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Existence of Multiple Positive Solutions of Quasilinear Elliptic Problems in $\mathbb{R}^N$

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Abstract

This paper concerns quasilinear elliptic equations of the form

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)u(x)|u|^{p-2}(1-|u|^{\gamma})$$

in $\mathbb{R}^N$ with $p > 1$ and $a(x)$ changes sign. We discuss the question of existence and multiplicity of solutions when $a(x)$ has some specific properties.

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1. Introduction

In this paper we study the problem of existence of solutions for quasilinear elliptic equations in $\mathbb{R}^N$ of the type

$$-\Delta_p u = \lambda a(x)u|u|^{p-2}(1-|u|^{\gamma}), \tag{1.1}$$

where $p > 1$, $\lambda > 0$, $\gamma < \dfrac{p^2}{N-p}$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $a(x)$ is a smooth weight function which changes sign in $\mathbb{R}^N$. Here we say a function $a(x)$ changes sign if the measures of the sets $\{x \in \mathbb{R}^N : a(x) > 0\}$ and $\{x \in \mathbb{R}^N : a(x) < 0\}$ are both positive.

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It is known [2] that when $a(x)$ satisfies proper conditions, the eigenvalue problem

$$-\Delta_p u = \lambda a(x) u|u|^{p-2}$$

in $\mathbb{R}^N$ allows positive eigenvalue λ_1^+ with positive eigenfunction u_1^+. Thus we can study the bifurcation problem when λ is near λ_1^+.

The bifurcation problem of this type on bounded domains has received extensive attention recently, and we refer to [4, 6] and [7, 8] for details. The variational method was used there to prove the main results. On the other hand, the study of the existence of global positive solutions of the p-Laplacian also sees great increase in number of papers published. We mention [9, 12, 13], to name a few. For the case $p = 2$, [11, 15] studied the bifurcation from the first eigenvalue in $\mathbb{R}^N$ and obtained the existence of bifurcating branches, where $a(x)$ was assumed positive. We note that topological degree arguments and fixed point theory are employed in [15] and [11] respectively. However, their principal operator is defined via a Green's function which is not available to the p-Laplacian.

Also we can mention the work of Alama and Tarantello [1], Berestycki et al. [3] and Ouyang [14]. Bifurcation results are also obtained in [1, 14]. For $p \neq 2$, Le and Schmitt [17] study this equation on bounded domain as an example in their more general framework and obtain the existence of nontrivial solutions.

In this work we investigate the situation where $a(x)$ decays as $|x| \to \infty$ and satisfies

$$\int_{\mathbb{R}^N} a(x)(u_1^+)^{p+\gamma} < 0.$$

Using variational arguments we prove that λ_1^+ is a bifurcation point of (1.1) and there exists $\lambda^* > \lambda_1^+$, such that (1.1) has at least two positive solutions for $\lambda \in (\lambda_1^+, \lambda^*)$. Moreover, under proper conditions, we give information about the bifurcating branches.

This paper is organized as follows: In Section 2 we introduce some assumptions and notations which we use in this paper. In Section 3 we prove the existence of multiple solutions in a certain range of λ. We then verify the case $\lambda = \lambda_1^+$ in this section.

2. Some Notations and Preliminaries

In this section we introduce some basic assumptions and notations which we will need in this paper. We assume first that $1 < p < N$ and $\gamma < \dfrac{p^2}{N-p}$. Write $a(x) = a_1(x) - a_2(x)$ with $a_1, a_2 \geq 0$, $a_1 \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$ and $a_2 \in L^\infty(\mathbb{R}^N)$. Let

$$\omega(x) = (1 + |x|)^{-p}, \;\; x \in \mathbb{R}^N,$$

$$W(x) = \max\{a_2(x), \omega(x)\} > 0, \;\; x \in \mathbb{R}^N.$$

The weight function $\omega(x)$ satisfies the inequality

$$\int (1 + |x|)^{-p} |u|^p \leq \left(\frac{p}{N-p}\right)^p \int |\nabla u_n|^p,$$

where here and henceforth the integrals are taken over $\mathbb{R}^N$ unless otherwise specified. We define as in [9], the norm

$$||u|| = \left(\int |\nabla u_n|^p + \int W(x)|u|^p \right)^{1/p}$$

and introduce the uniformly convex Banach space V by the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $||\cdot||$. We assume that $a(x)$ satisfies

(a_1) $|a(x)| \leq cW(x)$ for some $c > 0$,

(a_2) $a(x) \in L^{\gamma_1}(\mathbb{R}^N)$ where $\gamma_1 = \dfrac{p^*}{p^* - p - \gamma}$.

To introduce the last condition we first give the following result.

Proposition 2.1. Assume that above conditions are satisfied. The eigenvalue problem

$$-\Delta_p u = \lambda a(x) u|u|^{p-2} \tag{2.1}$$

has a pair of principal eigenvalue and eigenfunction (λ_1^+, u_1^+) with $\lambda_1^+ > 0$ and $0 < u_1^+ \in V$. Moreover, such λ_1^+ is simple, unique. If $a_2 \not\equiv 0$ and $a_i \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, $i = 1, 2$, then by symmetry there is also principal eigenpair (λ_1^-, u_1^-) with $\lambda_1^- < 0$ and $0 < u_1^- \in V$ with analogous properties. Moreover the principal eigenvalue λ_1^+ is isolated ([2,9]).

Also we have (from [9, Lemma 2.3, Theorem 4.1, 4.4 and 4.5]) the following result.

Proposition 2.2. There is a continuum C of positive decaying solutions of (1.1) such that $(\lambda_1^+, 0) \in \overline{C}$, and C is either unbounded in $E = \mathbb{R} \times V$, where E is equipped with the norm

$$||(\lambda, u)||_E = (|\lambda|^2 + ||u||^2)^{1/2}, \ \ (\lambda, u) \in E,$$

or there is another eigenvalue $\hat{\lambda} \neq \lambda_1^+$ such that $(\hat{\lambda}, 0) \in \overline{C}$. If for some $\delta > 0$ the problem (1.1) in λ_1^+ has no nonzero solution $u \in V$ for $0 < ||u|| < \delta$, then C is unbounded in E. Moreover, for any solution $u \in V$, $u \in L^Q(\mathbb{R}^N)$, where $p^* \leq Q \leq \infty$ and $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$.

Finally we assume that

(a_3) $\displaystyle\int a(x)(u_1^+)^{p+\gamma} < 0.$

Now we define the functionals $I_1, I_2, I_3 : V \to \mathbb{R}$ as follows: for $u \in V$

$$I_1(u) = \int |\nabla u|^p,$$

$$I_2(u) = \int a(x)|u|^p,$$

$$I_3(u) = \int a(x)|u|^{p+\gamma}.$$

Sometimes we split I_2 as $I_2 = I_2^+ - I_2^-$, where

$$I_2^+(u) = \int a_1(x)|u|^p,$$
$$I_2^-(u) = \int a_2(x)|u|^p.$$

The situation for I_3 is similar by symmetry. We use some properties of these operators in the next section.

By a (weak) solution of problem (1.1), we mean a function $u \in V$ such that for every $v \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int a(x)|u|^{p-2}uv + \lambda \int a(x)|u|^{p+\gamma-2}uv = 0. \tag{2.2}$$

Since the seminal work of Drábek and Huang [9], problems like (1.1) have captured great interest.

Now let us define the variational functional corresponding to problem (1.1). We set $J_\lambda : V \to \mathbb{R}$ as

$$J_\lambda(u) = \frac{1}{p}(I_1(u) - \lambda I_2(u)) - \frac{\lambda}{p+\gamma} I_3(u). \tag{2.3}$$

It is easy to see that $J \in C^1(V, \mathbb{R})$, and for all $v \in V$ we have

$$(J'_\lambda(u), v) = \int |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int a(x)|u|^{p-2}uv + \lambda \int a(x)|u|^{p+\gamma-2}uv = 0. \tag{2.4}$$

Since $C_0^\infty(\mathbb{R}^N) \subset V$, we know that critical points of $J_\lambda(u)$ are weak solutions of (1.1).

When J_λ is bounded below on V, J_λ has a minimizer on V which is a critical point of J_λ. In many problems such as (1.1), J_λ is not bounded below on V. In order to obtain an existence result in this case, motivated by Brown and Zhang [5], we introduce the Nehari manifold

$$S(\lambda) = \{u \in V : (J'_\lambda(u), u) = 0\}.$$

It is clear that $u \in S(\lambda)$ if and only if

$$\int |\nabla u|^p - \lambda \int a(x)|u|^p = \lambda \int a(x)|u|^{p+\gamma},$$

and so

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{p+\gamma}\right)(I_1(u) - \lambda I_2(u)) \\ &= \left(\frac{1}{p} - \frac{1}{p+\gamma}\right)(\lambda I_3(u)). \end{aligned}$$

It is useful to study $S(\lambda)$ it terms of the stationary points of the functions of the form $\varphi_u : t \to J_\lambda(tu)$ $(t > 0)$. Such maps are known as fibrering maps and were introduced by Drábek and Pohozaev in [10], and also mentioned in Brown and Zhang [5]. In this case we have

$$\begin{aligned}
\varphi_u(t) &= J_\lambda(tu) = \frac{1}{p}(I_1(tu) - \lambda I_2(tu)) - \frac{\lambda}{p+\gamma} I_3(tu) \\
&= \frac{t^p}{p}(I_1(u) - \lambda I_2(u)) - \frac{\lambda t^{p+\gamma}}{p+\gamma} I_3(u), \\
\varphi_u'(t) &= t^{p-1}(I_1(u) - \lambda I_2(u)) - \lambda t^{p+\gamma-1} I_3(u), \\
\varphi_u''(t) &= (p-1)t^{p-2}(I_1(u) - \lambda I_2(u)) - \lambda(p+\gamma-1)t^{p+\gamma-2} I_3(u).
\end{aligned}$$

Hence if we define

$$\begin{aligned}
S^+(\lambda) &= \{u \in S : \ (p-1)(I_1(u) - \lambda I_2(u)) > \lambda(p+\gamma) I_3(u)\}, \\
S^-(\lambda) &= \{u \in S : \ (p-1)(I_1(u) - \lambda I_2(u)) < \lambda(p+\gamma) I_3(u)\},
\end{aligned}$$

and

$$S^0(\lambda) = \{u \in S : \ (p-1)(I_1(u) - \lambda I_2(u)) = \lambda(p+\gamma) I_3(u)\},$$

then for $u \in S(\lambda)$ we have

(i) $\varphi_u'(1) = 0$.

(ii) $u \in S^+(\lambda), S^-(\lambda), S^0(\lambda)$ if $\varphi_u'(1) > 0, \varphi_u'(1) < 0, \varphi_u'(1) = 0$ respectively.

(iii) $S^+(\lambda)$ $(S^-(\lambda), S^0(\lambda), \text{resp.}) = \{u \in S(\lambda) : I_3(u) < (>, =, \text{resp.})\ 0\}$ so that $S^+(\lambda)$, $S^-(\lambda)$, $S^0(\lambda)$ correspond to minima, maxima and points of inflection of fibrering map, respectively.

(iv) The condition (a$_3$) on $a(x)$ implies that $u_1^+ \notin S^-(\lambda)$.

Remark 2.3. If $u \in S(\lambda)$ is a minimizer of J_λ on $S(\lambda)$, then $|u| \in S(\lambda)$ is also a minimizer of J_λ on $S(\lambda)$.

3. Properties of the Bifurcation Diagram

In this section we will consider the problem (1.1) in viewpoint of the bifurcation theory.

By using Proposition 2.2 we consider (u_n, λ_n) on the bifurcation diagram with $\lambda_n \to 0$ and $\lambda_n \le \lambda_1^+$, $\lambda_n \to \lambda_1^+$. By the means of the structure of the Nehari manifold $S(\lambda)$, we have

$$\int |\nabla u_n|^p - \lambda_n \int a(x)|u_n|^p = \lambda \int a(x)|u_n|^{p+\gamma}. \tag{3.1}$$

Let $v_n = \dfrac{u_n}{||u_n||}$. Observe that, by the uniform convexity of V, we may assume that $v_n \rightharpoonup v$ for some $v \in V$. By dividing (2.3) by $||u_n||^p$ we have

$$\int |\nabla v_n|^p - \lambda_n \int a(x)|v_n|^p = \lambda_n ||u_n||^\gamma \int a(x)|v_n|^{p+\gamma}.$$

Using the compactness argument mentioned in [10], we have $I_2^+(v_n) \to I_2^+(v)$ and so

$$\begin{aligned} 0 \le \int |\nabla v|^p - \lambda_1^+ \int a(x)|v|^p &\le \liminf_{n\to\infty} \left(\int |\nabla v_n|^p - \lambda_n \int a(x)|v_n|^p \right) \\ &= \lim_{n\to\infty} \left(\int |\nabla v_n|^p - \lambda_n \int a(x)|v_n|^p \right) = 0. \end{aligned}$$

Note that we use the variational characteristic of λ_1^+ in the first inequality. It then follows that $v = 0$ or $v = t(v)u_1^+$ for some positive constant $t(v)$. We show that the first is impossible. Suppose otherwise, then $I_2^+(v_n) \to 0$ and $0 \le \int |\nabla v|^p - \lambda_1^+ \int a(x)|v|^p \to 0$. So we conclude $\lambda_n \int a_2(x)|v_n|^p \to 0$, hence $\int a_2(x)|v_n|^p \to 0$. To obtain $||v_n|| \to 0$, that is our contradiction, it suffices to show that $\int W(x)|v_n|^p \to 0$. This follows from

$$0 \le \int W(x)|v_n|^p \le \int (\bar{a}_2(x) + \omega(x))|v_n|^p,$$

where $\bar{a}_2(x) = \max_{x\in\mathbb{R}^N} a_2(x)$, and Hardy's inequality. Hence $||v_n|| \to 0$, contradicting the fact $||v_n|| = 1$.

We now turn our attention to

$$0 \le \frac{1}{||u_n||^\gamma} \int |\nabla v_n|^p - \lambda_n \int a(x)|v_n|^p = \lambda_n \int a(x)|v_n|^{p+\gamma},$$

and conclude that $\int a(x)(u_1^+)^{p+\gamma} \ge 0$, contradicting ($a_3$).

Note that as $u_n \to 0$, $v_n \to t(v)u_1^+$ and

$$\frac{1}{||u_n||^{p+\gamma}} \int a(x)|u_n|^{p+\gamma} \to t(v)^{p+\gamma} \int a(x)(u_1^+)^{p+\gamma} \ge 0.$$

Thus $u_n \in S^+(\lambda)$. So we have proved the following result.

Theorem 3.1. The solution branch *C* bends to the right of λ_1^+ at $(\lambda_1^+, 0)$ and for (λ, u) close enough to $(\lambda_1^+, 0)$, we have $u \in S^+(\lambda)$.

Now we turn our attention to $S^-(\lambda)$ and investigate the behavior of J_λ on $S^-(\lambda)$. For $u \in S^-(\lambda)$ we have

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{p+\gamma} \right) (I_1(u) - \lambda I_2(u)) \\ &= \left(\frac{1}{p} - \frac{1}{p+\gamma} \right) (\lambda I_3(u)) > 0. \end{aligned}$$

So J_λ is bounded below by 0 on $S^-(\lambda)$. We now show that there exists a minimizer on $S^-(\lambda)$ which is a critical point of J_λ and so another nontrivial solution of (1.1).

Theorem 3.2. Suppose (a_1)–(a_3) hold. There exists $\delta > 0$ such that the problem (1.1) has two positive solutions whenever $\lambda_1^+ < \lambda < \lambda_1^+ + \delta$.

Proof. **Step 1.** First we claim that there exists $\delta > 0$ such that $S^-(\lambda)$ is closed in V and open in $S(\lambda)$ whenever $\lambda_1^+ < \lambda < \lambda_1^+ + \delta$.

Suppose otherwise. Then there exist λ_n and $u_n \in S^-(\lambda)$ such that $\lambda_n \to \lambda_1^+$ and $u_n \to u_0 \in S^-(\lambda)$, i.e.,

$$0 < \lambda_1^+ \int a(x)|u_n|^{p+\gamma} = \int (|\nabla u_n|^p - \lambda_n a(x)|u_n|^p) \to 0.$$

Let $v_n = \dfrac{u_n}{||u_n||}$. Then we can assume $v_n \rightharpoonup v_0$ in V for some $v_0 \in V$. By dividing the last relation by $||u_n||^p$ we get

$$\begin{aligned}\int (|\nabla v_n|^p - \lambda_n a(x)|v_n|^p) &= \lambda_n ||u_n||^\gamma \int a(x)|v_n|^{p+\gamma}\\ &= \lambda_n ||u_n||^{-p} \int a(x)|u_n|^{p+\gamma} \to 0.\end{aligned}$$

From the weak convergence of v_n to v_0 in V and $\int a(x)|v_n|^{p+\gamma} \to \int a(x)|v_0|^{p+\gamma}$, we conclude that

$$0 \le \int (|\nabla v_0|^p - \lambda_1^+ a(x)|v_0|^p) \le \lim_{n\to\infty} \left(\int (|\nabla v_n|^p - \lambda_n a(x)|v_n|^p) \right) = 0.$$

If $v_0 = 0$, we then derive that $\int a(x)|v_n|^p \to 0$ and $\int (|\nabla v_n|^p - \lambda_n a_1(x)|v_n|^p) \to 0$, the latter contradicting the fact that $||v_n|| = 1$. It then follows from the uniqueness of u_1^+ that $v_0 = t(v_0)u_1^+$ for some positive constant $t(v_0)$. We now have by the compactness argument

$$\lambda_1^+ \int a(x)|t(v_0)u_1^+|^{p+\gamma} = \lim_{n\to\infty} \lambda_n \int a(x)|v_n|^{p+\gamma} > 0,$$

which is impossible due to (a_2).

Step 2. Now we claim that there exist $M > 0$ and $\delta_1 > 0$ such that for all $u \in S^-(\lambda)$ and $\lambda_1^+ < \lambda < \lambda_1^+ + \delta_1$

$$\int (|\nabla u|^p - \lambda a(x)|u|^p) \ge M||u||^p. \tag{3.2}$$

We prove the claim by contradiction. Assume there exist $\lambda_n \to \lambda_1^+$ and $u_n \in S^-(\lambda_n)$ such that

$$\int (|\nabla u_n|^p - \lambda_n a(x)|u_n|^p) < \frac{1}{n}||u_n||^p.$$

Let $v_n = \dfrac{u_n}{||u_n||}$. We may assume $v_n \rightharpoonup v_0$ in V for some $v_0 \in V$. On the other hand $0 < \int (|\nabla v_n|^p - \lambda_n a(x)|v_n|^p) < \dfrac{1}{n} \to 0$. Note that the first inequality follows from the variational characteristic of λ_1^+. So

$$\int (|\nabla v_n|^p - \lambda_n a(x)|v_n|^p) \to 0. \tag{3.3}$$

Using the compactness argument we have $I_2^+(v_n) \to I_2^+(v_0)$. It then follows from $v_n \rightharpoonup v_0$ that

$$\begin{aligned} 0 \le \int (|\nabla v_0|^p - \lambda_1^+ a(x)|v_0|^p) &\le \liminf_{n\to\infty} \int (|\nabla v_n|^p - \lambda_n a(x)|v_n|^p) \\ &= \lim_{n\to\infty} \int (|\nabla v_n|^p - \lambda_n a(x)|v_n|^p) = 0. \end{aligned}$$

So $\int |\nabla v_0|^p = \lambda_1^+ \int a(x)|v_0|^p$. If $v_0 = 0$, then we arrive at a contradiction like in Step 1, and so the possibility of $v_0 = 0$ is excluded. Hence there exists a positive constant $t(v_0) \in (0, 1]$ such that $v_0 = t(v_0)u_1^+$. Again by the compactness argument, we obtain $I_3(v_n) \to I_3(v_0)$ and

$$\begin{aligned} 0 < \int (|\nabla u_n|^p - \lambda_n a(x)|u_n|^p) &= \lambda_n \int a(x)|u_n|^{p+\gamma} \\ &= \lambda_n ||u_n||^{p+\gamma} \int a(x)|v_n|^{p+\gamma} = 0. \end{aligned}$$

From $u_n \in S^-(\lambda_n)$, we get

$$\begin{aligned} 0 < \lambda_n \int a(x)|v_n|^{p+\gamma} &\to \lambda_1^+ \int a(x)|v_0|^{p+\gamma} \\ &= \lambda_1^+ t(v_0)^{p+\gamma} \int a(x)(u_1^+)^{p+\gamma} < 0, \end{aligned}$$

a contradiction.

Step 3. $J_\lambda(u)$ satisfies the Palais–Smale condition on $S^-(\lambda)$ for $\lambda_1^+ < \lambda < \lambda_1^+ + \delta_1$. Suppose there is a sequence $u_n \in S^-(\lambda)$ such that $J_\lambda(u_n) \le c$ and $J'_\lambda(u_n) \to 0$. Note that Step 2 implies that such sequence $\{u_n\}$ is bounded by $\dfrac{c}{M}\left(\dfrac{1}{p} - \dfrac{1}{p+\gamma}\right)^{-1}$, and so we may assume $u_n \rightharpoonup u_0$ in V for some $u_0 \in V$. Using the compactness argument we then derive that $I_2^+(u_n) \to I_2^+(u_0)$ and $I_3(u_n) \to I_3(u_0)$. Now we can estimate $(J'_\lambda(u_n) - J'_\lambda(u_m), u_n - u_m)$ as in the proof of [9, Lemmas 2.3 and 3.3] and derive that $I_1(u_n) \to I_1(u_0)$ and $I_2^-(u_n) \to I_2^-(u_0)$. We thus obtain by Hardy's inequality that $||u_n|| \to ||u_0||$ and hence a subsequence of u_n converges to u_0 strongly in V.

Step 4. Existence of a positive solution on $S^-(\lambda)$. From the above steps we obtain that J_λ has a nonnegative minimizer $u^* \in S^-(\lambda)$. Hence by the Lagrange multiplier theorem there exists $\mu \in \mathbb{R}$ such that

$$(J'_\lambda(u^*), \varphi) = \mu(I'_1(u^*) - \lambda I'_2(u^*) - \lambda I'_3(u^*), \varphi)$$

for all $\varphi \in V$. Taking $\varphi = u^*$ and using he fact that $u^* \in S^-(\lambda)$, we get

$$-\gamma\mu\lambda I_3(u^*) = 0,$$

which implies $\mu = 0$ and hence u^* is a solution of (1.1) on $S^-(\lambda)$. [16, Theorem 1.2] implies that $u^* > 0$ in $\mathbb{R}^N$. This concludes the proof. ■

Now we study the existence of positive solutions at the point λ_1^+.

Lemma 3.3. $S(\lambda_1^+) \setminus \{0\}$ is a closed nonempty set.

Proof. First we show that $S(\lambda_1^+) \setminus \{0\}$ is nonempty. Note that $a_1(x) \not\equiv 0$. So there exists a set $B \subset \mathbb{R}^N$ with $a(x) > 0$ in B. Take $u(x) \not\equiv 0$ such that $\emptyset \neq \text{supp} u \subset B$ and so

$$\lambda_1^+ \int_{\mathbb{R}^N} a(x)|u|^{p+\gamma} = \lambda_1^+ \int_B a(x)|u|^{p+\gamma} > 0.$$

Consider the auxiliary function

$$h(t) = |t|^p \int (|\nabla u|^p - \lambda_1^+ a(x)|u|^p) - |t|^{p+\gamma}\lambda \int a(x)|u|^{p+\gamma},$$

and observe that if $t \to \pm\infty$, we have $h(t) \to -\infty$. Using the facts that $h(0) = 0$ and $h'(0) = 0$ and considering the sign of $h'(t)$ when $t \to 0^+$, we obtain that $h(t_0) = 0$ for some $t_0 > 0$ and hence $0 \neq t_0 u \in S(\lambda_1^+)$ and the claim is proved.

Now suppose there exists a sequence $\{u_n\}$ in $S(\lambda_1^+) \setminus \{0\}$ such that $u_n \to 0$ in V. Since $u_n \in S(\lambda_1^+)$ we have

$$0 \leq \int (|\nabla u_n|^p - \lambda_1^+ a(x)|u_n|^p) = \lambda_1^+ \int a(x)|u_n|^{p+\gamma}. \tag{3.4}$$

The first inequality follows from the variational characteristic of λ_1^+. Dividing (3.4) by $||u_n||^{p+\gamma}$, we arrive at

$$0 \leq \int (|\nabla v_n|^p - \lambda_1^+ a(x)|v_n|^p) = \lambda_1^+ ||u_n||^\gamma \int a(x)|v_n|^{p+\gamma}, \tag{3.5}$$

where $v_n = \dfrac{u_n}{||u_n||}$. Without loss of generality we can assume $v_n \rightharpoonup v_0$ in V for some $v_0 \in V$. The compactness argument shows that $I_3(v_n) \to I_3(v_0)$ and so $\{I_3(v_n)\}$ is bounded. Hence the right-hand side of (3.5) tends to 0. From the weak convergence of v_n to v_0 in V and $I_2^+(v_n) \to I_2^+(v_0)$, we obtain that

$$0 \le \int (|\nabla v_0|^p - \lambda_1^+ a(x)|v_0|^p) \le \lim_{n\to\infty} \int (|\nabla v_n|^p - \lambda_1^+ a(x)|v_n|^p).$$

The possibility $v_0 = 0$ is excluded, as in Step 2 of Theorem 3.2. So there exists some positive constant $t(v_0)$ such that $v_0 = t(v_0)u_1^+$. On the other hand we have

$$0 \le ||u_n||^\gamma \int (|\nabla v_n|^p - \lambda_1^+ a(x)|v_n|^p) = \lambda_1^+ \int a(x)|v_n|^{p+\gamma}.$$

Letting $n \to \infty$ we conclude that $\int a(x)(u_1^+)^{p+\gamma} \ge 0$. This contradiction proves the lemma. ■

Theorem 3.4. Equation (1.1) has a positive solution at λ_1^+.

Proof. We do steps similar to those in the proof of Theorem 3.2. Indeed Step 2 with $\lambda = \lambda_1^+$ implies either $v_0 = 0$ or $v_0 = t(v_0)u_1^+$ for some nonzero $t(v_0)$. In either case we obtain a contradiction. Note that for $0 \ne u \in S(\lambda_1^+)$

$$J_{\lambda_1^+}(u) = \left(\frac{1}{p} - \frac{1}{p+\gamma}\right) \int (|\nabla u|^p - \lambda_1^+ a(x)|u|^p) \ge 0,$$

and so $J_{\lambda_1^+}(u)$ is bounded below and we can look for a nontrivial minimizer of this functional on $S(\lambda_1^+)$. Arguments similar to Steps 3 and 4 yield that the functional $J_{\lambda_1^+}$ satisfies the Palais–Smale condition on $S(\lambda_1^+) \setminus \{0\}$ and $\mu = 0$ as in Theorem 3.4. [16, Theorem 1.2] further implies that the solution is positive in $\mathbb{R}^N$. ■

Theorem 3.5. Let $0 \le \lambda < \lambda_1^+$. Then problem (1.1) has at least one solution.

Proof. To prove that J_λ satisfies the Palais–Smale condition on $S(\lambda)$ for $0 \le \lambda < \lambda_1^+$, we can do it as in Theorem 3.4, step by step. We omit the details. ■

Remark 3.6. Also in [5] with the condition (a_3), similar to these results by using a different approach in bounded domains are obtained for the case $p = 2$. In this paper we generalized the results in [5] for the more general cases $p > 1$ and whole of $\mathbb{R}^N$.

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