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On the Relationship between the Classical Linearization and Optimal Derivative[1]

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Abstract

The aim of this paper is to present the relationship between the classical linearization and the optimal derivative of a nonlinear ordinary differential equation. An application is presented using the quadratic error.

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1. Introduction

The study of stability of the equilibrium point of a nonlinear ordinary differential equation is an almost trivial problem if the function F which defines the nonlinear equation is sufficiently regular in the neighborhood of this point and if its linearization in this point is hyperbolic. In this case, we know that the nonlinear equation is equivalent to the

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linearized equation, in the sense that there exists a local diffeomorphism which transforms the neighboring trajectories of the equilibrium point to those neighbors of zero of the linear equation. On the other hand, the problem is all other when the nonlinear function is nonregular or the equilibrium point is the center.

Consider the nonregular case. Imagine the case when the only equilibrium point is nonregular. In this case, we cannot derive the nonlinear function and consequently we cannot study the linearized equation. A natural question arises then: Is it possible to associate another linear equation to the nonlinear equation which has the same asymptotic behavior?

The idea proposed by Benouaz and Arino is based on the method of approximation. In [2, 5–8], the authors introduced the optimal derivative, which is in fact a global approximation as opposed to the nonlinear perturbation of a linear equation, having a distinguished behavior with respect to the classical linear approximation in the neighborhood of the stationary point. The approach used is the least square approximation.

The aim of this paper is to present the relationship between the optimal derivative and Fréchet derivative in the equilibrium point. After a brief review of the optimal derivative procedure in the second section, the third section is devoted to the study of the relationship between the optimal derivative and Fréchet derivative in the equilibrium point in the scalar and vectorial case. In particular, in the scalar case, we prove for a class of functions that the optimal derivative can be computed even though the classical linearization in 0 does not exist. In the last section, we present applications and comparison using the quadratic error. The study shows, in particular, the influence of the choice of initial conditions. In the appendix, we have presented the details of the proof for the example in the case where the nonlinear function is not regular in 0.

2. The Optimal Derivative

2.1. The Procedure

Consider a nonlinear ordinary differential problem of the form

$$\frac{dx}{dt} = F(x), \quad x(0) = x_0, \tag{2.1}$$

where

- $x = (x_1, \ldots, x_n)$ is the unknown function,
- $F = (f_1, \ldots, f_n)$ is a given function on an open subset $\Omega \subset \mathbb{R}^n$,

with the assumptions

(H_1) $F(0) = 0$,

(H_2) the spectrum $\sigma(DF(x))$ is contained in the set $\{z : \mathrm{Re} z < 0\}$ for every $x \neq 0$, in a neighborhood of 0 for which $DF(x)$ exists,

(H_3) F is γ-Lipschitz continuous.

Consider $x_0 \in \mathbb{R}^n$ and the solution x of the nonlinear equation starting at x_0. With all linear $A \in \mathcal{L}(\mathbb{R}^n)$, we associate the solution y of the problem

$$\frac{dy}{dt} = Ay(t), \quad y(0) = y_0,$$

and we try to minimize the functional

$$G(A) = \int_0^\infty \|F(y(t)) - Ay(t)\|^2 \, dt \tag{2.2}$$

along a solution y. We obtain

$$\tilde{A} = \left(\int_0^\infty [F(x(t))][x(t)]^T dt \right) \left(\int_0^\infty [x(t)][x(t)]^T dt \right)^{-1}. \tag{2.3}$$

Precisely, the procedure is defined by the following scheme: Given x_0, we choose a first linear map. For example, if F is differentiable in x_0, then we can take $A_0 = DF(x_0)$ or the derivative value in a point in the vicinity of x_0. This is always possible if F is locally Lipschitz. If A_0 is an asymptotically stable map, then the solution starting from x_0 of the problem

$$\frac{dy}{dt} = A_0 y(t), \quad y(0) = y_0$$

tends to 0 exponentially. We can evaluate $G(A)$ using (2.2) and we minimize G for all matrices A. If F is linear, then the minimum is reached for the value $A = F$ (and we have $A_0 = F$). Generally, we can always minimize G, and the matrix which gives the minimum is unique. We call this matrix A_1 and replace A_0 by A_1, we replace y by the solution of the linearized equation associated to A_1, and we continue. The optimal derivative $\tilde{A}$ is the limit of the sequence build as such, and it is given by (2.3) (for details see [2, 5–7]).

2.2. Properties of the Procedure

We will now consider situations where the procedure converges.

Influence of the choice of the initial condition

Note that if we change $x(t)$ to z, then the relation (2.3) can be written as

$$\tilde{A} \oint_0^{x_0} z dz^T = \int_0^{x_0} F(z) dz^T,$$

where $\oint_0^{x_0}$ is the curvilinear integral along the orbit $\gamma(x_0) = \{e^{Bt} : t \geq 0\}$ of x_0. We obtain

$$\tilde{A} = \left(\oint_0^{x_0} F(z) dz^T \right) \left(\oint_0^{x_0} z dz^T \right)^{-1}.$$

It is clear that the optimal derivative depends on the initial condition x_0.

Case when F is linear

If F is linear with $\sigma(F)$ in the negative part of the complex plane, then the procedure gives F at the first iteration. Indeed, in this case, (2.3) reads

$$A\Gamma(x) = F\Gamma(x)$$

and it is clear that $A = F$ is a solution. It is unique if $\Gamma(x)$ is invertible. Therefore, the optimal approximation of a linear system is the system itself.

Case when F is the sum of a linear and nonlinear term

Consider the more general system of nonlinear equations with a nonlinearity of the form

$$F(x) = Mx + \tilde{F}(x), \quad x(0) = x_0,$$

where M is linear. The computation of the matrix A_1 gives

$$\begin{aligned} A_1 &= \left[\int_0^\infty [F(x(t))][x(t)]^T dt\right][\Gamma(x)]^{-1} \\ &= \left(M\Gamma(x) + \int_0^\infty [\tilde{F}(x(t))][x(t)]^T dt\right)[\Gamma(x)]^{-1} \\ &= M + \left(\int_0^\infty [\tilde{F}(x(t))][x(t)]^T dt\right)[\Gamma(x)]^{-1}. \end{aligned}$$

Hence, $A_1 = M + \tilde{A}_1$ with

$$\tilde{A}_1 = \left(\int_0^\infty [\tilde{F}(x(t))][x(t)]^T dt\right)[\Gamma(x)]^{-1}.$$

Then, for all j we have $A_j = M + \tilde{A}_j$ with

$$\tilde{A}_j = \left(\int_0^\infty [\tilde{F}(x_j(t))][x_j(t)]^T dt\right)\left[\Gamma(x_j)\right]^{-1}.$$

If, in particular, some components of F are linear, then the corresponding components of $\tilde{F}$ are zero, and the corresponding components of A_j are those of F. If f_k is linear, then the kth row of the matrix A_j is equal to f_k.

3. Relationship between the Optimal Derivative and the Classical Linearization in Zero

3.1. Scalar Case

Expression

Consider the scalar differential problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0 \tag{3.1}$$

with $f : \mathbb{R} \to \mathbb{R}$ and under the assumptions

(h$_1$) $f(0) = 0$,

(h$_2$) $f'(x) < 0$ in every point where f' exists in an interval $(-\alpha, \alpha)$ with $\alpha > 0$,

(h$_3$) f is absolutely continuous with respect to the Lebesgue measure.

The calculation is done in a way similar to that of the vectorial case. We start with the calculation of $a_0 = f'(x_0)$, then we calculate a_1 by solving the problem

$$\frac{dx}{dt} = a_0 x, \quad x(0) = x_0.$$

By changing F to f in (2.3), we have

$$a_1 = \frac{\int_0^\infty f(x(t))x(t)dt}{x_0 \int_0^\infty x^2(t)dt},$$

and by substituting $x = \exp(a_0 t)x_0$, we obtain

$$a_1 = \frac{\int_0^{x_0} f(x)dx}{\int_0^{x_0} xdx} = \frac{2}{x_0^2}\int_0^{x_0} f(x)dx.$$

Note that a_1 does not depend on a_0, and consequently, the procedure for the optimal derivative converges in the first step, namely

$$\tilde{a} = \tilde{a}(x_0) = \frac{2}{x_0^2}\int_0^{x_0} f(x)dx. \tag{3.2}$$

We remind the reader that it has been shown that $\tilde{a}(x_0)$ is a Lyapunov function [8] for the nonlinear problem (3.1). The scalar case is very interesting in the sense that we can write the optimal derivative as a function of the classical linearization of f in 0 (if f' exists in 0); so it is possible to find a limit when $x_0 \to 0$, namely $\tilde{a}(x_0)$, even though the derivative of f in 0 does not exist. The importance of the result lies in the possibility of using $\tilde{a}(x_0)$ for the description of the behavior of the solution and for the study of stability in the vicinity of 0 when the derivative in this point does not exist.

Case when the derivative of f in 0 exists

If f is continuous and if the derivative of f in 0 exists, then it is known [2] that $\tilde{a}(x_0)$ can be written as

$$\tilde{a}(x_0) = f'(0) + \frac{2}{x_0^2}\int_0^{x_0} z\varepsilon(z)dz, \quad \text{where} \quad \varepsilon(z) = \frac{f(z)}{z} - f'(0)$$

and that $\lim_{x_0 \to 0} \tilde{a}(x_0) = f'(0)$. This relation shows that the two quantities $\tilde{a}(x_0)$ and $f'(0)$ are almost equal and are equal in the limit as x_0 tends to 0.

Case when f is analytic in 0

Now assume that f is analytic in 0, i.e.,

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{3.3}$$

Then it is possible to give an expansion of $\tilde{a}(x_0)$ similar to the Taylor expansion of f in the neighborhood of 0. For this, we use the relation (3.2) and replace $f(z)$ by the expression given by relation (3.3) so that

$$\begin{aligned}\tilde{a}(x_0) &= \frac{2}{x_0^2} \int_0^{x_0} \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n dx \;=\; 2 \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{(n+1)!} x_0^{n-1} \\ &= f'(0) + \frac{1}{3} x_0 f'(0) + \ldots + \frac{2}{(n+1)!} x_0^{n-1} f^{(n)}(0) + \ldots,\end{aligned}$$

where this formula holds in the interval of convergence of the Taylor series in 0. Generally, if f is of class C^k with $k \in \mathbb{N}$ in the vicinity of 0 and $f(0) = 0$, then $\tilde{a}$ is of class C^{k-1} in this vicinity, and we obtain

$$\tilde{a}^{(j)}(0) = \frac{2}{(j+1)!} x_0^{j-1} f^{(j)}(0), \quad 0 \le j \le k-1.$$

Case when f is not regular in 0

We now consider the nonregular case, and more particularly the case that f is only nondifferentiable in 0. Writing $f(z)$ in the form

$$f(z) = -z g(z),$$

the relation (3.2) becomes

$$\tilde{a}(x_0) = -\frac{2}{x_0^2} \int_0^{x_0} z g(z) dz. \tag{3.4}$$

The chosen function

$$g_r(z) = p(|\ln z|^r),$$

where p is a bounded nonnegative periodic function of period 1 with $\bar{p} = \int_0^1 p(z) dz > 0$, is nondifferentiable in 0. The relation (3.4) is written for $r = 1$ and $0 < x_0 < 1$ as

$$\tilde{a}(x_0) = -\frac{2}{x_0^2} \int_0^{x_0} z p(|\ln z|) dz.$$

For all $\alpha \in (0, 1)$, we have

$$\begin{aligned}\tilde{a}(\alpha x_0) &= -\frac{2}{\alpha^2 x_0^2}\int_0^{\alpha x_0} zp(-\ln z)dz\\ &= -\frac{2}{\alpha^2 x_0^2}\int_0^{x_0} \alpha^2 zp(-\ln\alpha - \ln z)dz\\ &= -\frac{2}{x_0^2}\int_0^{x_0} zp(-\ln\alpha - \ln z)dz.\end{aligned}$$

So in particular, if $\ln\alpha = -1$, i.e., $\alpha = e^{-1}$, then $\tilde{a}(x_0/e) = \tilde{a}(x_0)$. In this case, $\tilde{a}(x_0)$ does not have a limit when $x_0 \to 0^+$. In the case $r > 1$, we obtain

$$\tilde{a}_r(x_0) = -2\int_0^1 zp((-\ln x_0 - \ln z)^r)dz.$$

Let us now consider the relation

$$\tilde{a}_r(x_0) = -\frac{2}{x_0^2}\int_0^{x_0} ug_r(u)du, \tag{3.5}$$

where $g_r(u) = p(|\ln u|^r)$. Note that $g_r(u)$ is nondifferentiable in 0. In this case, we will show that the optimal derivative (3.5) can exist even if the derivative of the function $g_r(u)$ in 0 does not exist. Then

$$\tilde{a}_r(x_0) \to -\bar{p} \quad \text{when} \quad x_0 \to 0 \quad \text{for every} \quad r > 1.$$

For more details, see the proof given in the appendix.

3.2. Vectorial Case

Let us suppose that the sequence A_j given by

$$A_j = \left(\int_0^\infty \left[F\left(e^{tA_{j-1}}x_0\right)\right]\left[e^{tA_{j-1}}x_0\right]^T dt\right)\left(\int_0^\infty \left[e^{tA_{j-1}}x_0\right]\left[e^{tA_{j-1}}x_0\right]^T dt\right)^{-1}$$

converges to the optimal matrix and that the derivative $DF(0)$ of F in 0 exists. In this case, we can write

$$F(x) = DF(0)x + \mathrm{o}(|x|). \tag{3.6}$$

Replacing the relation (3.6) in (2.3) and using the properties of the optimal derivative from [2, 5], we find

$$\begin{aligned}\tilde{A} &= \left[\int_0^\infty [DF(0)x(t) + \mathrm{o}(|x(t)|)]\,[x(t)]^T dt\right]\left[\int_0^\infty [x(t)][x(t)]^T dt\right]^{-1}\\
&= DF(0)\left[\int_0^\infty [x(t)][x(t)]^T dt\right]\left[\int_0^\infty [x(t)][x(t)]^T dt\right]^{-1}\\
&\quad+\left[\int_0^\infty [\mathrm{o}(|x(t)|)][x(t)]^T dt\right]\left[\int_0^\infty [x(t)][x(t)]^T dt\right]^{-1}\\
&= DF(0)+\left[\int_0^\infty [\mathrm{o}(|x(t)|)][x(t)]^T dt\right]\left[\int_0^\infty [x(t)][x(t)]^T dt\right]^{-1},\end{aligned}$$

where

$$\left[\int_0^\infty [\mathrm{o}(|x(t)|)][x(t)]^T dt\right]\left[\int_0^\infty [x(t)][x(t)]^T dt\right]^{-1} = \mathrm{o}(1),$$

i.e., a quantity which tends to 0 when $x_0 \to 0$, by supposing that $|x(t)|$ remains of the order of x_0.

4. Application

The precision of the optimal derivative is expressed in terms of the norm of the initial condition x_0 [8] and is given by

$$\|x(t) - \tilde{y}(t)\| < \mathrm{O}(\|x_0\|)^2.$$

The goal is to try to show for which initial conditions the precision is maintained. As long as $\|x_0\|$ is large in a certain sense, the approximation must be good. It becomes more difficult when approaching 0. Indeed, it is shown that the approach of 0 yields inversion of the quadratic error to the profit of the classical linearization. This shows, that the classical linearization is better near the origin when it exists. Let us present examples emphasizing the theoretical aspect in relation to the influence of the choice of the initial conditions on the quality of the approximation.

4.1. Computational Procedure

First of all let us point out briefly the iterative procedure allowing the calculation of the optimal derivative. Starting the calculus, the point x_0 is selected arbitrarily near the origin. The differential equations have been solved using the fourth order Runge–Kutta method [13, 16].

- Input x_0 and A_0.
- Level (I): Computation of A_1 in terms of A_0:

$$A_1 = \left[\int_0^\infty \left[F\left(e^{A_0 t}x_0\right)\right]\left[e^{A_0 t}x_0\right]^T dt\right]\left[\int_0^\infty \left[e^{A_0 t}x_0\right]\left[e^{A_0 t}x_0\right]^T dt\right]^{-1}.$$

- Level (II): Computation of A_j in terms of A_{j-1}:

$$A_j = \left[\int_0^{\infty} \left[F\left(e^{A_{j-1}t}x_0\right)\right]\left[e^{A_{j-1}t}x_0\right]^T dt\right] \times \left[\int_0^{\infty} \left[e^{A_{j-1}t}x_0\right]\left[e^{A_{j-1}t}x_0\right]^T dt\right]^{-1}.$$

- Level (III): Computation of

$$\left\| A_j - A_{j-1} \right\|.$$

- Level (IV): If

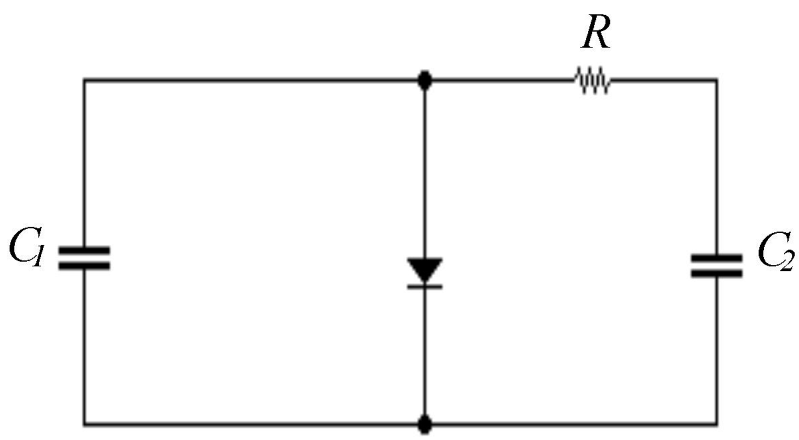
$$\left\| A_j - A_{j-1} \right\| < \varepsilon,$$

  where ε is the desired level of approximation, then set $\tilde{A} = A_j$. $\tilde{A}$ is the optimal derivative of F at x_0. Otherwise set $A_{j-1} = A_j$ and go to Level (II).

4.2. Example

The function of the electronic circuit (see [11]) in Figure 1 is represented by two variables of states (the voltage drop V_{c1} on the terminal of the first capacity and the voltage drop V_{c2} on the terminal of the second capacity). The nonlinearity is due to the use of a nonlinear diode.

Figure 1: Circuit used in the example



When a tension V_c is applied to the diode in the direct direction, the model of the diode is given by

$$f\left(V_{c1}\right) = \begin{cases} 0 & \text{if} \quad V_{c1} < 0 \\ aV_{c1} + bV_{c1}^2 + dV_{c1}^4 & \text{if} \quad V_{c1} \geq 0. \end{cases}$$

With the parameters

$$R = 33 \cdot 10^2 \Omega, \quad C_1 = 220 \cdot 10^{-4} F, \quad C_2 = 350 \cdot 10^{-4} F,$$
$$a = 10^{-4}, \quad b = 10^{-5}, \quad d = 10^{-6}$$

and starting from the laws of Kirchhoff relating to the nodes and the meshes of the circuit, we obtain the equations

$$\begin{cases} \dfrac{dV_{c1}}{dt} = -\dfrac{1}{C_1}\left(aV_{c1} + bV_{c1}^2 + dV_{c1}^4 + \dfrac{V_{c1} - V_{c2}}{R}\right) \\ \dfrac{dV_{c2}}{dt} = \dfrac{1}{RC_2}\left[V_{c1} - V_{c2}\right]. \end{cases} \tag{4.1}$$

Changing

$$x = V_{c1} \quad \text{and} \quad y = V_{c2},$$

the system (4.1) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{a}{C_1}x - \dfrac{b}{C_1}x^2 - \dfrac{d}{C_1}x^4 - \dfrac{1}{RC_1}x + \dfrac{1}{RC_1}y \\ \dfrac{dy}{dt} = \dfrac{1}{RC_2}x - \dfrac{1}{RC_2}y. \end{cases}$$

By replacing the parameters with their values, the system becomes

$$\begin{cases} \dfrac{dx}{dt} = -\left(1.8 \cdot 10^{-2}x + 4.55 \cdot 10^{-5}\left(10x^2 + x^4\right) - 1.38 \cdot 10^{-2}y\right) \\ \dfrac{dy}{dt} = 8.66 \cdot 10^{-3}(x - y). \end{cases} \tag{4.2}$$

Classical linearization

The classical linearization at the equilibrium point (0, 0) is obtained by calculating the Fréchet derivative of the nonlinear function of the system (4.2),

$$DF(0,0) = \begin{bmatrix} -1.8 \cdot 10^{-2} & 1.38 \cdot 10^{-2} \\ 8.66 \cdot 10^{-3} & -8.66 \cdot 10^{-3} \end{bmatrix}.$$

Optimal derivative

The optimal derivative is obtained by applying the algorithm proposed above, see Section 4.1. For the quadratic error, we use the relation

$$E_Q = \sum_{i=1}^{n} \|x_i(t) - \tilde{y}_i(t)\|^2,$$

where

- $x(t)$ represents a solution of the nonlinear system,
- $\tilde{y}(t)$ represents a solution of the optimal derivative.

Results of the method

We study the system using several initial conditions. The results obtained are exhibited in the following table, where E_{Qmax} (O.D.) and E_{Qmax} (C.L.) represent the maximum quadratic errors for the optimal derivative and the classical linearization, respectively. In the left column the initial conditions (x_0, y_0) are given. The second column represents the optimal derivative $\tilde{A}$.

(x_0, y_0)	$\tilde{A}$	E_{Qmax} (O.D.)	E_{Qmax} (C.L.)
$(8e-01, 5e-01)$	$\begin{bmatrix} -0.0187 & 0.0142 \\ 0.0087 & -0.0087 \end{bmatrix}$	$2.1302e-04$	$3.5140e-04$
$(8e-02, 5e-01)$	$\begin{bmatrix} -0.0181 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$7.5438e-06$	$1.0367e-05$
$(8e-02, 5e-02)$	$\begin{bmatrix} -0.0181 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$7.4729e-09$	$2.2644e-08$
$(8e-03, 5e-02)$	$\begin{bmatrix} -0.0180 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$8.5925e-10$	$1.0691e-09$
$(8e-03, 5e-03)$	$\begin{bmatrix} -0.0180 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$7.0425e-13$	$2.2132e-12$
$(8e-04, 5e-03)$	$\begin{bmatrix} -0.0179 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$9.0836e-14$	$1.0969e-13$
$(8e-04, 5e-04)$	$\begin{bmatrix} -0.0179 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$2.2657e-17$	$1.3572e-16$
$(8e-05, 5e-05)$	$\begin{bmatrix} -0.0178 & 0.0138 \\ 0.0087 & -0.0087 \end{bmatrix}$	$3.249e-21$	$3.481e-21$

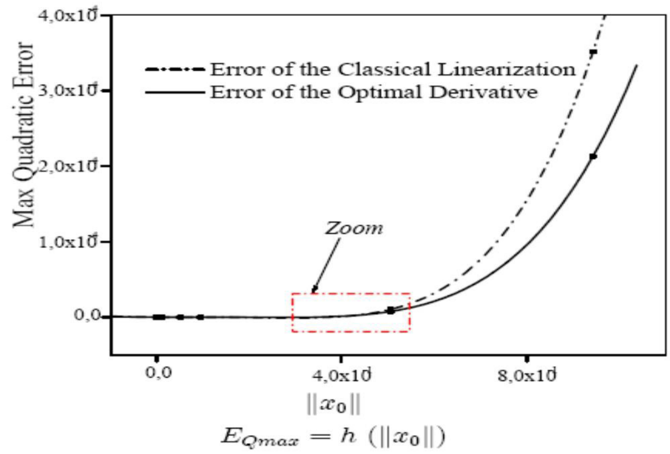
The curve

$$E_{\text{Qmax}} = h(\|x_0\|)$$

in Figure 2 is obtained starting from a smoothing polynomial using the Origin software. The determination of the value x_0 for which the curve of error changes behavior will be calculated is performed using the Matlab software.

In Figure 3, a zoom of the part where there is inversion of the quality of the approximation to the profit of the classical linearization is represented.

Figure 2: Max quadratic error with respect to the norm of the initial condition



5. Analysis of Results

The representation of the maximum quadratic error with respect to $\|x_0\|$ relating to the classical linearization and the optimal derivative enables us to divide our curve into two distinct parts:

- The first part, where the maximum quadratic error due to the classical linearization is lower than that due to the optimal derivative on an interval of $\|x_0\| < 0.43$. In this case the classical linearization gives a better approximation than the optimal derivative.

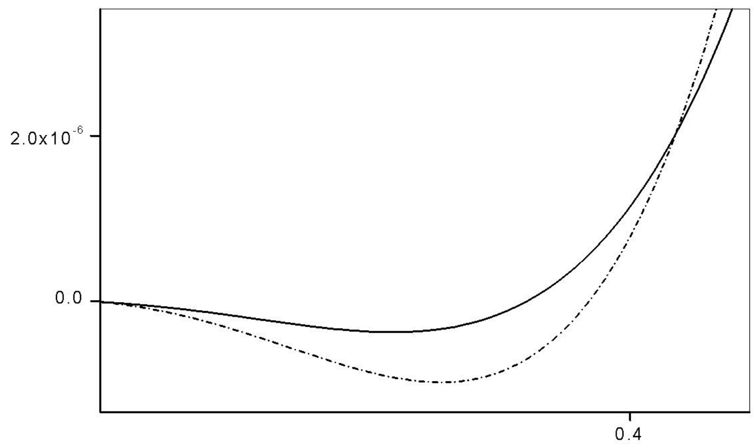
- The second part where the maximum quadratic error due to the classical linearization becomes definitely higher than that due to the optimal derivative on an interval of $\|x_0\| > 0.43$. Here it is the optimal derivative which is better. Namely, for a given initial condition x_0, approximation by the optimal derivative is better in a vicinity of the initial condition, while the classical linearization is better in the vicinity of the origin. These two aspects reflect the fact that the linearization by Fréchet derivative (when it exists and when it is hyperbolic) is the best approximation in the vicinity of the origin.

6. Appendix

By making the change of variable $v = |\ln u|^r$, we obtain

$$\tilde{a}_r(x_0) = -\frac{2}{rx_0^2}\int_{|\ln x_0|^r}^{\infty} e^{-2v^{1/r}} p(v)v^{1/r-1}dv.$$

Figure 3: Zoom of the part where there is inversion of the quality of the approximation



We start by replacing x_0 by a particular sequence of reals tending to 0, the sequence $e^{-k^{1/r}}$, with $k \in \mathbb{N}$, $k \to \infty$. For that, we note that if x_0 is rather small, then there is a unique k such that

$$k < |\ln u|^r < k+1.$$

After some calculations, we find that there is a constant C independent of x_0 such that

$$e^{-k^{1/r}-Ck^{1/r-1}} < x_0 < e^{-k^{1/r}}.$$

From this, we deduce in particular that

$$\frac{x_0^2}{e^{-2k^{1/r}}} \to 1 \quad \text{when} \quad x_0 \to 0. \tag{6.1}$$

We now will calculate the limit of the ratio, when $x_0 \to 0$, of

$$\frac{\tilde{a}_r\left(e^{-k^{1/r}}\right)}{\tilde{a}_r(x_0)} = \frac{-\frac{2}{re^{-2k^{1/r}}}\int_k^\infty e^{-2v^{1/r}}p(v)v^{1/r-1}dv}{-\frac{2}{rx_0^2}\int_{k+1}^\infty e^{-2v^{1/r}}p(v)v^{1/r-1}dv}.$$

Because of (6.1), the ratio of the terms except the integrals in the right-hand side of the equality tends to 1. We thus have

$$\frac{\tilde{a}_r\left(e^{-k^{1/r}}\right)}{\tilde{a}_r(x_0)} \sim \frac{\int_k^\infty e^{-2v^{1/r}}p(v)v^{1/r-1}dv}{\int_{k+1}^\infty e^{-2v^{1/r}}p(v)v^{1/r-1}dv} = 1 + \frac{\int_k^{k+1} e^{-2v^{1/r}}p(v)v^{1/r-1}dv}{\int_{k+1}^\infty e^{-2v^{1/r}}p(v)v^{1/r-1}dv}.$$

Suppose that $0 < m \le p(v) \le M < \infty$. Then

$$\frac{\int_k^{k+1} e^{-2v^{1/r}}p(v)v^{1/r-1}dv}{\int_{k+1}^\infty e^{-2v^{1/r}}p(v)v^{1/r-1}dv} \le \frac{M}{m}\frac{\int_k^{k+1} e^{-2v^{1/r}}v^{1/r-1}dv}{\int_{k+1}^\infty e^{-2v^{1/r}}v^{1/r-1}dv}. \tag{6.2}$$

The integrals on the right-hand side of (6.2) can be calculated using

$$\int_a^b e^{-2v^{1/r}} v^{1/r-1} dv = r \int_{a^{1/r}}^{b^{1/r}} e^{-2w} dw = \frac{r}{2}\left(e^{-2a^{1/r}} - e^{-2b^{1/r}}\right). \tag{6.3}$$

By (6.2) and (6.3), we obtain

$$\begin{aligned}\frac{\int_k^{k+1} e^{-2v^{1/r}} p(v) v^{1/r-1} dv}{\int_{k+1}^{\infty} e^{-2v^{1/r}} p(v) v^{1/r-1} dv} &\le \frac{M}{m} \frac{e^{-2k^{1/r}} - e^{-2(k+1)^{1/r}}}{e^{-2(k+1)^{1/r}}} \\ &= \frac{M}{m}\left(e^{2\left((k+1)^{1/r} - k^{1/r}\right)} - 1\right) \\ &\to 0 \quad \text{as} \quad k \to \infty\end{aligned}$$

since for $q(x) = x^{1/r}$ we have that for each $k \in \mathbb{N}$ there exists $\xi(k) \in (k, k+1)$ such that

$$(k+1)^{1/r} - k^{1/r} = \frac{q(k+1) - q(k)}{(k+1) - k} = q'(\xi(k)) = \frac{1}{r}(\xi(k))^{1/r-1} \to 0, \quad k \to \infty$$

due to $r > 1$. Thus

$$\frac{\tilde{a}_r\left(e^{-k^{1/r}}\right)}{\tilde{a}_r(x_0)} \to 1 \quad \text{as} \quad x_0 \to 0.$$

Hence $\tilde{a}_r\left(e^{-k^{1/r}}\right)$ and $\tilde{a}_r(x_0)$ have the same limit as $x_0 \to 0$. This leads us to the study of the behavior of

$$\tilde{a}_r\left(e^{-k^{1/r}}\right) = -\frac{2}{re^{-2k^{1/r}}} \int_k^{\infty} e^{-2v^{1/r}} p(v) v^{1/r-1} dv.$$

By using the Fourier series of the function $p(v) = \bar{p} + \tilde{p}(v)$, where $\bar{p} = \int_0^1 p(v) dv$ indicates the nonzero average value of $p(v)$, we find

$$\begin{aligned}\tilde{a}_r\left(e^{-k^{1/r}}\right) = &-\bar{p}\frac{2}{re^{-2k^{1/r}}} \int_k^{\infty} e^{-2v^{1/r}} v^{1/r-1} dv \\ &- \frac{2}{re^{-2k^{1/r}}} \int_k^{\infty} e^{-2v^{1/r}} \tilde{p}(v) v^{1/r-1} dv.\end{aligned}$$

According to the relation (6.3),

$$-\bar{p}\frac{2}{re^{-2k^{1/r}}} \int_k^{\infty} e^{-2v^{1/r}} v^{1/r-1} dv = -\bar{p}\frac{2}{re^{-2k^{1/r}}}\frac{r}{2} e^{-2k^{1/r}} = -\bar{p}$$

and thus

$$\tilde{a}_r\left(e^{-k^{1/r}}\right) = -\bar{p} - \frac{2}{r}\int_k^{\infty} e^{2\left(k^{1/r} - v^{1/r}\right)} \tilde{p}(v) v^{1/r-1} dv. \tag{6.4}$$

The second term on the right-hand side of (6.4) can be written as $-2B_k/r$, where

$$B_k = \int_k^\infty e^{2(k^{1/r}-v^{1/r})}\tilde{p}(v)v^{1/r-1}dv.$$

By making the change of variable $v = k + w$, we obtain

$$B_k = \int_0^\infty e^{2(k^{1/r}-(k+w)^{1/r})}\tilde{p}(w)(k+w)^{1/r-1}dw,$$

and with the change $w = kz$, we have

$$B_k = k^{1/r}\int_0^\infty e^{2k^{1/r}(1-(1+z)^{1/r})}\tilde{p}(kz)(1+z)^{1/r-1}dz. \tag{6.5}$$

For the study of B_k, we break up the integral in (6.5) into a sum of two integrals as $B_k = B_k^{(1)} + B_k^{(2)}$, where

$$B_k^{(1)} = k^{1/r}\int_0^\varepsilon e^{2k^{1/r}(1-(1+z)^{1/r})}\tilde{p}(kz)(1+z)^{1/r-1}dz,$$
$$B_k^{(2)} = k^{1/r}\int_\varepsilon^\infty e^{2k^{1/r}(1-(1+z)^{1/r})}\tilde{p}(kz)(1+z)^{1/r-1}dz.$$

We study initially $B_k^{(2)}$ and show that it tends to 0. By using the inequality

$$1-(1+z)^{1/r} \le -C(1+z)^{1/r} \quad \text{for} \quad z > \varepsilon > 0, \quad C > 0, \quad C = C(\varepsilon)$$

and by changing the variable $v = k^{1/r}C(1+z)^{1/r}$, we find

$$\begin{aligned}\left|B_k^{(2)}\right| &\le Mk^{1/r}\int_\varepsilon^\infty e^{-2k^{1/r}C(1+z)^{1/r}}(1+z)^{1/r-1}dz\\ &= \frac{Mr}{C}\int_{Ck^{1/r}(1+\varepsilon)^{1/r}}^\infty e^{-2v}dv\\ &= \frac{Mr}{2C}e^{-2C(1+\varepsilon)^{1/r}k^{1/r}}\\ &\to 0 \quad \text{as} \quad k\to\infty.\end{aligned}$$

It remains to evaluate $B_k^{(1)}$. We choose an antiderivative of $\tilde{p}$ denoted by $\tilde{P}$ obtained by formally integrating the Fourier series of $\tilde{p}$ (of which the average is zero). Integration of $\tilde{p}(kz)$ gives $\dfrac{1}{k}\tilde{P}(kz)$. Integration by parts gives

$$\begin{aligned}B_k^{(1)} &= \frac{1}{k}\tilde{P}(kz)k^{1/r}e^{2k^{1/r}(1-(1+z)^{1/r})}(1+z)^{1/r-1}\Big|_0^\varepsilon\\ &\quad - 2k^{2/r-1}\int_0^\varepsilon \tilde{P}(kz)\left\{-\frac{1}{r}(1+z)^{1/r-1}e^{2k^{1/r}(1-(1+z)^{1/r})}(1+z)^{1/r-1}\right\}dz\\ &\quad - k^{1/r-1}\int_0^\varepsilon \tilde{P}(kz)\left\{e^{2k^{1/r}(1-(1+z)^{1/r})}(1/r-1)(1+z)^{1/r-2}\right\}dz\end{aligned}$$

which tends to 0 as $k \to \infty$ provided

$$\frac{2}{r} - 1 < 0, \quad \text{i.e., for} \quad r > 2$$

since

$$e^{2k^{1/r}(1-(1+z)^{1/r})} \leq e^{-2Ck^{1/r}(1+z)^{1/r}} \to 0 \quad \text{as} \quad k \to \infty.$$

For the case $r < 2$, we reiterate the preceding calculation. We can define successive primitives of $\tilde{p}$, which we denote by

$$\tilde{P}_1, \quad \tilde{P}_2, \quad \ldots, \quad \tilde{P}_n = \tilde{P},$$

obtained by formally integrating the Fourier series of $\tilde{p}$ (whose average is zero). Successive integration of $\tilde{p}(kz)$ gives

$$\frac{1}{k}\tilde{P}_1(kz), \quad \frac{1}{k^2}\tilde{P}_2(kz), \quad \ldots, \quad \frac{1}{k^n}\tilde{P}_n(kz), \quad \ldots.$$

We stop as soon as we have $r > \dfrac{n}{n-1}$. With $r > 1$, we have the convergence to 0 of the term

$$2k^{1/r}\int_0^{\varepsilon} e^{2k^{1/r}(1-(1+z)^{1/r})}\tilde{p}(kz)(1+z)^{1/r-1}dz.$$

Finally,

$$\tilde{a}_r(x_0) \to -\bar{p} \quad \text{as} \quad x_0 \to 0, \quad \text{for every} \quad r > 1.$$

For more details concerning this proof, we refer to [3].

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