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Singular Nonlinear Boundary Value Problems Arising in Viscous Flow Behind a Shock Wave

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Abstract

A rigorous proof for existence and uniqueness of solutions to a class of singular nonlinear boundary value problems arising in viscous flow and heat transfer behind a shock wave is made and a theoretical estimate formula for the skin friction coefficient is presented. The formula can be successfully applied to estimate the value of the skin friction coefficient. The correctness of the analytical predictions is verified by the numerical results.

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1. Introduction

Fluid dynamicists have long known that the appearance of boundary layers was not restricted to the canonical problem of the motion of a body through a viscous fluid. A technologically important source of boundary layer phenomenon is the flow behind a shock wave traveling over smooth surfaces. When a plane shock wave advances into a stationary fluid, with a plane wall perpendicular to the wave front, a boundary layer is established along the wall behind the wave. This boundary layer is often important in the studies of phenomena involving nonstationary waves. For example, a shock-generated boundary layer occurs when a combustible mixture is ignited within a tube, in this case, a shock wave, followed by a flame front, the shock wave is particularly strong when ignition occurs at a closed end. For long tubes, the progress of the flame front will be related to the boundary layer development behind the shock, and the boundary may play a role in the acceleration of a low-speed flame to detonation wave in a long tube [8, 9].

Understanding the nature of the boundary layer behind shock wave by mathematical modeling with a view to predicting the flow and heat transfer has been the focus of considerable research. A principal reason for the interest in analysis of boundary layer flows along solid surfaces is the possibility of applying the theory to the efficient design of supersonic and hypersonic flights. For a list of the key references of a vast literature concerning this subject we refer to the references [3, 4, 6, 10–13].

All of the above-mentioned works have had attention paid to the analytical equations or numerical ones, so that the important questions touching on well-posedness, i.e., the qualitative properties of the solutions, such as the existence, uniqueness, and analyticity of the solutions remain unanswered.

This work is motivated by the desire to present an investigation for a class of singular nonlinear boundary value problems arising in viscous flow behind a shock wave. The mathematical model considered in the present paper has significance in studying several problems of engineering, meteorology, and oceanography [7, 14, 15].

2. Laminar Boundary Layer Equations

For the sake of simplicity, we restrict ourselves to the considerations of perfect gas. It will be assumed that μ (coefficient of viscosity), κ (thermal conductivity) are proportional to T and that C_p (specific heat at constant pressure) and Pr (Prandtl number, $\mu C_p/\kappa$) are independent of T. Consider a plane laminar flow with spatial coordinates (x, y), corresponding velocity components (u, v) and $dp/dx = 0$. For steady flow, the boundary layer equations for $x > 0$ can be written as [8, 9, 11]

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2.2}$$

$$\rho C_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y}\left(\kappa \frac{\partial u}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2, \tag{2.3}$$

$$P = \rho R T. \tag{2.4}$$

The boundary conditions are

$$u(x, 0) = u_w, u(x, \infty) = u_e, \tag{2.5}$$

$$v(x, 0) = 0, \tag{2.6}$$

$$T(x, 0) = T_w, T(x, \infty) = T_e. \tag{2.7}$$

3. Nonlinear Boundary Value Problems

3.1. Stream Function and Similarity Variables

We introduce a stream function Ψ and a similarity variable η by the expressions

$$\Psi = \sqrt{2u_e x \nu_w} f(\eta), \;\; \eta = \sqrt{\frac{u_e}{2x\nu_w}} \int_0^Y \frac{T_w}{T(x, y)} dy. \tag{3.1}$$

We substitute (3.1) into (2.1)–(2.7). The governing boundary layer equations of momentum and energy can be reduced to a set of ordinary differential equations by a suitable choice of moving coordinate system and by assuming that the fluid is an ideal gas having the viscosity and thermal conductivity both proportional to the temperature. The transformed momentum and energy equations may be reduced to the following differential equations:

Momentum equations

$$f'''(\eta) + f(\eta)f''(\eta) = 0, \quad 0 < \eta < +\infty, \tag{3.2}$$

$$f(0) = 0, \;\; f'(0) = \xi, \;\; f'(+\infty) = 1. \tag{3.3}$$

Energy equations

$$\overline{T}''(\eta) + Pr \cdot f(\eta)\overline{T}'(\eta) = -\frac{Pr \cdot u_e^2}{C_{pw}T_e}(f''(\eta))^2, \quad 0 < \eta < +\infty, \tag{3.4}$$

$$\overline{T}(0) = \lambda, \;\; \overline{T}(+\infty) = 1. \tag{3.5}$$

Here the prime denotes differentiation with respect to η, $\xi = f'(0) = \frac{u_w}{u_e}$, is the velocity ratio parameter, $\lambda = T_w/T_e$ is the temperature ration parameter, $Pr = \mu C_p/\kappa$ is the Prandtl number, and $1 < \xi < 6$ for a shock wave [6, 8, 9, 11].

3.2. Crocco Variables Transformation

Introducing a transformation as [6, 12]

$$g(z) = f''(\eta) \quad \text{(dimensionless shear stress)}, \tag{3.6}$$

$$z = f'(\eta) \quad \text{(dimensionless tangential velocity)}, \tag{3.7}$$

$$\theta(z) = \overline{T}(\eta) \quad \text{(dimensionless temperature)}, \tag{3.8}$$

and substituting (3.6)–(3.8) into (3.2)–(3.5), in terms of $f''(\eta) < 0$, $0 < \eta < +\infty$, $f''(+\infty) = 0$, and $(\gamma - 1)M_e^2 = u_e^2/(C_{p,w} \cdot T_e)$, we arrive at the following singular nonlinear two-point boundary value problems.

Momentum equations

$$g(z)g''(z) + z = 0, \quad 1 < z < \xi < 6, \tag{3.9}$$

$$g(1) = 0, \ g'(\xi) = 0. \tag{3.10}$$

Energy equations

$$\theta''(z) + (1 - Pr)\theta'(z)g'(z)/g = -Pr(\gamma - 1)M_e^2, \quad 1 < z < \xi < 6, \tag{3.11}$$

$$\theta(1) = 1, \ \theta(\xi) = \lambda. \tag{3.12}$$

Clearly, Eqs. (3.9)–(3.10) are de-coupled and may be considered firstly, the solutions then may be used to solve Eqs. (3.11)–(3.12). It may be seen from the derivation process that only the negative solutions of Eqs. (3.9)–(3.10) are physically significant.

3.3. The Solutions of Eqs. (3.9)–(3.10)

Let $t = \xi - z$, $w(t) = -g(\xi - t)$, then Eqs. (3.9)–(3.10) are changed into the nonlinear singular two-point boundary value problems

$$\begin{cases} w''(t) = \dfrac{t - \xi}{w(t)}, \ 0 < t < \xi - 1, \\ w'(0) = 0, \ w(\xi - 1) = 0. \end{cases} \tag{3.13}$$

In terms of negative solutions of Eqs. (3.9)–(3.10), it is seen that only the positive solutions of Eq. (3.13) is physically significant.

Since the problem is singular at $t = \xi - 1$, it is convenient by considering the boundary conditions without singularities

$$\begin{cases} w''(t) = \dfrac{t - \xi}{w(t)}, \ 0 < t < \xi - 1, \\ w'(0) = 0, \ w(\xi - 1) = h > 0. \end{cases} \tag{3.14}$$

Denote the solution of Eq. (3.14) by $w_h(t)$. We first show the following lemmas.

Lemma 3.1. If $h_1 > h_2 > 0$, then $w_{h_1}(t) \geq w_{h_2}(t)$.

Proof. If the inequality is not true, then there exists a point $t_0 \in [0, \xi - 1)$ such that $w_{h_1}(t_0) < w_{h_2}(t_0)$. We consider only two cases.

(i) $w_{h_1}(0) < w_{h_2}(0)$. Choose $t_0 = 0$, since $w_{h_1}(\xi - 1) > w_{h_2}(\xi - 1) > 0$. Then there exists a maximal interval $[0, k]$ $(k < \xi - 1)$ such that $w_{h_1}(t) < w_{h_2}(t)$ for $t \in [0, k)$, and $w_{h_1}(k) = w_{h_2}(k) = m > 0$. $w_{h_1}(t)$ and $w_{h_2}(t)$ are both positive solutions of the integral equation

$$w(t) = m + \int_0^k G_1(t, s)\frac{s - \xi}{w(s)} ds, \tag{3.15}$$

where Green's function $G_1(t, s)$ is defined as

$$G_1(t, s) = \begin{cases} t - k, & 0 \leq s \leq t < k, \\ s - k, & 0 \leq t \leq s < k. \end{cases}$$

Eq. (3.15) implies

$$0 < w_{h_2}(t) - w_{h_1}(t) = \int_0^k G_1(t, s)\left[\frac{s - \xi}{w_{h_2}(s)} - \frac{s - \xi}{w_{h_1}(s)}\right] ds < 0,$$

which is a contradiction.

(ii) $w_{h_1}(0) \geq w_{h_2}(0)$. Since $w_{h_1}(\xi - 1) > w_{h_2}(\xi - 1) > 0$, there exists a maximal interval $[a, b]$ $(0 \leq a < b < \xi - 1)$, which contains the point t_0 such that $w_{h_1}(a) = w_{h_2}(a)$ and $w_{h_1}(b) = w_{h_2}(b)$, and $w_{h_1}(t) < w_{h_2}(t)$ for $t \in (a, b)$. Let $w_{h_1}(a) = w_{h_2}(a) = \alpha$ and $w_{h_1}(b) = w_{h_2}(b) = \beta$. Then for $t \in [a, b]$, $w_{h_1}(t)$ and $w_{h_2}(t)$ are both positive solutions of the integral equation

$$w(t) = \frac{b\alpha - a\beta}{b - a} + \frac{\beta - \alpha}{b - a}t + \int_a^b G_2(t, s)\frac{s - \xi}{w(s)} ds, \tag{3.16}$$

where Green's function $G_2(t, s)$ is defined as

$$G_2(t, s) = \begin{cases} \dfrac{(b - t)(s - a)}{b - a}, & 0 \leq a \leq s \leq t \leq b < \xi - 1, \\ \dfrac{(b - s)(t - a)}{b - a}, & 0 \leq a \leq t \leq s \leq b < \xi - 1. \end{cases}$$

Eq. (3.16) implies

$$0 < w_{h_2}(t) - w_{h_1}(t) = \int_a^b G_2(t, s)\left[\frac{s - \xi}{w_{h_2}(s)} - \frac{s - \xi}{w_{h_1}(s)}\right] ds < 0,$$

which is also a contradiction. ∎

Lemma 3.2. For any fixed $h > 0$, Eq. (3.15) has at most one positive solution.

Proof. Suppose Eq. (3.15) has two positive solutions $w_1(t)$ and $w_2(t)$ for each fixed $h > 0$. Then, without loss of generality, we may assume that there exists a point $t_0 \in [0,\ \xi - 1]$ such that $w_1(t_0) > w_2(t_0)$. Since $w_1(\xi - 1) = w_2(\xi - 1) = h$, then there exists a maximal close interval $[a_1, b_1] \subseteq [0,\ \xi - 1]$ such that $w_1(t) > w_2(t)$ for $t \in [a_1, b_1]$.

(i) If $a_1 = 0$, then $w_1(t) > w_2(t)$ for $t \in [0, b_1] \subseteq [0,\ \xi - 1]$ and $w_1(b_1) = w_2(b_1)$.

(ii) If $a_1 \neq 0$, then $w_1(a_1) = w_2(a_1)$ and $w_1(b_1) = w_2(b_1)$ for $t \in [a_1, b_1] \subset [0,\ \xi - 1]$, and $w_1(t) > w_2(t)$ for $t \in (a_1, b_1)$.

Along the same lines as in the cases (i) and (ii) in Lemma 3.1, we may show that this is impossible. ■

Lemma 3.3. For any fixed $h > 0$, Eq. (3.15) has at least one positive solution $w_h(t)$.

Proof. For any fixed $h > 0$, if $w(t)$ is the positive solution of Eq. (3.15), then $w(t)$ is convex on $[0,\ \xi - 1]$ and must be a positive solution of the integral equation

$$w(t) = h + \int_0^{\xi-1} G_3(t,s)\frac{s-\xi}{w(s)}ds, \tag{3.17}$$

where Green's function $G_3(t, s)$ is defined as

$$G_3(t,s) = \begin{cases} t - \xi + 1, & 0 \le s \le t \le \xi - 1, \\ s - \xi + 1, & 0 \le t \le s \le \xi - 1. \end{cases}$$

We define a mapping T by

$$Tw(t) = h + \int_0^{\xi-1} G_3(t,s)\frac{s-\xi}{w(s)}ds,$$

where $\Omega = \{w(t) \in C\,[0,\ \xi - 1] : h \le w(t) \le (Th)(t)\}$, and $C\,[0,\ \xi - 1]$ is the set of all real-valued continuous functions defined on $[0,\ \xi - 1]$. Then T is a completely continuous mapping from Ω to Ω. The Schauder fixed point theorem [1, 2, 5] asserts that the mapping T has at least one fixed point $w_h(t)$ in Ω, which implies that $w_h(t)$ is a positive solution of Eq. (3.15). ■

We denote $w(0) = \sigma$ and consider the initial value problem

$$\begin{cases} w''(t) = \dfrac{t-\xi}{w(t)}, \ 0 < t < \xi - 1, \\ w(0) = \sigma > 0, \ w'(0) = 0. \end{cases} \tag{3.18}$$

Let $w(t)$ be the positive solution of Eq. (3.14) and $[0, t_\sigma^*)$ be the maximal interval of existence with $w(0; \xi) = \sigma$. Then we may established the following results.

Lemma 3.4.

(i) Let w_1 and w_2 be solutions for $\sigma = \sigma_1$ and $\sigma = \sigma_2$. If $\sigma_1 < \sigma_2$, then $t^*_{\sigma_1} < t^*_{\sigma_2}$.

(ii) t^*_σ is a continuous function of σ and $t^*_\sigma \to +\infty$ as $\sigma \to +\infty$.

The proof of this lemma is similar to that of [13, Lemmas 3.1, 3.2 and 3.3], so we omit it here.

Lemma 3.5. For any fixed $h > 0$, the positive solution w_h of Eq. (3.15) satisfies

$$w_h(0, \xi) > (\xi - 1)\sqrt{\frac{1 + 2\xi}{6}} \ (1 < \xi < 6).$$

Proof. In terms of Eq. (3.19), for $t \in (0, \xi - 1)$

$$w(t) < \sigma + \frac{1}{6\sigma}(t - 3\xi)t^2 < \sigma - \frac{1 + 2\xi}{6\sigma}t^2.$$

Let $f(t) = \sigma - \frac{1 + 2\xi}{6\sigma}t^2$. Then the positive solution of the initial value problem satisfies $w(t) < f(t)$ for $t \in (0, \xi - 1)$. In terms of Lemma 3.4, we assume that $f(t)$ intersects the t-axis at the point t^*_0. Then $t^*_0 = \sqrt{\frac{6\sigma^2}{1 + 2\xi}}$. Especially for $t^*_0 = \xi - 1$, this yields $\sigma = (\xi - 1)\sqrt{\frac{1 + 2\xi}{6}}$ for any ξ $(1 < \xi < 6)$.

Similar to Lemma 3.1, we may show the positive solution of Eq. (3.19) is increasing with σ, so the positive solutions $w(t, \sigma)$ of Eq. (3.19) cannot intersect the point $\xi - 1$ for $\sigma \le (\xi - 1)\sqrt{\frac{1 + 2\xi}{6}}$. This implies that for $\sigma \le (\xi - 1)\sqrt{\frac{1 + 2\xi}{6}}$, the positive solution of the initial value problem (3.18) satisfies $w(\xi - 1) < 0$. This shows that for any fixed $h > 0$, the positive solutions w_h of Eq. (3.15) satisfy the asserted inequality. ■

Theorem 3.6. Eq. (3.14) has a unique positive solution, i.e., (3.9)–(3.10) has a unique negative solution.

Proof. Lemma 3.2 and Lemma 3.3 show that for any $h > 0$, Eq. (3.15) has a unique positive solution. Then for any $h_2 > h_1 > 0$, in terms of Eq. (3.18) and Lemma 3.1

$$0 < w_{h_2}(t) - w_{h_1}(t) = h_2 - h_1 + \int_0^{\xi - 1} G_3(t, s)\left[\frac{s - \xi}{w_{h_2}(s)} - \frac{s - \xi}{w_{h_1}(s)}\right] ds \le h_2 - h_1.$$

This indicates the series of positive solutions $\{w_h\}$ converges to a limit uniformly with h on $[0,\ \xi - 1]$, denoted by w_0. Then

$$\lim_{h \to 0} w_h(t) = w_0(t),\ t \in [0,\ \xi - 1].$$

Lemma 3.5 implies $w_0(0,\xi) > (\xi - 1)\sqrt{\dfrac{1+2\xi}{6}}$ $(1 < \xi < 6)$. For any $h \geq 0$, by the convexity of $w_h(t)$, this yields

$$\begin{aligned} w_h(t) &\geq h + \frac{w_h(0,\xi) - h}{1-\xi}\,[t-(\xi-1)] \\ &= h + \frac{w_h(0,\xi)}{1-\xi}\,[t-(\xi-1)] - \frac{h}{1-\xi}\,[t-(\xi-1)] \\ &\geq h + \frac{w_h(0,\xi)}{1-\xi}\,[t-(\xi-1)] - h \\ &\geq -\sqrt{\frac{1+2\xi}{6}}\,[t-(\xi-1)]. \end{aligned} \tag{3.19}$$

It follows from Eq. (3.18) that

$$G_3(t,s)\frac{s-\xi}{w_h(s)} \leq -G_3(t,s)\frac{s-\xi}{[s-(\xi-1)]\sqrt{\frac{1+2\xi}{6}}}. \tag{3.20}$$

Letting $h \to 0^+$ in the integral equation, we get

$$w_h(t) = h + \int_0^{\xi-1} G_3(t,s)\frac{s-\xi}{w_h(s)}ds.$$

By using the monotone convergence theorem [1,2,5], we obtain

$$w_0(t) = \lim_{h\to 0}\int_0^{\xi-1} G_3(t,s)\frac{s-\xi}{w_h(s)}ds = \int_0^{\xi-1}\lim_{h\to 0} G_3(t,s)\frac{s-\xi}{w_h(s)}ds,$$

i.e.,

$$w_0(t) = \int_0^{\xi-1} G_3(t,s)\frac{s-\xi}{w_0(s)}ds. \tag{3.21}$$

The above arguments indicate that Eq. (3.14) has a unique positive solution w_0. Furthermore, the convexity of w_0 and Lemma 3.5 imply that

$$(\xi-1)\sqrt{\frac{1+2\xi}{6}} < w(0,\xi) < \frac{\sqrt{6}(\xi^2-1)}{2\sqrt{1+2\xi}} \quad (1 < \xi < 6).$$

This proves that (3.9)–(3.10) has a unique negative solution g, satisfying

$$-\frac{\sqrt{6}(\xi^2-1)}{2\sqrt{1+2\xi}} < g(\xi) < -(\xi-1)\sqrt{\frac{1+2\xi}{6}} \quad (1 < \xi < 6). \tag{3.22}$$

This completes the proof. ■

In order to illustrate the reliability and efficiency of the proposed theoretical results and the estimate formula, Eqs. (3.9)–(3.10) are solved numerically by using the shooting technique. Fig. 1 shows the characteristics of dimensionless shear stress for $\xi = 2.0$ to 5.0. The results indicate that the skin friction $g(\xi)$ decreases with increasing the velocity ratio parameter ξ, and for each fixed value of parameter ξ, shear stress $g(z)$ decreases with increasing z in $[1, \xi]$.

Denote the skin friction coefficient $g(\xi)$ $(1 < \xi < 6)$ obtained numerically by $\sigma_{\text{com}} = g(\xi)\,(1 < \xi < 6)$, and the estimated results obtained by estimation formula (3.22) by $\sigma_{\text{lower-bound}} = -\dfrac{\sqrt{6}(\xi^2 - 1)}{2\sqrt{1+2\xi}}$ and $\sigma_{\text{upper-bound}} = -(\xi - 1)\sqrt{\dfrac{1+2\xi}{6}}$, respectively.

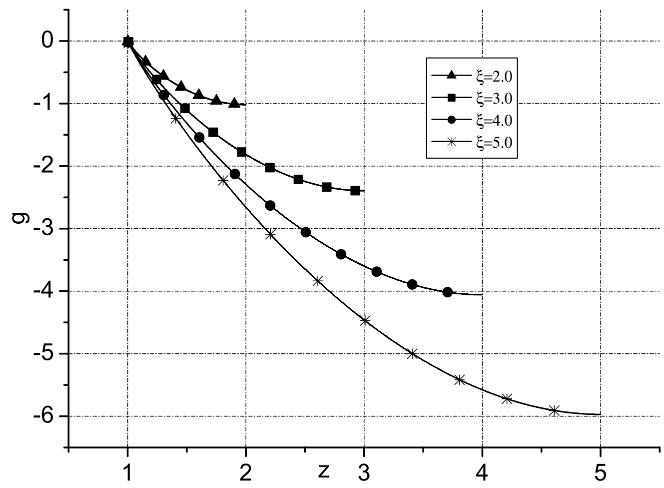

Fig. 1 Dimensionless shear stress profiles for $\xi = 2.0$ to 5.0

A comparison is presented in Table 1. The reliability and efficiency of the theoretical prediction are verified by numerical results.

Table 1 The skin friction coefficient obtained numerically and by formula (3.22)

ξ	$\sigma_{\text{lower-bound}}$	$\sigma_{\text{com}} = g(\xi)$	$\sigma_{\text{upper-bound}}$
ξ=2.0	-1.643168	-1.019003	-0.9128709
ξ=3.0	-3.70328	-2.395098	-2.160247
ξ=4.0	-6.123724	-4.057916	-3.674235
ξ=5.0	-8.862587	-5.970288	-5.416026

3.4. The Solutions of Eqs. (3.11)–(3.12)

Utilizing the unique analytical solution of Eqs. (3.9)–(3.10), the solution of Eqs. (3.11)–(3.12) is established and represented as follows:

$$\begin{aligned}\theta(z) = & -Pr\cdot(\gamma-1)M_e^2\int_1^z |g(s)|^{1-Pr}ds\int_1^z |g(s)|^{Pr-1}ds \\ & + \frac{-1+\lambda-Pr\cdot(\gamma-1)M_e^2\int_1^\xi |g(s)|^{1-Pr}\left(\int_1^s |g(\varsigma)|^{Pr-1}d\varsigma\right)ds}{\int_1^\xi |g(s)|^{Pr-1}ds}\int_1^z |g(s)|^{Pr-1}ds \\ & + Pr\cdot(\gamma-1)M_e^2\int_1^\xi |g(s)|^{1-Pr}ds\int_1^z |g(s)|^{Pr-1}ds \\ & + Pr\cdot M_e^2\int_1^z |g(s)|^{1-Pr}\left(\int_1^s |g(x)|^{Pr-1}dx\right)ds+1. \end{aligned} \tag{3.23}$$

For $\gamma = 1.4$, the ordinary shock-theory gives $M_e^2 = \dfrac{5}{6\xi-1}$. The solution (3.23) may be reduced to a very simple form, it was affected only by parameters of Pr, ξ and λ. In particular, for $Pr = 1$, Eq. (3.23) yields

$$\theta(z) = -\frac{1}{6\xi-1}(z-1)^2+\left(\frac{-1+\lambda}{\xi-1}+\frac{\xi-1}{6\xi-1}\right)(z-1)+1.$$

This shows that the temperature distribution $\theta(z)$ has a parabolic distribution with tangential velocity z, the solution reveals the characteristics of parameters ξ and λ on $\theta(z)$ in detail.

4. Conclusions

This paper presents a theoretical analysis for the boundary layer flow and heat transfer behind a shock wave. The boundary layer equations are reduced into a singular nonlinear two-point boundary value of ordinary differential equation when Crocco variables were introduced. Sufficient conditions for existence and uniqueness of positive solutions are established. Furthermore, a theoretical estimate formula for the skin friction coefficient is given. The reliability and efficiency of the theoretical prediction are verified by numerical results.

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