that $L_{\omega t}$ erects and converges to the y-axis when ε tends to ε_0. To be more precise, for any $t > 0$ (including $t \to 0$), there is $\varepsilon_t > 0$ such that if $\varepsilon \in (\varepsilon_t, \varepsilon_0)$, then $x = \omega(y) < t^2 \ll t$, $|\omega'(y)| < t^2 \ll t$ for any $(x, y) \in L_{\omega t}$. Therefore there is $\tilde{\varepsilon} = \max(\check{\varepsilon}, \varepsilon_t)$ so that $\bar{F}^{-J+1}(AH) \subset \tilde{U}$. In doing so one can fix $\tilde{\varepsilon}$ so close to ε_0 that $|\psi_J'(x)| < 0.01$, $|\psi_J''(x)| < 5$ at those points $(x, y) \in \bar{F}^{-J+1}(AH)$ whose $x \in [0, \tilde{\delta}]$. Thus $y = \psi_J(x)$ is the same as $y = \psi(x)$ of Lemma 2.7.

Let $\aleph$ be a family of C^2-smooth functions of the kind $y = \psi(x)$ and $y = \varphi_m(x)$ be restrictions of $y = \psi_{J+m}(x)$, $m = 1, 2, \ldots$ to $[0, \tilde{\delta}]$. We show that $y = \varphi_m(x)$ belongs to $\aleph$.

Lemma 2.8. Let a, ε, $y = \varphi_m(x)$ be such that Lemmas 2.1–2.7 are valid. Then

1) $\{y = \varphi_m(x)\}_{m=1}^{\infty}$ converges to its limit $y = \varphi_*(x)$ as $m \to \infty$ uniformly with respect to x and

2) $\mathrm{graph}(\varphi_*) = \mathrm{Cl}(\tilde{W}_+^s)$.

Proof. The maps $\bar{F}^{-j}$ shrink $AH\bar{F}(H)\bar{F}(A)$ to the y-axis. Therefore $\bar{F}^{-j}$ $(AH\bar{F}(H)\bar{F}(A))$ is a sequence of nested curvilinear quadrangles. Their sides $\bar{F}^{-2j}(AH)$ and $\bar{F}^{-2j+1}(AH)$ form two monotone sequences, one of which is increasing while the other is decreasing. As is shown above $\bar{F}^{-J+1}(AH) \subset \tilde{U}$. It is clear too that we can without loss of generality assume that $\bar{F}^{-J}(AH) \subset \tilde{U}$. It follows from the latter inclusion that $\bar{F}^{-j}(AH) \subset \tilde{U}$ for all $j \geq J - 1$. Due to Lemma 2.7, this means that

$y = \varphi_m(x)$ belongs to $\aleph$. In view of the compactness of $\aleph$ as a subset of the C^1-smooth function set, there are C^1-smooth (invariant with respect to Φ) functions which are limits of the sequences $\{\varphi_{J+2m}\}_{m=1}^{\infty}$ and $\{\varphi_{J+2m-1}\}_{m=1}^{\infty}$. Denote these functions by φ_* and φ^*, respectively. Obviously φ_* and φ^* are one-valued functions. By definition, their graphs contain P_+ and S_+. It is easy to see that $y = \varphi_*(x)$, $y = \varphi^*(x)$ should be solutions of (2.8). However if a neighbourhood of S_+ is small enough, then, as we know, there exists only one curve invariant with respect to $\bar{F}$ whose slope is less than 0.01. This is $W^s_{\text{loc}}(S_+)$. Therefore, within the given neighbourhood, the graphs of $y = \varphi_*(x)$, $y = \varphi^*(x)$ coincide with $W^s_{\text{loc}}(S_+)$.

Let us show that $\text{graph}(\varphi_*) = \text{graph}(\varphi^*) = \text{Cl}(\tilde{W}^s_+)$. Denote $\det(D\bar{F}) = (1 - 2\varepsilon)(y^2 - x^2)$. Estimating $|\det(D\bar{F})|$, we observe $|\det(D\bar{F})| > 1$ in $\tilde{U}$ provided ε and σ are small enough. This implies $|\det(D\bar{F}^{-1})| < 1$ in $\tilde{U}$. Due to the smoothness and the invariance of $\text{graph}(\varphi_*)$, $\text{graph}(\varphi^*)$ with respect to $\bar{F}^{-1}$, the latter means $\text{graph}(\varphi_*) = \text{graph}(\varphi^*)$. Obviously $\text{Cl}(\tilde{W}^s_+) \subset \text{graph}(\varphi_*)$. Since $y = \varphi_*(x)$ is a one-valued function, $\text{Cl}(\tilde{W}^s_+) \setminus \tilde{W}^s_+$ is a fixed point. But inside $AH\bar{F}(H)\bar{F}(A)$ except S_+ there is one fixed point only. It is P_+. Since P_+ belongs to the y-axis, we infer that $\text{graph}(\varphi_*) = \text{Cl}(\tilde{W}^s_+)$. As is well known [1], analytic maps have analytic stable and unstable manifolds at their hyperbolic fixed points. Therefore, $y = \varphi_*(x)$ is an analytic function. ■

The method presented above can be used in a neighbourhood to the right of S_+, i.e., for $x > \delta$. Indeed let J be fixed in such a manner that the arc $\bar{F}^{-J}(AH)$ coincides with the graph of the function $y = \psi_J(x)$, where $\psi_J \in \aleph$. Choose ζ so that $0 < \delta < \zeta^3 \ll \zeta < \sigma$ and $\bar{F}^{-J}(AH)$ can be prolonged inside the domain $U \equiv \{(x, y) : 1/(1-2\varepsilon_0) - \sigma < y < 1/(1-2\varepsilon_0) + \sigma, 0 \le x \le \zeta\}$. Let $\bar{F}^{-J}(B)$ be the first intersection point of prolongation of $\bar{F}^{-J}(AH)$ with $x = \zeta$. Thus $\bar{F}^{-J}(BH)$ is a simply connected arc in U. This arc contains $\bar{F}^{-J}(AH)$, intersects $x = \zeta$, and is the graph of a function which belongs to $\aleph$ at $[0, \zeta]$. Denote this function by $y = \Psi_0(x)$. Without loss of generality, we can assume that the domain of $\bar{F}^{-1}$ includes U. Then we can, with the aid of the map Φ, define functions $y = \Psi_m(x)$ which are restrictions of $y = \Psi_{J+m}(x)$ to $[0, \zeta]$. For the sake of simplicity, the restriction of $\bar{F}^{-1}$ to U will be denoted by the same symbol $\bar{F}^{-1}$ in the sequel.

Consider $\displaystyle\bigcup_{j=0}^{\infty} \bar{F}^{-j}\left(W^s_{\text{loc}}(S_+) \cap U \cap \{(x, y) : x \ge \delta\}\right)$ and let $\hat{W}^s_+ \subset U$ be its simply connected component such that $S_+ \in \hat{W}^s_+$.

Lemma 2.9. Let a, ε, $y = \Psi_m(x)$ be such that Lemmas 2.1–2.7 are valid. Then

1) $\{y = \Psi_m(x)\}_{m=1}^{\infty}$ converges to its limit $y = \Psi_*(x)$ as $m \to \infty$ uniformly with respect to x and

2) $\text{graph}(\Psi_*) = \text{Cl}(\hat{W}^s_+)$.

Proof. Let $\Psi_0 \in \aleph$. Repeating the computations produced in the proof of Lemma 2.8 it is not difficult to check that Φ transforms $y = \Psi_0(x)$ into another function, say, $y = \Psi_1(x)$ such that $\Psi_1 \in \aleph$ too. That is, $\Phi : \aleph \longmapsto \aleph$. Repeating the computations again and again, one can as above construct two function sequences $\{\Psi_{2m}\}_{m=1}^{\infty}$ and $\{\Psi_{2m+1}\}_{m=1}^{\infty}$. The same reasons as above show that these sequences both are convergent. Their limits are continuous functions. Let us denote these functions by $y = \Psi_*(x)$ and $y = \Psi^*(x)$, respectively. We show that the limits coincide and determine a finite size fraction of $\hat{W}^s_+$. In order to do this, it is necessary to prove that the domain of $y = \Psi_m(x)$, $m = 1, 2, \ldots$ contains the interval $[0, \zeta]$. We shall do it for $\varepsilon = \varepsilon_0$. A general case ensues from it by continuity.

So, let $\varepsilon = \varepsilon_0$. Then $S_+ = P_+$ and $1/(1-2\varepsilon_0) = a-2$. Observe first of all that $\bar{F}^{-1}$ transforms a line $x = \zeta$ into a hyperbola $x = \dfrac{\zeta}{(a-2)y}$. The hyperbola intersects the line at a point $T = (\zeta, a-2)$. It is clear that there is no loss of generality in assuming $y = \Psi_0(x)$ such that $\Psi_0(x) > a-2$ for all $x \in [0, \zeta]$. Denote $T_f \equiv \text{graph}(\Psi_0) \cap \{(x, y) : x = \zeta\}$ and let $p \in (0, \sigma)$ be such that $a - 2 + p$ is the y-coordinate of T_f. Assume additionally that σ is chosen so that, in $\text{Cl}(U)$, the action of $\bar{F}^{-1}$ along the y-axis is a linear (accurate to the second order quantity of smallness) contraction towards the line $y = a - 2$. Obviously this can be done. Then the shrinkage coefficient is equal to $1/(a-2)$. Therefore, if y_f denotes the value of the y-coordinate of $\bar{F}^{-1}(T_f)$, then y_f can be estimated by $y_f = a-2-p/(a-2)+\mathcal{O}(p^2)$. Let us compute now x_f, the value of the x-coordinate of $\bar{F}^{-1}(T_f)$. Denote the angle between $x = \zeta$ and $x = \dfrac{\zeta}{(a-2)y}$ at T by h. Then $\text{tg}(h) = -\zeta/(a-2)$. Since σ is small enough, the hyperbola arc which lies in $\text{Cl}(U)$ coincides (accurate to the second order quantity of smallness) with its tangent at T. Therefore, accurate to $\mathcal{O}(p^2)$, $x_f = \zeta - \dfrac{p\,\text{tg}(h)}{a-2} = \zeta\left[1 + p(a-2)^{-2}\right] \equiv \zeta'$. So, we find that the domain of $y = \Psi_1(x)$ is approximately equal to $[0, \zeta']$ which is larger than $[0, \zeta]$.

In order to treat $y = \Psi_2(x)$, it is enough to apply the aforesaid reasons to $T'_f \equiv \bar{F}^{-1}(T_f)$. In doing so, one should keep in mind that $\zeta' > \zeta$. Let us denote the intersection point of $x = \dfrac{\zeta'}{(a-2)y}$ with $x = \zeta$ by T', the angle between $x = \dfrac{\zeta'}{(a-2)y}$ and $x = \zeta$ at T' by h', the coordinates of $\bar{F}^{-1}(T'_f)$ by x'_f, y'_f. Making computations similar to those presented above, we find accurate to $\mathcal{O}(p^3)$ that $y'_f = a - 2 + p(a-2)^{-2}$, $\text{tg}(h') = -\zeta/(a-2)\left[1 + p(a-2)^{-2}\right]$, $x'_f = \zeta\left[1 + p(a-2)^{-2}\right]\left[1 - p(a-2)^{-3}\right] = \zeta\left[1 + p(a-3)(a-2)^{-3}\right] \equiv \zeta''$. Thus, we see that the domain of $y = \Psi_2(x)$ is approximately equal to the interval $[0, \zeta'']$ which is smaller than $[0, \zeta']$ while larger than $[0, \zeta]$. All this takes place provided a is close enough to 4, and so on. As a result we find that the domains of $y = \Psi_m(x)$, $m = 1, 2, \ldots$ all are larger than $[0, \zeta]$. By continuity, the same is true for ε, a sufficiently close to ε_0, 4, respectively. Since σ was fixed independently of values of the parameters a and ε, there exist limits of the sequences of $\{\Psi_{2m}\}_{m=1}^{\infty}$, $\{\Psi_{2m+1}\}_{m=1}^{\infty}$, and these limits coincide with the arc $\hat{W}^s_+$. Actually it is

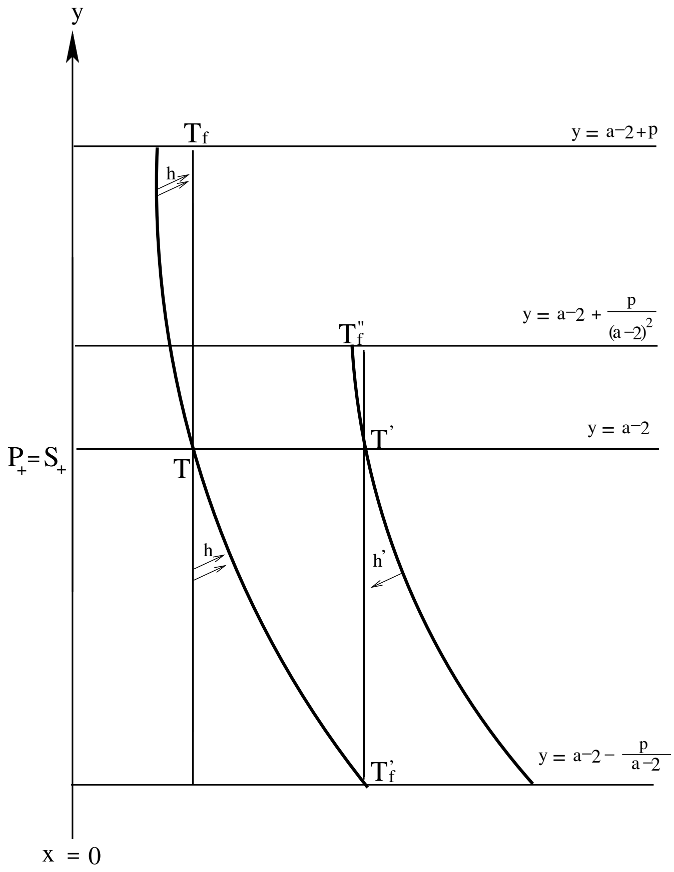

Figure 3.

easy to see that $y = \Psi_*(x)$ and $y = \Psi^*(x)$ should be the solutions of (2.8). However as we know, the only curve which is invariant with respect to $\bar{F}$ and has the slope less than 0.01 is $\hat{W}^s_{\mathrm{loc}}(S_+)$. Hence, in the given neighbourhood, the graphs graph(Ψ_*), graph$(\Psi, *)$ coincide with the arc of $\hat{W}^s_+$. We show that this arc stretches throughout the whole interval $[0, \zeta]$. Indeed, it is evident that $\hat{W}^s_+ \subset \mathrm{graph}(\Psi_*) \cap \mathrm{graph}(\Psi^*)$. If $\hat{W}^s_+$ does not intersect $x = \zeta$, then $\mathrm{Cl}(\hat{W}^s_+)$ is inside $\mathrm{Cl}(U) \cap \{(x, y) : \delta \le x \le \zeta\}$. Since $y = \Psi_*(x)$, $y = \Psi^*(x)$ are one-valued functions, it is clear that $\mathrm{Cl}(\hat{W}^s_+) \setminus \hat{W}^s_+$ is a one point set. Due to its invariance with respect to $\bar{F}^{-1}$, it should be a fixed point. But as we know, if $\varepsilon \in (\hat{\varepsilon}, \varepsilon_0)$, then except S_+ there are no other fixed points of F in $\mathrm{Cl}(U \cap \{(x, y) : \delta \le x \le \zeta\})$. Therefore, our assumption is false. ■

Lemma 2.10. There is $\hat{a} \in (1+\sqrt{4+2\sqrt{2}}, 4)$ such that, for any $a \in (\hat{a}, 4)$ and arbitrary $U(P_+)$, $U(\Lambda_a)$, one can find $\varepsilon_a \in (0, \varepsilon_0)$ and then $J = J(a, U(P_+), U(\Lambda_a), \varepsilon_a)$ such that, for all $\varepsilon \in (\varepsilon_a, \varepsilon_0)$, the following is fulfilled:

$\imath)$ $\bigcup_{j=0}^{J} F^j \left(W^u_{\mathrm{loc}}(S_+)\right) \subset U(\Lambda_a)$;

$\imath\imath)$ $\left[\bigcup_{j=0}^{J} F^j \left(W^u_{\mathrm{loc}}(S_+)\right) \cap W^s_{\mathrm{loc}}(S_+)\right] \setminus \{S_+\} \neq \emptyset$;

$\imath\imath\imath)$ the set $\left[\bigcup_{j=0}^{J} F^j \left(W^u_{\mathrm{loc}}(S_+)\right) \cap W^s_{\mathrm{loc}}(S_+)\right] \setminus \{S_+\}$ contains points of transverse intersection of the manifolds $W^s_{\mathrm{loc}}(S_+)$ and $W^u(S_+)$.

Proof. Let $\zeta \in (0, \sigma)$ be small enough. Choose $\varepsilon_F > 0$ so that Lemmas 2.8 and 2.9 are satisfied for any $\varepsilon \in (\varepsilon_F, \varepsilon_0)$. The latter permits us to consider ζ as a size of $W^s_{\mathrm{loc}}(S_+)$. Lemmas 2.8 and 2.9 imply that $\imath)$ $W^s_{\mathrm{loc}}(S_+) \subset U$, $\imath\imath)$ $\mathrm{Cl}\big(W^s_{\mathrm{loc}}(S_+)\big) \cap \{(x, y) : x = 0\} \neq \emptyset$, $\mathrm{Cl}\big(W^s_{\mathrm{loc}}(S_+)\big) \cap \{(x, y) : x = \zeta\} \neq \emptyset$, $\imath\imath\imath)$ the slope of $W^s_{\mathrm{loc}}(S_+)$ to the x-axis does not exceed 0.01 at all points of $W^s_{\mathrm{loc}}(S_+)$. Here, U is the same as in Lemma 2.9.

Since $\bar{F}^2 \equiv F^2$ and $\bar{F}^{j+1}\left(W^u_{\mathrm{loc}}(S_+)\right) \supset \bar{F}^j\left(W^u_{\mathrm{loc}}(S_+)\right)$ for all natural j, the unstable manifold of F at S_+ coincides with that of $\bar{F}$ at S_+. Therefore, one can study the $\bar{F}$-images of $W^u_{\mathrm{loc}}(S_+)$ instead of the F-images of $W^u_{\mathrm{loc}}(S_+)$. We shall do this in the sequel.

Fix $t > 0$ so that $\sqrt[4]{t} < \zeta$. Without loss of generality, one can assume that ε_F is chosen so that $\sqrt[4]{\delta} < t$ and Lemma 2.4 is fulfilled. This means that there exists the arc $L_{\omega t}$ of $W^u(S_+)$ which stretches out of $\bar{F}^2(Z_t)$ to $\bar{F}(Z_t)$ (see the designations preceding Lemma 2.4) and such that: $\jmath)$ the distance between $L_{\omega t}$ and Λ_a is less than t; $\jmath\jmath)$ the slope of $W^u(S_+)$ to the x-axis exceeds $1/t \gg 1$ at all points of $L_{\omega t}$. Repeating the reasons and the computations presented after the proof of Lemma 2.4, we can find once more a fraction of $W^u(S_+)$. This fraction is situated on the other side of Λ_a, possesses the same properties as $L_{\omega t}$ and stretches out of $F^3(K)$ to $F(K)$. Continuing computations and repeating reasons similar to those aforesaid, at the next step, we can obtain once more a fraction of $W^u(S_+)$ which is located on the same side of Λ_a as $L_{\omega t}$, possesses the same properties as $L_{\omega t}$ and stretches out approximately of $F(K)$ to $F^3(K)$. It is easy to verify the validity of the last statements. To do this, it is necessary to localize $\bar{F}$-images of a few remarkable points. For example, let us consider $Z_\pm$ and Z_x, the points of intersection of $L_{\omega t}$ with the critical set K and with the x-axis, respectively. It is not difficult to see that if it is known where $\bar{F}$-images of the points of $Z_\pm$ and Z_x are, then $\bar{F}(L_{\omega t})$ can be localized good enough. In particular because $\jmath)$ we have continuous dependence of $L_{\omega t}$ on a and ε; $\jmath\jmath)$ $Z_t \approx Z_+$ when a, ε close to 4, ε_0, respectively; $\jmath\jmath\jmath)$ $\lim\limits_{\varepsilon \nearrow \varepsilon_0} \mathrm{dist}(Z_\pm, O) = \lim\limits_{\varepsilon \nearrow \varepsilon_0} \mathrm{dist}(Z_x, O) = 0$ for a close to 4; $\jmath\jmath\jmath\jmath)$ $\lim\limits_{a \nearrow 4} \lim\limits_{\varepsilon \nearrow \varepsilon_0} \mathrm{dist}(\bar{F}(L_{\omega\tau}), [\bar{F}(O), \bar{F}^2(O)]) = 0$. Analogously in order to localize $\bar{F}^2(L_{\omega t})$, it is necessary to find points of intersection of $\bar{F}(L_{\omega t})$ with the critical set, with its $\bar{F}$-image and with the coordinate axes, respectively, and then to compute and to localize $\bar{F}$-images of these points, and so on. In doing so, the parameter values can be fixed such that the end points of $\bar{F}^j(L_{\omega t})$, $j = \overline{1, J}$ both belong to a neighbourhood of $F^2(O)$. Obviously the neighbourhood size (resp. natural J) can be chosen as small (resp. large) as desirable. Indeed, since $F^2(O) = F^j(O)$, $j \geq 2$ when $a = 4$ and $L_{\omega t} \cap K \to O$ as $\varepsilon \nearrow \varepsilon_0$, due to the continuous dependence of F on parameters, there exist $\hat{a} \in (1 + \sqrt{4 + 2\sqrt{2}}, 4)$ and $\varepsilon_a \in (\varepsilon_F, \varepsilon_0)$ such that the aforesaid assertion is valid for all $j = \overline{1, J}$ provided $\varepsilon \in (\varepsilon_a, \varepsilon_0)$. As for slopes of the fractions of $\bar{F}^j(L_{\omega t})$ under consideration, one can show that these slopes exceed $1/t \gg 1$ at the points of $\bar{F}^j(L_{\omega t})$ which lie between $F(K)$ and $F^{j+2}(K)$, $j = 1, 2, \ldots$. This can be done in the same way as it is done above where the slopes of $\bar{F}(L_{\omega t})$ are studied.

It is much simpler to collect and to pick up successive steps of the algorithm that is presented above in figures than to describe those steps by words. That is why we depict several first steps of the given algorithm in Figs. 4–5.

Studying $\bar{F}^j(L_{\omega t})$, $j = 2, 3, \ldots$, it is easy to observe that $\bar{F}^2(L_{\omega t})$ is such that $\bar{F}^2\left(L_{\omega t} \setminus W^u_{\text{loc}}(S_+)\right)$ contains an arc which has the necessary properties and intersects $W^s_{\text{loc}}(S_+)$ transversely. Denote by $\bar{F}^2\left(L_{\omega t} \setminus W^u_{\text{loc}}(S_+)\right) \cap W^s_{\text{loc}}(S_+) \stackrel{\text{def}}{=} \Theta$ the intersection point. There is an alternative: either *a*) $\Theta = S_+$ or *b*) $\Theta \neq S_+$. Let us discuss the alternative.

If (a) is satisfied, then there is a point $\tilde{\Theta} \in W^u_{\text{loc}}(S_+) \setminus \{S_+\}$ such that: 1) $\bar{F}^{-m}(\tilde{\Theta}) \in W^u_{\text{loc}}(S_+)$ for all natural m; 2) $\lim\limits_{m\to\infty} \bar{F}^{-m}(\tilde{\Theta}) = S_+$; 3) there is natural M such that $\bar{F}^M(\tilde{\Theta}) = \Theta = S_+$.

If (b) is fulfilled, then there is a point $\bar{\Theta} \in W^u_{\text{loc}}(S_+) \setminus \{S_+\}$ such that: 1) $\bar{F}^{-m}(\bar{\Theta}) \in W^u_{\text{loc}}(S_+)$ for all natural m; 2) $\lim\limits_{m\to\infty} \bar{F}^{-m}(\bar{\Theta}) = S_+$; 3) there is natural M such that $\bar{F}^M(\bar{\Theta}) = \Theta$; 4) $\bar{F}^m(\Theta) \in W^s_{\text{loc}}(S_+)$ for all natural m and $\lim\limits_{m\to\infty} \bar{F}^m(\Theta) = S_+$.

Denote $\left(\bigcup\limits_{n=0}^{\infty} \bar{F}^{-n}(\breve{\Theta})\right) \cup \left(\bigcup\limits_{n=0}^{N} \bar{F}^{-n}(\breve{\Theta})\right)$, the orbit of $\breve{\Theta}$, by $\text{Orb}(\breve{\Theta})$. Here $\breve{\Theta}$ is $\tilde{\Theta}$ or $\bar{\Theta}$, and N is a natural or countable number. Considering the orbits $\text{Orb}(\tilde{\Theta})$ and $\text{Orb}(\bar{\Theta})$, we can see that these orbits both are homoclinic to S_+. Taking into account that $\bar{F}^2\left(L_{\omega t} \setminus W^u_{\text{loc}}(S_+)\right)$ intersects $W^s_{\text{loc}}(S_+)$ transversely, we infer that the given orbits both are transverse homoclinic too.[1] (We notice that if (a) takes place, then there must be a neighbourhood $\tilde{V}$ of $\tilde{\Theta}$ in $W^u(S_+)$ such that $\bar{F}^M(\tilde{V}) = W^u_{\text{loc}}(S_+)$ because

[1] Recall that the orbit $\text{Orb}(\breve{\Theta})$ is termed homoclinic to the fixed point S_+ if it is biasymptotic to the given point, i.e., when $\text{Orb}(\breve{\Theta}) \subset W^s(S_+) \cap W^u(S_+)$. The homoclinic orbit $\text{Orb}(\breve{\Theta})$ is called the transverse homoclinic one when, first, S_+ is the hyperbolic fixed point and, second, for any sufficiently large natural m and n such that $\bar{F}^{-m}(\breve{\Theta}) \in W^u_{\text{loc}}(S_+)$, $\bar{F}^n(\breve{\Theta}) \in W^s_{\text{loc}}(S_+)$, there is a disc of $W^u_{\text{loc}}(S_+)$ possessing the following properties: $\jmath$) the disc contains the point $\bar{F}^{-m}(\breve{\Theta})$; $\jmath\jmath$) $\bar{F}^{m+n}$ is a diffeomorphism of the disc into its $\bar{F}^{m+n}$-image; $\jmath\jmath\jmath$) the aforementioned $\bar{F}^{m+n}$-image of the disc intersects $W^s_{\text{loc}}(S_+)$ at $\bar{F}^n(\breve{\Theta})$ transversely.

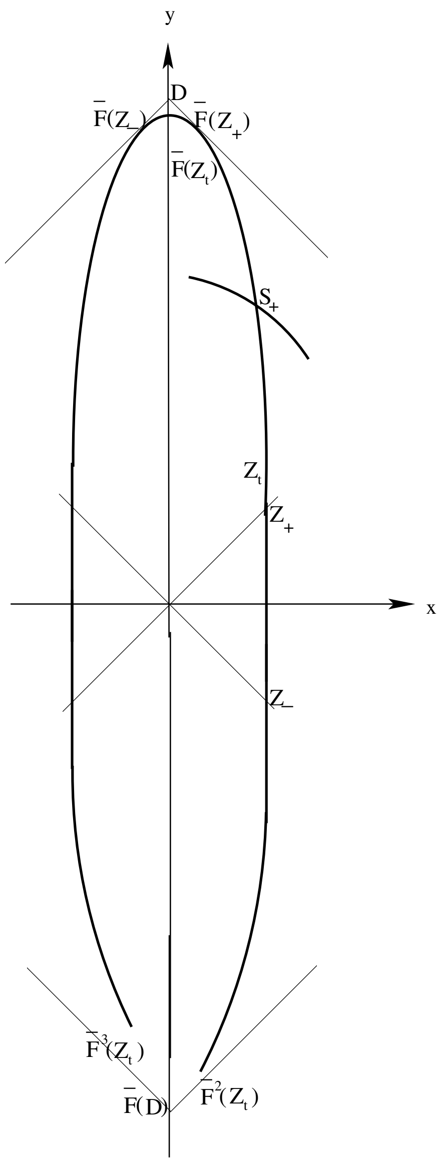

Figure 4.

S_+ is a hyperbolic saddle and there is only one 1-dimensional expansion direction of $\bar{F}$ at S_+.) ■

Proof of Theorem 1.1. Theorem 1.1 is a corollary of Lemma 2.10. Indeed, according to [5,9], any neighbourhood of an orbit which is transverse homoclinic to a hyperbolic fixed point contains a nontrivial topologically transitive set chaotic in the sense of Li and Yorke [7]. This set is invariant with regard to F^2, and periodical points are everywhere dense on the given set. Because $F(S_+) = S_-$, $F(W^u_{\text{loc}}(S_+)) = W^u_{\text{loc}}(S_-)$, $F(W^s_{\text{loc}}(S_+)) = W^s_{\text{loc}}(S_-)$, there exists another set which is symmetric with respect to the y-axis to the

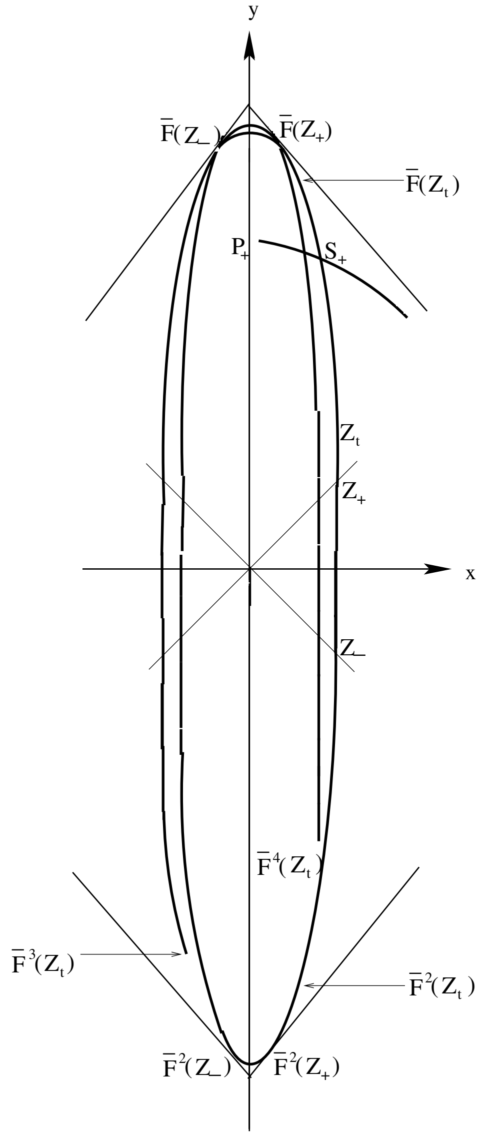

Figure 5.

aforesaid one and has the same properties. Their union gives us a set possessing all the necessary properties which is invariant with respect to F. ■

Concluding the study of F, let us show that, at the bifurcation value $\varepsilon = \varepsilon_0$, the phase pattern of F in a vicinity of $U(P_+)$ does not differ from that for $\varepsilon \in (\varepsilon_0, 1/2)$.

Lemma 2.11. Given $\varepsilon = \varepsilon_0$ and $a > 3$. Then F has the 1-dimensional local stable manifold $W^s_{\mathrm{loc}}(P_+)$ at the point P_+.

Proof. Let us transfer the origin of coordinates at the point P_+ and introduce coordinates

$\hat{x} = x$, $\hat{y} = y - a + 2$. Then, for $\varepsilon = \varepsilon_0$, the map F takes the form

$$\hat{F} : \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} -\hat{x} - \hat{x}\hat{y}/(a-2) \\ -(a-2)\hat{y} - [\hat{x}^2 + \hat{y}^2]/2 \end{pmatrix}.$$

In what follows, we will omit "hats" above the coordinates $\hat{x}$, $\hat{y}$. The eigenvalues of $D\hat{F}$ at $(0, 0)$ are equal to -1 and $2 - a$. We show that there is a curve $y = \varphi(x)$ which is invariant with respect to $\hat{F}$ and that this curve is a locally stable manifold of $\hat{F}$ at $(0, 0)$.

Assume the curve exists. Because of its invariance with respect to F, the inclusion $F\,(\text{graph}(\varphi)) \subset \text{graph}(\varphi)$ holds. In other words, we have the functional identity

$$-(a-2)\varphi(x) - \{x^2 + [\varphi(x)]^2\}/2 = \varphi\left(-x - x\varphi(x)/(a-2)\right), \varphi(0) = 0.$$

Differentiating this identity three times, we find $\varphi'(0) = 0$, $\varphi''(0) = -1/(a-1)$, $\varphi'''(0) = 0$. Thus, if the function-solution exists, then $y = -x^2/[2(a-1)] + \mathcal{O}(x^4)$. We show that this indeed holds. Denote $y = -x^2/[2(a-1)] \stackrel{\text{def}}{=} \varphi_0(x)$ and consider its $\hat{F}$-image. Due to

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \equiv \hat{F}\left(\begin{pmatrix} x \\ \varphi_0(x) \end{pmatrix}\right) = \begin{pmatrix} -x + \dfrac{x^3}{2(a-1)(a-2)} \\ -\dfrac{x^2}{2(a-1)}\left[1 + \dfrac{x^2}{4(a-1)}\right] \end{pmatrix},$$

the image of $y = \varphi_0(x)$ is a curve $\bar{y} = \varphi_1(\bar{x})$, where $\bar{x} = -x + \dfrac{x^3}{2(a-1)(a-2)}$, $\bar{y} = -\dfrac{x^2}{2(a-1)}\left[1 + \dfrac{x^2}{4(a-1)}\right]$. Since $|\bar{x}| < |x|$ and $y(\bar{x}) > y(x) > \bar{y}(\bar{x})$ for $(x, y) \neq (0, 0) \neq (\bar{x}, \bar{y})$, it is clear that this curve is located under $y = \varphi_0(x)$. Denote $\Pi_0 = \{(x, y) : \varphi_1(x) \leq y \leq \varphi_0(x), -\gamma \leq x \leq \gamma\}$, where $\gamma > 0$ is a small constant. Consider $\hat{F}(\Pi_0)$. Since $\hat{F}$ is a diffeomorphism inside a neighbourhood of $(0, 0)$, $\hat{F}$-images of the curves $y = \varphi_j(x)$, $j = 0, 1$ localize $\hat{F}(\Pi_0)$ completely. Computing, we find

$$\begin{pmatrix} \breve{x} \\ \breve{y} \end{pmatrix} \equiv \hat{F}\left(\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}\right) = \begin{pmatrix} x - \dfrac{x^3}{(a-1)(a-2)} + \mathcal{O}(x^5) \\ -\dfrac{x^2}{2(a-1)}\left[1 - \dfrac{(a-3)x^2}{4(a-1)} - \dfrac{x^2}{a-2}\right] + \mathcal{O}(x^6) \end{pmatrix}.$$

Hence the image of $\bar{y} = \varphi_1(\bar{x})$ is a curve $\breve{y} = \varphi_2(\breve{x})$, where

$$\breve{x} = x - x^3(a-1)^{-1}(a-2)^{-1} + \mathcal{O}(x^5),$$

$$\breve{y} = -x^2/[2(a-1)]\left\{1 - (a-3)x^2/[4(a-1)] - x^2/(a-2)\right\} + \mathcal{O}(x^6).$$

We show that $\breve{y} = \varphi_2(\breve{x})$ is located above $y = \varphi_0(x)$. Computing $\varphi_0(\breve{x})$ we obtain $\varphi_0(\breve{x}) = -x^2/[2(a-1)] + x^4(a-2)^{-1}(a-1)^{-2} + \mathcal{O}(x^6)$. Since

$$\frac{1}{(a-2)(a-1)^2} < \frac{a-3}{8(a-1)^2} + \frac{1}{2(a-2)(a-1)} \quad \text{when } a > 3,$$

for x small enough, we obtain 1) the curve $y = \varphi_2(x)$ lies under $y = 0$ but above $y = \varphi_0(x)$; 2) the cone Π_0 is located inside the cone $\hat{F}(\Pi_0)$. Notice that Dom_j, the domains of $y = \varphi_j(x)$, shrink slightly under the action of $\hat{F}$. That is $\mathrm{Dom}_0 \supset \mathrm{Dom}_1 \supset \mathrm{Dom}_2 \supset \cdots \supset \mathrm{Dom}_j \supset \ldots$.

Choose $\gamma > 0$ so that

$$\Pi_0 \cap \{(x, y) : -\gamma/2 \le x \le \gamma/2\} \subset \hat{F}(\Pi_0) \cap \{(x, y) : -\gamma/2 \le x \le \gamma/2\}$$

and consider a sequence of $\{\Pi_{-m}\}_{m=1}^{\infty}$, where $\Pi_{-m} = \hat{F}^{-1}(\Pi_{-m+1}) \cap \Pi_0$, $m = 1, 2, \ldots$. What is said above means that $\{\Pi_{-m}\}_{m=1}^{\infty}$ constitute a set of the nested cones. Denote $\Pi = \bigcap\limits_{m=1}^{\infty} \Pi_{-m}$. This is an invariant set. Because $|\det(D\hat{F}^{-1}(P_+))| = 1/(a-2) < 1$, there is $\gamma > 0$ so that $|\det(D\hat{F}^{-1}((x, y)))| < 1$ for any $(x, y) \in U_\gamma((0, 0))$. In view of the invariance of Π, the latter means that $\mathrm{mes}(\Pi) = 0$. Therefore, the sides of Π which are limits of sequences of graphs of the functions $y = \varphi_{-2m}(x)$ and $y = \varphi_{-2m+1}(x)$ coincide. Thus there is only one curve $y = \varphi(x)$ which is invariant with respect to $\hat{F}^{-1}$. It is easy to check that this is a smooth arc. In fact, the graphs of $y = \varphi_0(x)$, $y = \varphi_1(x)$ touch each other at $x = 0$ and $\varphi_0'(0) = \varphi_1'(0) = 0$, $\varphi_0''(0) = \varphi_1''(0) = -1/(a-2)$. The graphs of $y = \varphi_{-j}(x)$, $j = 2, 3, \ldots$, by construction, all are situated between them and $\varphi_{-j}'(0) = 0$, $\varphi_{-j}''(0) = -1/(a-2)$ for all natural j. Therefore the function $y = \varphi(x)$ is at least twice differentiable at $x = 0$ and has a finite continuous first derivative in a neighbourhood of $x = 0$. So, there is the neighbourhood of P_+ in which the graph of the function $y = \varphi(x)$ is a C^1-smooth arc. Thus, the C^1-smooth locally stable manifold of F at P_+ prolongs to exist even when $\varepsilon = \varepsilon_0$. This means that the phase space pattern bifurcation does still not take place although the exponential rate of convergence to P_+ along $W^s_{\mathrm{loc}}(P_+)$ is already lost. ∎

3. Proof of Theorem 1.3

Proof of Theorem 1.3. The main idea is to find a suitable family of diffeomorphisms so that the results which are just presented for the endomorphisms can be applied to these diffeomorphisms. Since our approach consists in a successive use of continuity and continuous dependence, we restrict ourselves to indicate the proof scheme in considerable detail.

Consider the families Υ_0 and Υ_b. Obviously, the plane $x = z = 0$ is invariant with respect to Υ_0, the arc $\Lambda(\Upsilon_0) \equiv \{(0, 0, y, t) : y = [a(a-2) - t^2]/2,\ f_a^2(0) \le t \le f_a(0)\}$, where $f_a(0) \equiv a(a-2)/2$, $f_a^2(0) \equiv a(a-2)[a(a-2)-4]/8$ belongs to this plane, $P_+(\Upsilon_0) = (0, 0, a-2, a-2) \in \Lambda(\Upsilon_0)$ is a fixed point. It is evident too that $\Lambda(\Upsilon_0)$ is a topologically transitive set for $a \in (\hat{a}, 4) \cap \mho$. Indeed, $\Upsilon_0((0, 0, y, t)) \in \Lambda(\Upsilon_0)$ for $y \in [f_a^2(0), f_a(0)]$ as $\Upsilon_0((0, 0, y, t)) = (0, 0, [a(a-2) - y^2]/2, y)$. Since there exists $\hat{y} \in [f_a^2(0), f_a(0)]$ such that $\mathrm{Cl}\left(\bigcup\limits_{j=1}^{\infty} f_a^j(\hat{y})\right) = [f_a^2(0), f_a(0)]$ when $a \in$

$(\hat{a}, 4) \cap \mho$, projections of $\bigcup_{j=1}^{\infty} \Upsilon_0^j((0, 0, \hat{y}, t))$ are dense on the segment $[f_a^2(0), f_a(0)]$ of the y-axis and on the same segment of the t-axis. The latter means that $\Lambda(\Upsilon_0)$ is a topologically transitive set. In a vicinity of $P_+(\Upsilon_0)$, there are two periodic of period 2 points $S_\pm(\Upsilon_0) \equiv (\pm x_s(0), \pm z_s(0), y_s(0), t_s(0))$ when $\varepsilon \in (\varepsilon_a, \varepsilon_0)$. Here $x_s(0) = -z_s(0) = \sqrt{a(a-2) - (3-4\varepsilon)(1-2\varepsilon)^{-2}}$, $y_s(0) = t_s(0) = 1/(1-2\varepsilon)$. $\lim_{\varepsilon \nearrow \varepsilon_0} S_\pm(\Upsilon_0) = P_+(\Upsilon_0)$.

Denote by xOy the factor space $\mathbb{R}^4/\sim$ which is constituted by identification of the points of $\mathbb{R}^4$ whose coordinates x,y are identical. That is $xOy \equiv \mathbb{R}^4/\sim = \{(x, \cdot, y, \cdot)\}$ or simply $\{(x, y)\}$. Obviously, the action of Υ_0 restricted to xOy coincides with that of F.

Consider

$$\{(x, z, y, t) : 0 \le x \le \zeta, -\infty < z < \infty, y = \Omega(x), -\infty < t < \infty\} \subset \mathbb{R}^4,$$

"a suspension" over a fraction of the stable manifold $W^s(P_+) \subset xOy$ of F at S_+. Denote it by $W^s_{\mathrm{loc}}(S_+(\Upsilon_0))$. It is not difficult to check that $W^s_{\mathrm{loc}}(S_+(\Upsilon_0))$ is a fraction of the stable manifold of Υ_0 at the point $S_+(\Upsilon_0)$ adjoining to $S_+(\Upsilon_0)$. Indeed, since $\Upsilon_0^j((x_0, z, \Omega(x_0), t)) = \big(x_j, x_{j-1}, \Omega(x_j), \Omega(x_{j-1})\big)$, where

$$x_j = -(1-2\varepsilon)x_{j-1}\Omega(x_{j-1}), \Omega(x_j) = \frac{a(a-2) - \Omega^2(x_{j-1}) - x_{j-1}^2}{2}, \quad j = 1, 2, \ldots,$$

it is clear that

$$\lim_{j\to\infty} \Upsilon_0^{2j}((x_0, z, \Omega(x_0), t)) = S_+(\Upsilon_0), \ \lim_{j\to\infty} \Upsilon_0^{2j-1}((x_0, z, \Omega(x_0), t)) = S_-(\Upsilon_0).$$

It is obvious that

$$\Upsilon_0\left(W^s_{\mathrm{loc}}(S_+(\Upsilon_0))\right) \subset W^s_{\mathrm{loc}}(S_-(\Upsilon_0)), \ \Upsilon_0^2\left(W^s_{\mathrm{loc}}(S_+(\Upsilon_0))\right) \subset W^s_{\mathrm{loc}}(S_+(\Upsilon_0)).$$

Consider the set

$$\begin{aligned}\{(x, z, y, t) : \quad & x = (1-2\varepsilon)w\omega(w), y = \left[a(a-2) - \omega^2(w) - w^2\right]/2, \\ & z = -\omega(w), t = w, 0 < w < a(a-2)/2\}.\end{aligned}$$

Computing its Υ_0-image, one can easily verify that the given set is a semi-locally unstable manifold $W^u_{\mathrm{sl}}(S_+(\Upsilon_0))$ of Υ_0 at $S_+(\Upsilon_0)$. In fact, the equalities

$$x = (1-2\varepsilon)w\omega(w), \ y = \left[a(a-2) - \omega^2(w) - w^2\right]/2,$$

where $\omega(t)$ is the solution of the equation (2.2), are fulfilled for any $(x, z, y, t) \in W^u_{\mathrm{sl}}(S_+(\Upsilon_0))$. Therefore $x = \omega(y)$. The latter means that

$$\Upsilon_0((x, z, y, t)) = (-(1-2\varepsilon)\bar{w}\omega(\bar{w}), \omega(\bar{w}), [a(a-2) - \omega^2(\bar{w}) - \bar{w}^2]/2, \bar{w})$$

and

$$\Upsilon_0^2\,((x,z,y,t)) = (-(1-2\varepsilon)\breve{w}\omega(\breve{w}), \omega(\breve{w}), \left[a(a-2)-\omega^2(\breve{w})-\breve{w}^2\right]/2, \breve{w}),$$

where $\bar{w} = y$ and $\breve{w} = \left[a(a-2)-\omega^2(\bar{w})-\bar{w}^2\right]/2$. Thus, $W^u_{\mathrm{sl}}(S_+(\Upsilon_0))$ is an invariant set with respect to Υ_0^2.

Let zOt be a factor space similar to xOy, namely $zOt = \{(\cdot, z, \cdot, t)\}$ or simply $\{(z,t)\}$. Because projections of $W^u_{\mathrm{sl}}(S_+(\Upsilon_0))$ on xOy and zOt are stretched under the action of Υ_0, the set $W^u_{\mathrm{sl}}(S_+(\Upsilon_0))$ is stretched under the action of Υ_0 too. Therefore, $\Upsilon_0\left(W^u_{\mathrm{sl}}(S_+(\Upsilon_0))\right) \supset W^u_{\mathrm{sl}}(S_-(\Upsilon_0))$ and $\Upsilon_0^2\left(W^u_{\mathrm{sl}}(S_+(\Upsilon_0))\right) \supset W^u_{\mathrm{sl}}(S_+(\Upsilon_0))$.

Denote by $W^u(S_+(\Upsilon_0)) = \bigcup_{j=0}^{\infty} \Upsilon_0^{2j}\left(W^u_{\mathrm{sl}}(S_+(\Upsilon_0))\right)$ the globally unstable manifold of Υ_0 at $S_+(\Upsilon_0)$. Let us show that there is a point where $W^u(S_+(\Upsilon_0))$ intersects $W^s_{\mathrm{loc}}(S_+(\Upsilon_0))$ transversely. Since

$$\det\left(D\Upsilon_0^2(S_\pm(\Upsilon_0)) - \nu(0)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$
$$= \nu^2(0)\cdot\det\left(DF_0^2(S_\pm) - \nu(0)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right),$$

we observe that two eigenvalues ν_j, $j = 1, 2$ of $D\Upsilon_0^2$ at $S_\pm(\Upsilon_0)$ are equal to 0 and the other two eigenvalues are nonzero and coincide with those of DF^2 at $S_\pm$. Therefore $|\nu_3(0)| < 1$, $|\nu_4(0)| > 1$. Thus, the dimension of the locally stable manifold of Υ_0 adjoining to the point $S_\pm(\Upsilon_0)$ is equal to 3 and that of the local unstable manifold of Υ_0 at $S_\pm(\Upsilon_0)$ is equal to 1. Further, considering the manifold projections on xOy, we find those that coincide with $W^s(S_+)$ and $W^u(S_+)$, respectively. According to Lemma 2.8, in any vicinity of $U(S_+)\setminus\{S_+\}$, there are points of transverse intersection of $W^s(S_+)$ and $W^u(S_+)$. Let $(x_\Theta, y_\Theta) \in U(S_+)\setminus\{S_+\}$ be one of such points. Because $(x_\Theta, y_\Theta) \in W^u(S_+)$, there exist w, j such that $F^j((\check{x}_\Theta, \check{y}_\Theta)) = (x_\Theta, y_\Theta)$, where $(\check{x}_\Theta, \check{y}_\Theta) \in W^u_{\mathrm{loc}}(S_+)$, $\check{x}_\Theta = (1-2\varepsilon)w\omega(w)$ and $\check{y}_\Theta = \left[a(a-2)-\omega^2(w)-w^2\right]/2$. Consider a point $\Upsilon_0^j\left((\check{x}_\Theta, \check{z}_\Theta, \check{y}_\Theta, \check{t}_\Theta)\right)$, where $\check{z}_\Theta = -\omega(w)$, $\check{t}_\Theta = w$. It is obvious that $\Upsilon_0^j\left((\check{x}_\Theta, \check{z}_\Theta, \check{y}_\Theta, \check{t}_\Theta)\right) = (x_\Theta, z_\Theta, y_\Theta, t_\Theta) \in W^u(S_+(\Upsilon_0))$. On the other hand, $(x_\Theta, y_\Theta) \in W^s(S_+)$ implies $y_\Theta = \Omega(x_\Theta)$. Therefore $(x_\Theta, z_\Theta, y_\Theta, t_\Theta) \in W^s_{\mathrm{loc}}(S_+(\Upsilon_0))$. It is easy to verify that this is a point of transverse intersection of $W^s_{\mathrm{loc}}(S_+(\Upsilon_0))$ and $W^u(S_+(\Upsilon_0))$.

Indeed, as is known, a curve determined by the equations $x = x(w)$, $z = z(w)$, $y = y(w)$, $t = t(w)$ is tangent to the hypersurface $N(x,z,y,t) = 0$ at the point $(x(\hat{w}), z(\hat{w}), y(\hat{w}), t(\hat{w}))$ if the following two conditions are fulfilled: $N(x(\hat{w}), z(\hat{w}), y(\hat{w}), t(\hat{w})) = 0$ and

$$\left(\frac{\partial N}{\partial x}x' + \frac{\partial N}{\partial z}z' + \frac{\partial N}{\partial y}y' + \frac{\partial N}{\partial t}t'\right)|_{w=\hat{w}} = 0. \tag{3.1}$$

In doing so, at least one of the derivatives $\dfrac{\partial N}{\partial x}, \dfrac{\partial N}{\partial y}, \dfrac{\partial N}{\partial z}$ must be nonzero. Applying the aforesaid reasons to the case under examination, we find that $N(x, z, y, t) = y - \Omega(x)$, $\partial N/\partial x = -\Omega'(x)$, $\partial N/\partial z = 0$, $\partial N/\partial y = 1$, $\partial N/\partial t = 0$. Without loss of generality, one can assume that $\hat{w}$ is such that $(x_\Theta, z_\Theta, y_\Theta, t_\Theta) \equiv (x(\hat{w}), z(\hat{w}), y(\hat{w}), t(\hat{w}))$. The equality (3.1) reduces then to

$$\frac{dy(\hat{w})}{dw} - \frac{d\Omega(x(\hat{w}))}{dx}\frac{dx(\hat{w})}{dw} = 0.$$

Since the projection of $W^u(S_+(\Upsilon_0))$ on xOy coincides with $W^u(S_+)$, at least one of the derivatives $dx(\hat{w})/dw, dy(\hat{w})/dw$ should be nonzero. Let it be $dx(\hat{w})/dw$. Taking into account that $\left|\dfrac{dy(\hat{w})/dw}{dx(\hat{w})/dw}\right| = \left|\dfrac{1}{\omega'(y_\Theta)}\right| > \dfrac{1}{\tau} \gg 1$, we find that

$$\frac{dy(\hat{w})}{dw} - \frac{d\Omega(x(\hat{w}))}{dx}\frac{dx(\hat{w})}{dw} = \frac{dx(\hat{w})}{dw}\left[\frac{1}{\omega'(y_\Theta)} - \Omega'(x_\Theta)\right] \neq 0$$

because $|\Omega'(x_\Theta)| \ll 1$ at (x_Θ, y_Θ). Therefore, $(x_\Theta, z_\Theta, y_\Theta, t_\Theta)$ is a point where $W^u(S_+(\Upsilon_0))$ intersects $W^s_{\mathrm{loc}}(S_+(\Upsilon_0))$ transversely.

Consider now Υ_b. Since $D\Upsilon_b = (1-2\eta)b^2 \neq 0$, Υ_b is a diffeomorphism of $\mathbb{R}^4$ into itself.

$$P_+(\Upsilon_b) \equiv \left(0, 0, \sqrt{a(a-2) + (1-b^2)} - (1-b), \sqrt{a(a-2) + (1-b^2)} - (1-b)\right)$$

is a fixed point of Υ_b and $\Lambda(\Upsilon_b) \equiv \mathrm{Cl}\left(W^u(P_+(\Upsilon_b)) \cap \{(x, z, y, t) : z = t = 0\}\right)$. At $\varepsilon = \varepsilon_{ab\eta}$, where $\varepsilon_{ab\eta}$ is a solution of the equation $[1 - (1-2\eta)b]/(1 - 2\varepsilon_{ab\eta}) = \sqrt{a(a-2) + (1-b^2)} - (1-b)$, the point $P_+(\Upsilon_b)$ bifurcates in a direction transverse to the plane $x = z = 0$ and two periodical of period 2 points $S_\pm(\Upsilon_b) \equiv (\pm x_s, \pm z_s, y_s, t_s)$ appear. Here

$$x_s = -z_s = \sqrt{a(a-2) - [1-(1-2\eta)b]^2(1-2\varepsilon)^{-2} - 2(1-2\eta)[1-(1-2\eta)b]/(1-2\varepsilon)}$$

and $y_s = t_s = [1-(1-2\eta)b]/(1-2\varepsilon)$. The points $S_\pm(\Upsilon_b)$ exist as long as the values of a, ε, b and η satisfy the inequalities

$$\begin{aligned} -(1-b) - \sqrt{a(a-2) + (1-b^2)} &\leq [1-(1-2\eta)b]/(1-2\varepsilon) \\ &\leq \sqrt{a(a-2) + (1-b^2)} - (1-b). \end{aligned}$$

Of course, $\lim\limits_{\varepsilon \to \varepsilon_{ab\eta}} S_\pm(\Upsilon_b) = P_+(\Upsilon_b)$. It is clear that Υ_b converges to Υ_0 uniformly as $|b| \to 0$ and $\lim\limits_{b \to 0} S_\pm(\Upsilon_b) = S_\pm(\Upsilon_0)$. The latter implies $\lim\limits_{b \to 0} \nu_j(b) = \nu_j(0)$, where $\nu_j(b)$ are the eigenvalues of $D\Upsilon_b^2$ at $S_\pm(\Upsilon_b)$. Therefore, for $|b|$ small enough, Υ_b has the 3-dimensional stable $W^s(S_+(\Upsilon))$ manifold and the 1-dimensional unstable $W^u(S_+(\Upsilon_b))$

manifold. We show there exist points where these manifolds intersect each other transversely.

Let a, ε be such that Lemma 2.10 is valid for F. As is known [8], a stable/unstable manifold of diffeomorphism at its hyperbolic fixed point depends continuously (even C^k-smoothly, where k is the degree of diffeomorphism smoothness) on the diffeomorphism. Analyzing the proof of the given fact which is presented in [8], it is not difficult to observe that this proof remains true for a wide map class. This is because what is really used there is a non-degeneracy of differential on its contracting/expanding tangent subspace rather than in the whole tangent space. Since $\nu_4(0)$ is the only eigenvalue of $D\Upsilon_0^2$ whose absolute value exceeds 1, we see that $D\Upsilon_0^2\,(S_+(\Upsilon_0))$ is an operator which is non-degenerate on its expanding tangent subspace and, therefore, the aforementioned theorem is true for $W^u\,(S_+(\Upsilon_0))$ too. The latter means that an arbitrary compact piece of $W^u\,(S_+(\Upsilon_0))$ can be with any desirable accuracy approximated (as $b \to 0$) by compact pieces of $W^u\,(S_+(\Upsilon_b))$. Unfortunately, a similar general result with regard to properties of stable manifolds at the hyperbolic fixed point of endomorphisms whose differential is degenerate on its tangent contracting subspace is unknown to the author. Therefore, in what follows, the continuous dependence of the stable manifold of $W^s\,(S_+(\Upsilon_0))$ on the map will be shown. In other words, we show that $W^s_{\text{loc}}\,(S_+(\Upsilon_b))$ tends uniformly to $W^s_{\text{loc}}\,(S_+(\Upsilon_0))$ as $b \to 0$.

Let us look for the locally stable manifold of Υ_b at $S_+(\Upsilon_b)$ as a hypersurface $y = \wp_b(x, z, t)$. Due to symmetry of $W^s\,(S_+(\Upsilon_b))$, $W^s\,(S_-(\Upsilon_b))$ with respect to opposite values of x and z, it is clear that $\wp_b(x, z, t) = \wp_b(-x, -z, t)$. Let $(x, z, y, t) \in W^s\,(S_+(\Upsilon_b))$. As $\Upsilon_b\left(W^s\,(S_+(\Upsilon_b))\right) = W^s\,(S_-(\Upsilon_b))$, we have $(\breve{x}, \breve{z}, \breve{y}, \breve{t}) \in W^s\,(S_-(\Upsilon_b))$, where $\breve{x} = (1-2\varepsilon)x\wp_b(x, z, t) + (1-2\eta)bz$, $\breve{z} = x$, $\breve{y} = \left[a(a-2) - \wp_b^2(x, z, t) - x^2\right]/2 + bt$, $\breve{t} = \wp_b(x, z, t)$ and, in doing so, $\breve{y} = \wp_b(\breve{x}, \breve{z}, \breve{t})$. Since $\breve{y} = \wp_b(\breve{x}, \breve{z}, \breve{t}) = \wp_b(-\breve{x}, -\breve{z}, \breve{t})$, what is said above can be expressed in view of an equation with respect to $\wp_b$:

$$\frac{a(a-2)}{2} + bt - \frac{x^2 + \wp_b^2(x, z, t)}{2} = \wp_b\left((1-2\varepsilon)x\wp_b(x, z, t) + (1-2\eta)bz, -x, \wp_b(x, z, t)\right).$$

Differentiation of the given equation with respect to x, z, t gives us three equations involving $\frac{\partial\wp_b}{\partial x}$, $\frac{\partial\wp_b}{\partial z}$, $\frac{\partial\wp_b}{\partial t}$. Substituting the coordinates of $S_+(\Upsilon_b)$ to these equations, we obtain equations for

$$\frac{\partial\wp_b(x_s, z_s, t_s)}{\partial x}, \quad \frac{\partial\wp_b(x_s, z_s, t_s)}{\partial z}, \quad \frac{\partial\wp_b(x_s, z_s, t_s)}{\partial t}.$$

Solving them and taking then limits, we find that $\lim\limits_{b\to 0} \frac{\partial\wp_b(x_s, z_s, t_s)}{\partial x} = \Omega'(x_s(0))$, $\lim\limits_{b\to 0} \frac{\partial\wp_b(x_s, z_s, t_s)}{\partial z} = 0$, $\lim\limits_{b\to 0} \frac{\partial\wp_b(x_s, z_s, t_s)}{\partial t} = 0$. The latter means that, in a small neighbourhood of $S_+(\Upsilon_0)$, a piece of $W^s\,(S_+(\Upsilon_0))$ adjoining to $S_+(\Upsilon_0)$ can be with any desirable accuracy approximated as $b \to 0$ by $W^s_{\text{loc}}(S_+(\Upsilon_b))$. In view of structural stability of transverse intersections, this implies that if there is a point of transverse intersection of $W^s_{\text{loc}}(S_+(\Upsilon_0))$ with $W^u(S_+(\Upsilon_0)) \setminus \{S_+(\Upsilon_0)\}$, then one can find $\tilde{b} > 0$ such that the manifolds $W^s_{\text{loc}}(S_+(\Upsilon_b))$ and $W^u(S_+(\Upsilon_b)) \setminus \{S_+(\Upsilon_b)\}$ intersect each other transversely in a neighbourhood of the given point when $b \in (-\tilde{b}, \tilde{b})$. The existence of an orbit of F which is transverse homoclinic to S_+ results in the existence of a similar orbit of Υ_b which is transverse homoclinic to $S_+(\Upsilon_b)$. The latter means that Theorem 1.3 is valid for sufficiently small $|b|$. ∎

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