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Convergence of the Homotopy Decomposition Method for Solving Nonlinear Equations

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Abstract

A new definition of the homotopy analysis method is given by means of the decomposition method in this paper. The convergence of the homotopy decomposition method is proved under some reasonable hypotheses, which provide the theoretical basis of the homotopy decomposition method for solving nonlinear problems.

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It is complex and significant to obtain analytical approximations of nonlinear equations. There are some analytic techniques for nonlinear problems, such as perturbation techniques that are well known and widely applied. In the eighties, Adomian [2–4] provided an efficient numerical technique for solving large classes of nonlinear equations. The mathematical technique was used to solving problems of science and engineering [12,13]. The Adomian technique is very simple in its principles. However, Adomian's decomposition method has some restrictions. It is due to the fact that approximate solutions often contain polynomials, and the difficulties consist in proving the convergence of series solutions. Some attempts to prove convergence have been made in [5–7]. These proofs were given in very particular cases.

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In recent years, Shijun Liao [8–11] developed a new analytic method for nonlinear problems in general, namely the homotopy analysis method. Unlike previous Adomian's method, the homotopy analysis method provides us with a simple way to control and adjust the convergence region and convergence rate of solution series of nonlinear problems. The homotopy analysis method provides great freedom to choose proper initial approximations, auxiliary linear operator and auxiliary functions. However, there are no rigorous theories to direct us to choose the initial approximation, the auxiliary linear operator, and the auxiliary function. Therefore, it is necessary for us to seek some mathematical technique to choose them.

In this paper, we propose a new definition of the homotopy analytic method by the decomposition technique. The convergence of the solution series is proved based on suitable and reasonable hypotheses. Furthermore, a mathematical theorem is presented in this paper to direct us to apply the proposed method.

1. The Homotopy Decomposition Method

The homotopy analysis method is proposed by means of homotopy. This method is rather general and valid for nonlinear differential equations. Above all, the so-called homotopy is constructed by introducing an embedding parameter. Then homotopy analytical solutions are obtained by zero-order deformation equations and high-order deformation equations. However, it is difficult to understand the so-called homotopy and n-order of deformation equations. In the following section, we will propose a new definition of the homotopy analytic method — the homotopy decomposition method.

We consider a general nonlinear differential equation

$$F(u) = Lu + Ru + Nu - g = 0, \tag{1.1}$$

where F represents a general nonlinear operator involving both linear and nonlinear terms, L is the highest order derivative, R is the linear differential operator of less order than L, Nu represents the nonlinear terms, g is the source term, and u is an unknown function with t as an independent variable. We introduce a nonzero auxiliary parameter h, a nonzero auxiliary function $H(t)$ and an auxiliary linear operator L_F to construct such a new kind of auxiliary equation

$$L_F u = hH(t)F(u) + L_F u. \tag{1.2}$$

Assuming that a nonlinear problem has a unique solution, we decompose u into

$$u = \sum_{n=0}^{\infty} u_n. \tag{1.3}$$

Eq. (1.2) is changed into

$$L_F \sum_{n=0}^{\infty} u_n = hH(t)F\left(\sum_{n=0}^{\infty} u_n\right) + L_F \sum_{n=0}^{\infty} u_n. \tag{1.4}$$

Let u_0 denote the initial approximation of u according to the initial condition of Eq. (1.1). The homotopy decomposition method employs the recursive relation

$$\begin{aligned}
L_F(u_1) &= hH(t)A_0, \\
L_F(u_2) &= hH(t)A_1 + L_F(u_1), \\
&\cdots\cdots \\
L_F(u_{n+1}) &= hH(t)A_n + L_F(u_n), \qquad (1.5)
\end{aligned}$$

where A_n ($n = 0, 1, 2, \ldots$) represent decomposition polynomials. A_0 and A_n are given by

$$\begin{aligned}
A_0 &= F(u_0) \\
A_n(u_0, u_1, \ldots, u_n) &= \sum_{k_1+2k_2+\cdots+nk_n=n} F_n\left(\frac{u_1^{k_1}}{k_1!}\cdot\frac{u_2^{k_2}}{k_2!}\cdot\cdots\frac{u_n^{k_n}}{k_n!}\right), \quad n > 0, \qquad (1.6)
\end{aligned}$$

where

$$\begin{aligned}
F_n &= L_n + R_n + N_n, \\
L_n(t) &= L\left(\frac{d^{(k_1+k_2+\cdots+k_n)}u}{du^{(k_1+k_2+\cdots+k_n)}}|_{u=u_0}\cdot t\right), \\
R_n(t) &= R\left(\frac{d^{(k_1+k_2+\cdots+k_n)}u}{du^{(k_1+k_2+\cdots+k_n)}}|_{u=u_0}\cdot t\right), \\
N_n(t) &= \frac{d^{(k_1+k_2+\cdots+k_n)}N(u)}{du^{(k_1+k_2+\cdots+k_n)}}|_{u=u_0}\cdot t. \qquad (1.7)
\end{aligned}$$

As long as the solution series (1.3) given by the homotopy decomposition method is convergent, it must be the solution of the considered nonlinear problem. A series is often of no use if it is convergent in a rather restricted region, and thus proving convergence of the solution series is very important.

2. Convergence of the Homotopy Decomposition Method

Let us reconsider the formula

$$\begin{aligned}
A_0 &= F(u_0) \\
A_n(u_0, u_1, \ldots, u_n) &= \sum_{k_1+2k_2+\cdots+nk_n=n} F_n\left(\frac{u_1^{k_1}}{k_1!}\cdot\frac{u_2^{k_2}}{k_2!}\cdot\cdots\frac{u_n^{k_n}}{k_n!}\right), \quad n > 0. \qquad (2.1)
\end{aligned}$$

In Eq. (2.1) we have to sum over all solutions of the equation

$$k_1 + 2k_2 + \cdots + nk_n = n, \quad k_i \geq 0, \ \ i = 0, 1, 2, \ldots. \qquad (2.2)$$

Let $P(n)$ be the number of solutions of Eq. (2.1). Abbaoui and Cherruault [1] showed that

$$p(n) < \exp\left(n\pi\sqrt{\frac{2}{3}}\right), \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.3}$$

Now, let us consider a Banach space E, and $\| \cdot \|$ denotes the norm in E. Return to the nonlinear equation mentioned above

$$F(u) = 0, \tag{2.4}$$

where F represents a general nonlinear operator involving both linear and nonlinear terms. From (2.3), we deduce the following result.

Theorem 2.1. If we assume that

(a) $F(u)$ is analytic in a neighborhood of u_0, and $\| F_n \| \leq M'$ for any n;

(b) $\| H(t) \| \leq M''$, $\| L_F u_0 \| \leq M'''$ and

$$\left\| \sum_{k_1+2k_2+\cdots+nk_n=n} F_n\left(\frac{u_1^{k_1}}{k_1!}\cdot\frac{u_2^{k_2}}{k_2!}\cdots\frac{u_n^{k_n}}{k_n!}\right)\right\| \leq M^n \exp\left(n\pi\sqrt{\frac{2}{3}}\right)$$
$$(0 < M < 1) \text{ for } n = 1, 2, 3, \ldots;$$

(c) there exists a constant $k > 0$ such that $\| L_f(u_{n+1} - u_n) \| \geq k \| u_{n+1} - u_n \|$ for $u_n \in E$ $(n = 0, 1, 2, \ldots)$,

then the solution series $\sum_{n=0}^{\infty} u_n$ by the scheme Eqs. (1.5) is absolutely convergent.

Proof. We know that

$$A_n(u_0, u_1, \ldots, u_n) = \sum_{k_1+2k_2+\cdots+nk_n=n} F_n\left(\frac{u_1^{k_1}}{k_1!}\cdot\frac{u_2^{k_2}}{k_2!}\cdot\cdots\frac{u_n^{k_n}}{k_n!}\right), \quad n > 0. \tag{2.5}$$

By using the hypothesis (b), one can find that

$$\| A_n \| \leq M' M^n \exp\left(n\pi\sqrt{\frac{2}{3}}\right). \tag{2.6}$$

Eq. (1.5) can be written as

$$\begin{aligned} L_F(u_1 - u_0) &= hH(t)A_0 - L_F u_0, \\ L_F(u_{n+1} - u_n) &= hH(t)A_n, \quad n > 0. \end{aligned} \tag{2.7}$$

Applying (2.6) and the hypothesis (c), it is easy to see that

$$\| u_{n+1} - u_n \| \leq \left| \frac{1}{k} h M' M'' M^n \exp\left(n\pi\sqrt{\frac{2}{3}}\right) + (M''')^{1-\chi_n} \right| \quad \text{for any } n, \tag{2.8}$$

where

$$\chi_n = \begin{cases} 0, & \text{when } n = 0 \\ 1, & \text{otherwise.} \end{cases} \tag{2.9}$$

Then, we have

$$\| (n+1)(u_n - u_{n+1}) \| \leq (n+1) \left| \frac{1}{k} h M' M'' M^n \exp\left(n\pi\sqrt{\frac{2}{3}}\right) + (M''')^{1-\chi_n} \right|, \tag{2.10}$$

which is the general term of a convergent series. But we know that

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} (n+1)(u_n - u_{n+1}) \tag{2.11}$$

and thus $\sum_{n=0}^{\infty} u_n$ is convergent in E.

3. Conclusions

In this paper, we give a new definition of the homotopy analysis method called the homotopy decomposition method, which is proposed by means of Adomian's method. Consequently, it becomes possible to prove the convergence of the homotopy analysis method with reasonable assumptions. At the same time, these reasonable assumptions provide mathematical theorems for us to choose the initial approximation, the auxiliary linear operator, and the auxiliary function.

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