Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 2 Number 1 (2007), pp. 13–30
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http://www.ripublication.com/adsa.htm

Oscillation of Certain Third Order Nonlinear Functional Differential Equations

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Abstract

In this paper we shall investigate the oscillatory properties of the equations

$$\frac{d^2}{dt^2}\left(\frac{1}{a(t)}\left(\frac{dx(t)}{dt}\right)^{\alpha}\right) + q(t)f(x[g(t)]) = 0$$

and

$$\frac{d^2}{dt^2}\left(\frac{1}{a(t)}\left(\frac{dx(t)}{dt}\right)^{\alpha}\right) = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]),$$

where α is the ratio of two positive odd integers.

AMS subject classification: 34C10.
Keywords: Functional differential equations, oscillation, nonoscillation, comparison.

Received March 30, 2007; Accepted April 4, 2007

1. Introduction

Consider the third order nonlinear functional differential equations of the form

$$\frac{d^2}{dt^2}\left(\frac{1}{a(t)}\left(\frac{dx(t)}{dt}\right)^{\alpha}\right)+q(t)f(x[g(t)])=0 \tag{1.1}$$

and

$$\frac{d^2}{dt^2}\left(\frac{1}{a(t)}\left(\frac{dx(t)}{dt}\right)^{\alpha}\right)=q(t)f(x[g(t)])+p(t)h(x[\sigma(t)]), \tag{1.2}$$

where

(i) α is the ratio of two positive odd integers,

(ii) $a, p, q \in C([t_0,\infty),[0,\infty))$ such that $\sup\{p(t) : t \geq T\} > 0$ and $\sup\{q(t) : t \geq T\} > 0$ for any $T \geq t_0 \geq 0$, and $a(t) > 0$, $t \geq t_0$,

(iii) $g, \sigma \in C^1([t_0,\infty),\mathbb{R})$ satisfying $g'(t) \geq 0$, $\sigma'(t) \geq 0$, $g(t) < t$, $\sigma(t) > t$ and $\lim_{t\to\infty} g(t) = \infty$,

(iv) $f, h \in C(\mathbb{R},\mathbb{R})$, $xf(x) > 0$, $xh(x) > 0$, $f'(x) \geq 0$ and $h'(x) \geq 0$ for $x \neq 0$.

By a solution of equation (1.1) or (1.2) we mean a function $x \in C^1([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$ which has the property that $(1/a)(x')^{\alpha} \in C^2([T_x,\infty),\mathbb{R})$ and satisfies equation (1.1) or (1.2) for all large $t \geq T_x$. A solution is said to be *oscillatory* if it has a sequence of zeros clustering at $t = \infty$, otherwise, a solution is said to be *nonoscillatory*. An equation is said to be *oscillatory* if all its solutions are oscillatory.

In the last three decades there has been an increasing interest in studying the oscillatory and nonoscillatory behavior of solutions of functional differential equations. Most of the work on the subject, however, has been restricted to first and second order equations, equations of type (1.1) and (1.2) with $\alpha = 1$ as well as higher order equations and half-linear equations of the form

$$\frac{d}{dt}\left(\frac{1}{a(t)}\left(\frac{dx(t)}{dt}\right)^{\alpha}\right)+\delta q(t)f(x[g(t)])=0,$$

where $\delta = \pm 1$. For recent contributions, we refer to [1, 4] and the references cited therein.

It appears that only little is known regarding the oscillation of equations (1.1) and (1.2). Therefore, the main goal here is to present asymptotic study of the oscillation of all solutions of equations (1.1) and (1.2). Moreover, we shall establish some new criteria for the oscillation of similar type equations, namely,

$$\frac{d}{dt}\left(\frac{1}{a(t)}\left(\frac{d^2x(t)}{dt^2}\right)^{\alpha}\right)+q(t)f(x[g(t)])=0 \tag{1.3}$$

and
$$\frac{d}{dt}\left(\frac{1}{a(t)}\left(\frac{d^2x(t)}{dt^2}\right)^{\alpha}\right) = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]). \tag{1.4}$$

The obtained results extend, improve and correlate many known criteria which have appeared recently in the literature.

2. Oscillation of Equation (1.1)

We shall assume throughout that

$$\int^{\infty} a^{1/\alpha}(s)ds = \infty. \tag{2.1}$$

We define the operators

$$\begin{aligned} L_0x(t) &= x(t), \quad L_1x(t) = \frac{1}{a(t)}\left(\frac{d}{dt}L_0x(t)\right)^{\alpha}, \\ L_2x(t) &= \frac{d}{dt}L_1x(t), \quad L_3x(t) = \frac{d}{dt}L_2x(t). \end{aligned} \tag{2.2}$$

Thus, equation (1.1) can be written as

$$L_3x(t) + q(t)f(x[g(t)]) = 0.$$

If we let x be an eventually positive solution of equation (1.1), then $L_3x(t) \le 0$ eventually, and hence $L_ix(t)$, $i = 0, 1, 2$ are eventually of one sign.

There are two possibilities to consider:

(I) $L_ix(t) > 0$, $i = 0, 1, 2$ eventually, or

(II) $L_0x(t) > 0$, $L_1x(t) < 0$ and $L_2x(t) > 0$ eventually.

Case (I) Let $L_ix(t) > 0$, $i = 0, 1, 2$ for $t \ge t_0 \ge 0$. Then, it follows that

$$L_1x(t) = \int_{t_0}^{t} L_2x(s)ds \ge (t - t_0)L_2x(t) \quad \text{for} \quad t \ge t_0,$$

or

$$x'(t) \ge a^{1/\alpha}(t)(t - t_0)^{1/\alpha}L_2^{1/\alpha}x(t) \quad \text{for} \quad t \ge t_0.$$

Integrating the above inequality from t_0 to t, we have

$$x(t) \ge \left(\int_{t_0}^{t} a^{1/\alpha}(s)(s - t_0)^{1/\alpha}ds\right) L_2^{1/\alpha}x(t) \quad \text{for} \quad t \ge t_0. \tag{2.3}$$

Case (II) Let $L_0x(t) > 0$, $L_1x(t) < 0$ and $L_2x(t) > 0$ for $t \geq t_0 \geq 0$. Then, for $t \geq s \geq t_0$,

$$L_1x(t) - L_1x(s) = \int_s^t L_2x(u)du$$

and so,

$$-L_1x(s) = -\frac{1}{a(s)}(x'(s))^\alpha \geq (t-s)L_2x(t),$$

or

$$-x'(s) \geq a^{1/\alpha}(s)(t-s)^{1/\alpha}L_2^{1/\alpha}x(t).$$

Thus, it follows that

$$x(s) \geq \left(\int_s^t a^{1/\alpha}(u)(t-u)^{1/\alpha}du\right)L_2^{1/\alpha}x(t) \quad \text{for} \quad t \geq s \geq t_0. \tag{2.4}$$

In what follows we shall use the following notations. For $t \geq s \geq T \geq t_0 \geq 0$, we let

$$A_1[t,T] = \int_T^t a^{1/\alpha}(s)(s-t_0)^{1/\alpha}ds$$

and

$$B_1[t,s] = \int_s^t a^{1/\alpha}(u)(t-u)^{1/\alpha}du.$$

Thus, inequalities (2.3) (of Case (I)) and (2.4) (of Case (II)) can be written as

$$x(t) \geq A_1[t,t_0]L_2^{1/\alpha}x(t), \quad t \geq t_0 \tag{2.5}$$

and

$$x(s) \geq B_1[t,s]L_2^{1/\alpha}x(t), \quad t \geq s \geq t_0. \tag{2.6}$$

For equation (1.3), we define

$$\begin{gathered}\overline{L}_0x(t) = x(t), \quad \overline{L}_1x(t) = \frac{d}{dt}\overline{L}_0x(t),\\ \overline{L}_2x(t) = \frac{1}{a(t)}\left(\frac{d}{dt}L_1x(t)\right)^\alpha, \quad \overline{L}_3x(t) = \frac{d}{dt}\overline{L}_2x(t).\end{gathered} \tag{2.7}$$

Then, the equation (1.3) takes the form

$$\overline{L}_3x(t) + q(t)f(x[g(t)]) = 0.$$

If x is an eventually positive solution of equation (1.3), then the cases (I) and (II) are to be considered with L replaced by $\overline{L}$. For Case (I) one can easily conclude that

$$x(t) \geq A_3[t,t_0]\overline{L}_2^{1/\alpha}x(t) \quad \text{for} \quad t \geq t_0, \tag{2.8}$$

where
$$A_3[t,T] = \int_T^t \left(\int_T^s a^{1/\alpha}(u)du \right) ds \quad \text{for} \quad t \geq T \geq t_0$$
and if Case (II) holds, then for $t \geq s \geq t_0$, we see that
$$x(s) \geq B_3[t,s]\overline{L}_2^{1/\alpha} x(t), \tag{2.9}$$
where
$$B_3[t,s] = \int_s^t \left(\int_u^t a^{1/\alpha}(\tau)d\tau \right) du.$$
We are now ready to prove oscillatory criteria for the equation (1.1). For this, we shall assume that
$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0, \tag{2.10}$$
$$\frac{f(u^{1/\alpha})}{u} \geq k > 0, \quad k \text{ is a real constant}, \quad u \neq 0 \tag{2.11}$$
and
$$\int_0^{\pm\epsilon} \frac{du}{f(u^{1/\alpha})} < \infty \quad \text{for every } \epsilon > 0. \tag{2.12}$$
Theorem 2.1. Let conditions (i)–(iv), (2.1), (2.10) and (2.11) hold. If for $t \geq t_0 \geq 0$,
$$\limsup_{t\to\infty} \int_{g(t)}^t q(s)f(A_1[g(s),t_0])ds > \frac{1}{k} \tag{2.13}$$
and
$$\limsup_{t\to\infty} \int_{g(t)}^t q(s)f(B_1[g(t),g(s)])ds > \frac{1}{k}, \tag{2.14}$$
then equation (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.1). Then, $L_3x(t) \leq 0$ eventually and hence $L_ix(t)$, $i = 1,2,3$ are eventually of one sign. This leads to the two possibilities (I) and (II). For Case (I), we obtain (2.5). Now there exists a $T \geq t_0$ such that
$$x[g(t)] \geq A_1[g(t),t_0]L_2^{1/\alpha}x[g(t)] \quad \text{for} \quad t \geq T. \tag{2.15}$$
Integrating equation (1.1) from $g(t)$ to $t(\geq T)$, we have
$$\begin{aligned} -L_2x(t) + L_2x[g(t)] &= \int_{g(t)}^t q(s)f(x[g(s)])ds \\ &\geq \int_{g(t)}^t q(s)f(A_1[g(s),t_0]L_2^{1/\alpha}x[g(s)])ds \\ &\geq \int_{g(t)}^t q(s)f(A_1[g(s),t_0])f(L_2^{1/\alpha}x[g(s)])ds \\ &\geq f(L_2^{1/\alpha}x[g(t)]) \int_{g(t)}^t q(s)f(A_1[g(s),t_0])ds. \end{aligned}$$

Thus, it follows that

$$L_2x[g(t)] \geq f(L_2^{1/\alpha}x[g(t)]) \int_{g(t)}^{t} q(s)f(A_1[g(s), t_0])ds,$$

or

$$\frac{L_2x[g(t)]}{f(L_2^{1/\alpha}x[g(t)])} \geq \int_{g(t)}^{t} q(s)f(A_1[g(s), t_0])ds.$$

Taking lim sup of both sides of the above inequality as $t \to \infty$, we arrive at a contradiction to condition (2.13).

Next for the Case (II), we obtain (2.6). Substituting $g(s)$ and $g(t)$ for s and t respectively, we have

$$x[g(s)] \geq B_1[g(t), g(s)]L_2^{1/\alpha}x[g(t)] \quad \text{for} \quad t \geq s \geq t_0. \tag{2.16}$$

Integrating equation (1.1) from $g(t)$ to t, we find

$$\begin{aligned} L_2x[g(t)] &\geq -L_2x(t) + L_2x[g(t)] \\ &= \int_{g(t)}^{t} q(s)f(x[g(s)])ds \\ &\geq \int_{g(t)}^{t} q(s)f(B_1[g(t), g(s)]L_2^{1/\alpha}x[g(t)])ds \\ &\geq f(L_2^{1/\alpha}x[g(t)]) \int_{g(t)}^{t} q(s)f(B_1[g(t), g(s)])ds, \end{aligned}$$

or

$$\frac{L_2x[g(t)]}{f(L_2^{1/\alpha}x[g(t)])} \geq \int_{g(t)}^{t} q(s)f(B_1[g(t), g(s)])ds.$$

Taking lim sup of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (2.14). This completes the proof. ■

The following corollary is immediate.

Corollary 2.2. Let conditions (i)–(iv), (2.1), (2.10) and (2.11) hold. If condition (2.14) holds, then all bounded solutions of equation (1.1) are oscillatory.

Theorem 2.3. Let conditions (i)–(iv), (2.1), (2.10) and (2.12) hold. If for $t \geq t_0 \geq 0$,

$$\int^{\infty} q(s)f(A_1[g(s), t_0])ds = \infty \tag{2.17}$$

and

$$\int^{\infty} q(s)f(B_1[g(t), g(s)])ds = \infty, \tag{2.18}$$

then equation (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.1). Proceeding as in the proof of Theorem 2.1 we obtain (2.15) for Case (I) and (2.16) for Case (II). Now for Case (I), from equation (1.1), we obtain

$$\begin{aligned} -\frac{d}{dt}L_2x(t) &= q(t)f(x[g(t)]) \\ &\geq q(t)f(A_1[g(t),t_0]L_2^{1/\alpha}x[g(t)]) \\ &\geq q(t)f(A_1[g(t),t_0])f(L_2^{1/\alpha}x(t)), \end{aligned}$$

or

$$\frac{-\frac{d}{dt}L_2x(t)}{f(L_2^{1/\alpha}x(t))} \geq q(t)f(A_1[g(t),t_0]) \quad \text{for} \quad t \geq T \geq t_0.$$

Integrating the above inequality from T to t, we have

$$\int_{L_2x(t)}^{L_2x(T)} \frac{du}{f(u^{1/\alpha})} \geq \int_T^t q(s)f(A_1[g(s),t_0])ds.$$

Taking limit of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (2.17).

Next for Case (II), from equation (1.1), we see that

$$-L_3x(s) = q(s)f(x[g(s)]) \geq q(s)f(B_1[g(t),g(s)])f(L_2^{1/\alpha}x(s)) \text{ for } t \geq s \geq T \geq t_0,$$

or

$$\frac{-\frac{d}{ds}L_2x(s)}{f(L_2^{1/\alpha}x(s))} \geq q(s)f(B_1[g(t),g(s)]).$$

The rest of the proof is similar to that of Case (I) and hence omitted. This completes the proof. ■

The following corollary is immediate.

Corollary 2.4. Let conditions (i)–(iv), (2.1) and (2.10) hold. If

$$\frac{u}{f(u^{1/\alpha})} \to 0 \quad \text{as} \quad u \to 0 \tag{2.19}$$

and

$$\limsup_{t\to\infty} \int_{g(t)}^t q(s)f(B_1[g(t),g(s)])ds > 0, \tag{2.20}$$

then all bounded solutions of equation (1.1) are oscillatory.

Next, we present the following comparison result for the oscillation of equation (1.1).

Theorem 2.5. Let conditions (i)–(iv), (2.1) and (2.10) hold. If the first order delay equations

$$y'(t) + q(t)f(A_1[g(t),t_0])f(y^{1/\alpha}[g(t)]) = 0, \quad t_0 \geq 0 \tag{2.21}$$

and

$$z'(t) + q(t) f\left(B_1\left[\frac{t+g(t)}{2}, g(t)\right]\right) f\left(z^{1/\alpha}\left[\frac{t+g(t)}{2}\right]\right) = 0 \tag{2.22}$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.1). Proceeding as in the proof of Theorem 2.1 we obtain (2.15) for Case (I) and (2.6) for Case (II). Now for Case (I), using (2.10) and (2.15) in equation (1.1), we have

$$\begin{aligned} -L_3 x(t) &= q(t) f(x[g(t)]) \\ &\geq q(t) f(A_1[g(t), t_0] L_2^{1/\alpha} x[g(t)]) \\ &\geq q(t) f(A_1[g(t), t_0]) f(L_2^{1/\alpha} x[g(t)]) \quad \text{for} \quad t \geq T \geq t_0. \end{aligned}$$

Setting $y(t) = L_2 x(t) > 0$ for $t \geq T$, we obtain

$$y'(t) + q(t) f(A_1[g(t), t_0]) f(y^{1/\alpha}[g(t)]) \leq 0 \quad \text{for} \quad t \geq T.$$

Integrating the above inequality from $t (\geq T)$ to u and letting $u \to \infty$, we have

$$y(t) \geq \int_t^\infty q(s) f(A_1[g(s), t_0]) f(y^{1/\alpha}[g(s)]) ds, \quad t \geq T.$$

As in [5] it is easy to conclude that there exists a positive solution y of equation (2.21) with $\lim_{t\to\infty} y(t) = 0$, which contradicts the fact that equation (2.21) is oscillatory.

Next for Case (II), substituting $g(t)$ and $(t+g(t))/2$ for s and t, respectively in (2.6), we have

$$x[g(t)] \geq B_1\left[\frac{t+g(t)}{2}, g(t)\right] L_2^{1/\alpha} x\left[\frac{t+g(t)}{2}\right] \quad \text{for} \quad t \geq T.$$

Using this inequality in equation (1.1) and proceeding as in Case (I), we obtain

$$z'(t) + q(t) f\left(B_1\left[\frac{t+g(t)}{2}, g(t)\right]\right) f\left(z^{1/\alpha}\left[\frac{t+g(t)}{2}\right]\right) \leq 0 \quad \text{for} \quad t \geq T,$$

where $z(t) = L_2^{1/\alpha} x(t)$, $t \geq T$. The rest of the proof is similar to that of Case (I) above and hence omitted. ■

The following corollary is immediate.

Corollary 2.6. Let conditions (i)–(iv), (2.1), (2.10) and (2.11) hold. If

$$\liminf_{t\to\infty} \int_{g(t)}^t q(s) f(A_1[g(s), t_0]) ds > \frac{1}{ke}, \quad t_0 \geq 0 \tag{2.23}$$

and

$$\liminf_{t\to\infty}\int_{(t+g(t))/2}^{t} q(s)f\left(B_1\left[\frac{s+g(s)}{2}, g(s)\right]\right)ds > \frac{1}{ke}, \tag{2.24}$$

then equation (1.1) is oscillatory.

Remark 2.7. We note that identical results as those presented above for the oscillation of equation (1.3) can be easily obtained by replacing A_1 and B_1 with A_3 and B_3, respectively. The details are left to the reader.

3. Oscillation of Equation (1.2)

In this section, we shall give some new criteria for the oscillation of equations (1.2) and (1.4). For $t \geq s \geq T \geq t_0$, we let

$$A_2[t,s] = \int_s^t a^{1/\alpha}(u)(u-s)^{1/\alpha}du,$$

$$B_2[t,T] = \int_T^t a^{1/\alpha}(u)(t-u)^{1/\alpha}du,$$

$$A_4[t,s] = \int_s^t \left(\int_s^u a^{1/\alpha}(\tau)d\tau\right)du,$$

and

$$B_4[t,T] = \int_T^t \left(\int_s^t a^{1/\alpha}(u)du\right)ds.$$

Using (2) in equation (1.2), we get

$$L_3x(t) = q(t)f(x[g(t)]) + p(t)h(x[\sigma(t)]).$$

Now, if x is an eventually positive solution of equation (1.2), then $L_3x(t) \geq 0$ eventually and hence $L_ix(t)$, $i = 0, 1, 2$ are eventually of one sign. We shall distinguish the following two cases:

(I_1) $L_ix(t) > 0$, $i = 0, 1, 2$ eventually,

(II_1) $L_0x(t) > 0$, $L_1x(t) > 0$ and $L_2x(t) < 0$ eventually.

Case (I_1) Let $L_ix(t) > 0$, $i = 0, 1, 2$ for $t \geq t_0 \geq 0$. Then, for $t \geq s \geq t_0$,

$$L_1x(t) - L_1x(s) = \int_s^t L_2x(u)du$$

and so

$$L_1x(t) \geq (t-s)L_2x(s),$$

or

$$x'(t) \geq a^{1/\alpha}(t)(t-s)^{1/\alpha} L_2^{1/\alpha} x(s).$$

Thus, it follows that

$$x(t) \geq \left(\int_s^t (a(u)(u-s))^{1/\alpha}\, du \right) L_2^{1/\alpha} x(s) = A_2[t,s] L_2^{1/\alpha} x(s), \quad t \geq s \geq t_0. \tag{3.1}$$

Case (II_1) Let $L_0 x(t) > 0$, $L_1 x(t) > 0$, $L_2 x(t) < 0$ for $t \geq t_0 \geq 0$. Then, for $t \geq s \geq t_0$,

$$-L_1 x(s) \leq L_1 x(t) - L_1 x(s) = \int_s^t L_2 x(u) du$$

and so,

$$L_1 x(s) \geq \int_s^t (-L_2 x(u)) du,$$

or

$$x'(s) \geq (a(s)(t-s))^{1/\alpha} (-L_2^{1/\alpha} x(t)).$$

Thus, we have

$$x(t) \geq \left(\int_{t_0}^t (a(s)(t-s))^{1/\alpha}\, ds \right) (-L_2^{1/\alpha} x(t)) = B_2[t,t_0](-L_2^{1/\alpha} x(t)), \quad t \geq t_0. \tag{3.2}$$

Next using (2) in equation (1.4), we see that

$$\overline{L}_3 x(t) = q(t) f(x[g(t)]) + p(t) h(x[\sigma(t)]).$$

Now if x is an eventually positive solution of equation (1.4), then the Cases (I_1) and (II_1) are considered with $\overline{L}$ replacing L.

Now for Case (I_1) one can easily see that for $t \geq s \geq t_0$,

$$x(t) \geq A_4[t,s] L_2^{1/\alpha} x(s) \tag{3.3}$$

and for Case (II_1), we obtain

$$x(t) \geq B_4[t,t_0](-L_2^{1/\alpha} x(t)) \quad \text{for} \quad t \geq t_0. \tag{3.3}$$

We shall assume that

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \quad \text{for} \quad xy > 0 \tag{3.4}$$

and

$$\frac{h(u^{1/\alpha})}{u} \geq k_1 > 0, \quad \text{where } k_1 \text{ is a constant}, \quad u \neq 0. \tag{3.5}$$

Theorem 3.1. Let conditions (i)–(iv), (2.1), (2.10), (2.11), (3.4) and (3.5) hold. If

$$\limsup_{t\to\infty}\int_t^{\sigma(t)} p(s)h(A_2[\sigma(s),\sigma(t)])ds > \frac{1}{k_1} \tag{3.6}$$

and

$$\limsup_{t\to\infty}\int_{g(t)}^{t} q(s)f(B_2[g(s),t_0])ds > \frac{1}{k_1} \quad \text{for} \quad t_0 \geq 0, \tag{3.7}$$

then equation (1.2) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.2). Then, $L_3x(t) \geq 0$ eventually and hence $L_ix(t)$, $i = 0, 1, 2$ are eventually of one sign. Next, we distinguish the two Cases (I_1) and (II_1). For Case (I_1) we obtain (3.1). Letting $s = \sigma(t)$ and $t = \sigma(s)$ in (3.1), we have

$$x[\sigma(s)] \geq A_2[\sigma(s),\sigma(t)]L_2^{1/\alpha}x[\sigma(t)], \quad s \geq t. \tag{3.8}$$

From equation (1.2), we find that

$$\begin{aligned} L_3x(s) &\geq p(s)h(x[\sigma(s)]) \\ &\geq p(s)h(A_2[\sigma(s),\sigma(t)]L_2^{1/\alpha}x[\sigma(t)]) \\ &\geq p(s)h(A_2[\sigma(s),\sigma(t)])h(L_2^{1/\alpha}x[\sigma(t)]). \end{aligned}$$

Integration of the above inequality on $[t,\sigma(t)]$ yields

$$L_2x[\sigma(t)] \geq \left(\int_t^{\sigma(t)} p(s)h(A_2[\sigma(s),\sigma(t)])ds\right)h(L_2^{1/\alpha}x[\sigma(t)]),$$

or

$$\frac{L_2x[\sigma(t)]}{h(L_2^{1/\alpha}x[\sigma(t)])} \geq \int_t^{\sigma(t)} p(s)h(A_2[\sigma(s),\sigma(t)])ds.$$

Taking lim sup of both sides as $t \to \infty$, we have a contradiction to condition (3.6).

Next for Case (II_1), we obtain (3.2) for $t \geq t_0 \geq 0$. There exists a $T \geq t_0$ such that

$$x[g(t)] \geq B_2[g(t),t_0](-L_2^{1/\alpha}x[g(t)]) \quad \text{for} \quad t \geq T. \tag{3.9}$$

It follows from equation (1.2) that

$$L_3x(t) \geq q(t)f(x[g(t)]) \geq q(t)f(B_2[g(t),t_0])f(-L_2^{1/\alpha}x[g(t)]).$$

Integrating the above inequality on $[g(t),t]$, we find

$$-L_2x[g(t)] \geq \left(\int_{g(t)}^{t} q(s)f(B_2[g(s),t_0]ds\right)f(-L_2^{1/\alpha}x[g(t)]).$$

The rest of the proof is similar to that of Case (I_1) above and hence omitted. ■

Next, we replace conditions (2.10) and (2.11) by

$$\frac{f^{1/\alpha}(u)}{u} \geq m > 0, \quad m \quad \text{is a constant, for } u \neq 0 \tag{3.10}$$

and conditions (3.4) and (3.5) by

$$\frac{h^{1/\alpha}(u)}{u} \geq m_1 > 0, \quad m_1 \quad \text{is a constant, for } u \neq 0 \tag{3.11}$$

and prove the following result.

Theorem 3.2. Let conditions (i)–(iv), (2.1), (3.10) and (3.11) hold. If

$$\limsup_{t\to\infty} \int_t^{\sigma(t)} \left(a(\eta) \int_t^{\eta} \int_t^{\beta} p(s)dsd\beta\right)^{1/\alpha} d\eta > \frac{1}{m_1} \tag{3.12}$$

and

$$\limsup_{t\to\infty} B_2[g(t), t_0] \left(\int_t^{\infty} q(s)ds\right)^{1/\alpha} > \frac{1}{m} \quad \text{for} \quad t_0 \geq 0, \tag{3.13}$$

then equation (1.2) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.2). As in the proof of Theorem 3.1 the Cases (I_1) and (II_1) are considered. Suppose (I_1) holds. It follows from equation (1.2) that for $\beta \geq t \geq t_0$,

$$L_2x(\beta) \geq \int_t^{\beta} p(s)h(x[\sigma(s)])ds \geq \left(\int_t^{\beta} p(s)ds\right) h(x[\sigma(t)])$$

and for $\eta \geq t \geq t_0$, we have

$$L_1x(\eta) \geq \int_t^{\eta} L_2x(\beta)d\beta \geq \left(\int_t^{\eta} \int_t^{\beta} p(s)dsd\beta\right) h(x[\sigma(t)]),$$

or

$$x'(\eta) \geq \left(a(\eta) \int_t^{\eta} \int_t^{\beta} p(s)dsd\beta\right)^{1/\alpha} h^{1/\alpha}(x[\sigma(t)]).$$

Now for $\xi \geq t \geq t_0$, we find

$$x(\xi) \geq \int_t^{\xi} x'(\eta)d\eta \geq \left(\int_t^{\xi} \left(a(\eta) \int_t^{\eta} \int_t^{\beta} p(s)dsd\beta\right)^{1/\alpha} d\eta\right) h^{1/\alpha}(x[\sigma(t)]).$$

Putting $\xi = \sigma(t)$ in the above inequality, we get

$$\frac{x[\sigma(t)]}{h^{1/\alpha}(x[\sigma(t)])} \geq \int_t^{\sigma(t)} \left(a(\eta) \int_t^{\eta} \int_t^{\beta} p(s)dsd\beta\right)^{1/\alpha} d\eta.$$

Taking lim sup of the above as $t \to \infty$, we obtain a contradiction to condition (3.12).

Suppose (II_1) holds. It follows from equation (1.2) that

$$-L_2x(t) \geq \int_t^\infty q(s)f(x[g(s)])ds \geq \left(\int_t^\infty q(s)ds\right) f(x[g(t)]), \quad t \geq T \geq t_0.$$

Using inequality (3.9) and the fact that $-L_2x(t)$ is nonincreasing, we have

$$\begin{aligned} x[g(t)] &\geq B_2[g(t), t_0](-L_2^{1/\alpha}x[g(t)]) \\ &\geq B_2[g(t), t_0](-L_2^{1/\alpha}x(t)) \\ &\geq B_2[g(t), t_0]\left(\int_t^\infty q(s)ds\right)^{1/\alpha} f^{1/\alpha}(x[g(t)]), \end{aligned}$$

or

$$\frac{x[g(t)]}{f^{1/\alpha}(x[g(t)])} \geq B_2[g(t), t_0]\left(\int_t^\infty q(s)ds\right)^{1/\alpha}, \quad t \geq T \geq t_0.$$

The rest of the proof is similar to that of Case (I_1) and hence omitted. ■

In what follows for $t \geq s \geq T \geq t_0$, we let

$$C[t, s] = \int_s^t a^{1/\alpha}(u)du.$$

Now we shall prove the following comparison results.

Theorem 3.3. Let conditions (i)–(iv), (2.1), (2.10) and (3.4) hold. If all unbounded solutions of the second order advanced equation

$$\frac{d^2y(t)}{dt^2} - p(t)h\left(C\left[\sigma(t), \frac{t+\sigma(t)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{t+\sigma(t)}{2}\right]\right) = 0 \tag{3.14}$$

and all bounded solutions of the second order delay equation

$$\frac{d^2z(t)}{dt^2} - q(t)f(C[g(t), t_0])f(z^{1/\alpha}[g(t)]) = 0, \quad t_0 \geq 0 \tag{3.15}$$

are oscillatory, then equation (1.2) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.2). As in the proof of Theorem 3.1, we have the Cases (I_1) and (II_1) to consider. For Case (I_1) we have $L_i x(t) > 0$, $i = 1, 2$ for $t \geq t_0$. Thus, for $s \geq t \geq t_0$ it follows that

$$\begin{aligned} x(s) &= x(t) + \int_t^s x'(u)du = x(t) + \int_t^s a^{1/\alpha}(u) L_1^{1/\alpha} x(u)du \\ &\geq \left(\int_t^s a^{1/\alpha}(u)du\right) L_1^{1/\alpha} x(t) = C(s,t) L_1^{1/\alpha} x(t). \end{aligned}$$

Let $y(t) = L_1 x(t)$. Substituting $\sigma(t)$ and $(t + \sigma(t))/2$ for s and t respectively in the above inequality, we obtain

$$x[\sigma(t)] \geq C\left[\sigma(t), \frac{t+\sigma(t)}{2}\right] y^{1/\alpha}\left[\frac{t+\sigma(t)}{2}\right] \quad \text{for} \quad t \geq T \geq t_0. \tag{3.16}$$

Using (3.16) in equation (1.2), we have

$$\frac{d^2 y(t)}{dt^2} \geq p(t)h(x[\sigma(t)]) \geq p(t)h\left(C\left[\sigma(t), \frac{t+\sigma(t)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{t+\sigma(t)}{2}\right]\right) \quad \text{for} \quad t \geq T.$$

By a comparison theorem in [2, 3], we see that equation (3.14) has an unbounded eventually positive solution, which is a contradiction.

For Case (II_1) we have $L_0 x(t) > 0$, $L_1 x(t) > 0$ and $L_2 x(t) < 0$ for $t \geq t_0$. Thus, for $t \geq t_0 \geq 0$ it follows that

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t x'(u)du = x(t_0) + \int_{t_0}^t a^{1/\alpha}(u) L_1^{1/\alpha} x(u)du \\ &\geq \left(\int_{t_0}^t a^{1/\alpha}(u)du\right) L_1^{1/\alpha} x(t) = C[t, t_0] L_1^{1/\alpha} x(t). \end{aligned}$$

Let $z(t) = L_1 x(t)$. Substituting $g(t)$ for t in the above inequality, we get

$$x[g(t)] \geq C[g(t), t_0] z^{1/\alpha}[g(t)] \quad \text{for} \quad t \geq T_1 \geq t_0. \tag{3.17}$$

Using (2.10) and (3.17) in equation (1.2), we obtain

$$\frac{d^2 z(t)}{dt^2} \geq q(t) f(x[g(t)]) \geq q(t) f(C[g(t), t_0]) f(z^{1/\alpha}[g(t)]) \quad \text{for} \quad t \geq T_1.$$

Once again, by a comparison result in [2, 3], one can easily see that equation (3.15) has a bounded eventually positive solution, which is a contradiction. This completes the proof. ■

Theorem 3.4. Let conditions (i)–(iv), (2.1), (2.10) and (3.4) hold. If the first order advanced equation

$$\frac{dy(t)}{dt} - p(t)h\left(A_2\left[\sigma(t), \frac{t+\sigma(t)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{t+\sigma(t)}{2}\right]\right) = 0 \tag{3.18}$$

and the first order delay equation

$$\frac{dz(t)}{dt} + q(t)f(B_2[g(t), t_0])f(z^{1/\alpha}[g(t)]) = 0, \quad t_0 \geq 0 \tag{3.19}$$

are oscillatory, then equation (1.2) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.2). As in the proof of Theorem 3.1, we consider the Cases (I_1) and (II_1). If Case (I_1) holds, then from (3.1), we have

$$x[\sigma(t)] \geq A_2\left[\sigma(t), \frac{t+\sigma(t)}{2}\right] L_2^{1/\alpha} x\left[\frac{t+\sigma(t)}{2}\right] \quad \text{for} \quad t \geq T \geq t_0. \tag{3.20}$$

Using (3.20) and (3.4) in equation (1.2), we get

$$L_3 x(t) \geq p(t)h(x[\sigma(t)]) \geq p(t)h\left(A_2\left[\sigma(t), \frac{t+\sigma(t)}{2}\right]\right) h\left(L_2^{1/\alpha} x\left[\frac{t+\sigma(t)}{2}\right]\right).$$

Setting $L_2 x(t) = y(t)$ for $t \geq T$, we find

$$\frac{dy(t)}{dt} \geq p(t)h\left(A_2\left[\sigma(t), \frac{t+\sigma(t)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{t+\sigma(t)}{2}\right]\right) \quad \text{for} \quad t \geq T.$$

As in [2,3], we see that the equation (3.18) has an eventually positive solution, which is a contradiction.

Next if (II_1) holds, then as in the proof of Theorem 3.1, we obtain (3.9) for $t \geq T$. Now using (3.9) and (2.10) in equation (1.2), we find

$$L_3 x(t) \geq q(t)f(x[g(t)]) \geq q(t)f(B_2[g(t), t_0])f(-L_2^{1/\alpha} x[g(t)]), \quad t \geq T.$$

Putting $z(t) = -L_2 x(t), t \geq T$ we have

$$\frac{dz(t)}{dt} + q(t)f(B_2[g(t), t_0])f(z^{1/\alpha}[g(t)]) \leq 0.$$

The rest of the proof is similar to that of Theorem 2.5 and hence omitted. ■

From Theorem 3.4, one can easily deduce the following corollaries.

Corollary 3.5. Let conditions (i)–(iv), (2.1), (2.10), (2.11), (3.4) and (3.5) hold. If

$$\liminf_{t\to\infty} \int_t^{(t+\sigma(t))/2} p(s)h\left(A_2\left[\sigma(s), \frac{s+\sigma(s)}{2}\right]\right) ds > \frac{1}{ek_1} \tag{3.21}$$

and

$$\liminf_{t\to\infty} \int_{g(t)}^t q(s)f(B_2[g(s), t_0])ds > \frac{1}{ek}, \tag{3.22}$$

then equation (1.2) is oscillatory.

Corollary 3.6. Let conditions (i)–(iv), (2.1), (2.10), (2.12) and (3.4) hold and

$$\int_{\pm\epsilon}^{\pm\infty} \frac{du}{h(u^{1/\alpha})} < \infty \quad \text{for} \quad \epsilon > 0. \tag{3.23}$$

If

$$\int^{\infty} p(s)h\left(A_2\left[\sigma(s), \frac{s+\sigma(s)}{2}\right]\right) ds = \infty \tag{3.24}$$

and

$$\int^{\infty} q(s)f(B_2[g(s), t_0])ds = \infty, \quad t_0 \geq 0, \tag{3.25}$$

then equation (1.2) is oscillatory.

Remark 3.7. We note that identical results as those obtained above for the oscillation of equation (1.4) can be easily established by replacing A_2 and B_2 by A_4 and B_4 respectively and equations (3.14) and (3.15) in Theorem 3.3 by

$$\frac{d}{dt}\left(\frac{1}{a(t)}\left(\frac{dy(t)}{dt}\right)^{\alpha}\right) - p(t)h\left(C\left[\sigma(t), \frac{t+\sigma(t)}{2}\right]\right) h\left(y^{1/\alpha}\left[\frac{t+\sigma(t)}{2}\right]\right) = 0 \tag{3.26}$$

and

$$\frac{d}{dt}\left(\frac{1}{a(t)}\left(\frac{dz(t)}{dt}\right)^{\alpha}\right) - q(t)f(C[g(t), t_0])f(z^{1/\alpha}[g(t)]) = 0, \quad t_0 \geq 0 \tag{3.27}$$

respectively. The details are left to the reader.

4. Examples and Remarks

Remark 4.1. By applying Theorem 2.1 to the equation

$$\frac{d^2}{dt^2}\left(\frac{dx(t)}{dt}\right)^{\alpha} + qx^{\alpha}[t-\tau] = 0, \tag{4.1}$$

where α is as in (i), q and τ are positive constants, we find that equation (4.1) is oscillatory if

$$q > \frac{\alpha+2}{\tau^2}\left(\frac{\alpha+1}{\alpha\tau}\right)^{\alpha}. \tag{4.2}$$

Also, we see that the equation

$$\frac{d^2}{dt^2}\left(\frac{dx}{dt}\right)^{\alpha} + qx^{\beta}[t-\tau] = 0 \tag{4.3}$$

is oscillatory by Theorem 2.3 provided that α, q and τ are as in equation (4.1) and β is the ratio of two positive odd integers with $0 < \beta < \alpha$.

Remark 4.2. By applying Theorem 3.1 to the equation

$$\frac{d^2}{dt^2}\left(\frac{dx(t)}{dt}\right)^{\alpha} = qx^{\alpha}[t-\tau] + px^{\alpha}[t+\sigma], \tag{4.4}$$

where α is as in (i), p, q, τ and σ are positive real constants, it follows that equation (4.4) is oscillatory if

$$p > \frac{\alpha+2}{\sigma^2}\left(\frac{\alpha+1}{\alpha\sigma}\right)^{\alpha}. \tag{4.5}$$

Remark 4.3. By applying Theorem 3.3 to equation (4.4), one may conclude that equation (4.4) is oscillatory if all unbounded solutions of the advanced second order equation

$$y''(t) - p\left(\frac{\sigma}{2}\right)^{\alpha} y\left[t+\frac{\sigma}{2}\right] = 0 \tag{4.6}$$

and all bounded solutions of the second order delay equation

$$z''(t) - q(t-\tau-t_0)^{\alpha} z[t-\tau] = 0, \quad t_0 \geq 0 \tag{4.7}$$

are oscillatory.

We note that if we apply Theorem 3.4 to equation (4.4), then we can easily see that equation (4.4) is oscillatory if the first order equation with advanced argument

$$y'(t) - \frac{\sigma}{2}p\left(\frac{\alpha\sigma}{2(\alpha+1)}\right)^{\alpha} y\left[t+\frac{\sigma}{2}\right] = 0 \tag{4.8}$$

and the first order delay equation

$$z'(t) + q\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}(t-\tau-t_0)^{\alpha+1} z[t-\tau] = 0, \quad t_0 \geq 0 \tag{4.9}$$

are oscillatory.

Next, by applying Corollary 3.5 to equation (4.4), we see that equation (4.4) is oscillatory for any $q > 0$ and

$$p > \frac{1}{e}\left(\frac{2}{\sigma}\right)^2\left(\frac{2(\alpha+1)}{\alpha\sigma}\right)^{\alpha},$$

and by Corollary 3.6, we see that the equation

$$\frac{d^2}{dt^2}\left(\frac{dx(t)}{dt}\right)^{\alpha} = qx^{\beta}[t-\tau] + px^{\gamma}[t+\sigma] \tag{4.10}$$

is oscillatory provided that p, q, τ and σ are positive constants, α, β and γ are ratios of two positive odd integers with $0 < \beta < \alpha < \gamma$.

Remark 4.4. Similar oscillation results as those presented above can be obtained for equations (1.3) and (1.4) with constant coefficients and deviations. The details are left to the reader.

We note that our results in this paper are new even for the special cases of the equations considered with constant coefficients and deviations.

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