Existence of Positive Periodic Solutions of Mutualism Systems with Several Delays

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Abstract
In this paper, by using Mawhin coincidence degree, we study the global existence of positive periodic solutions for a class of mutualism systems with several delays and obtain a new and interesting criterion, which is much different from those of Yang etc. [8]. Moreover, our arguments for obtaining bounds of solutions to the operator equation $Lx = \lambda Nx$ are much different from those used in [8]. So some new arguments are employed for the first time.

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1. Introduction
Traditional Lotka–Volterra type predator-prey model or competitive model has received great attention from both theoretical and mathematical biologists and has been studied extensively (for example, see [1–5, 9, 11–22]). But there are few papers considering the mutualism system. Goh [6] discussed the stability in models of mutualism of the form

\begin{align*}
\frac{dx_1(t)}{dt} &= x_1(t)[r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t)],
\frac{dx_2(t)}{dt} &= x_2(t)[r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t)].
\end{align*}

(1.1)

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Then Gopalsamy and He [7] studied the following system with discrete delay
\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t - \tau) + a_{12}(t)x_2(t - \tau)\right], \\
\dot{x}_2(t) &= x_2(t)\left[r_2(t) + a_{21}(t)x_1(t - \tau) - a_{22}(t)x_2(t - \tau)\right].
\end{align*}
\]
(1.2)

Some sufficient conditions were obtained for the persistence and global attractivity of system (1.2). Then Yang etc. [8] obtained some sufficient conditions for the existence of positive periodic solutions of the system
\[
\begin{align*}
\dot{y}_1(t) &= y_1(t)\left[r_1(t) - a_{11}(t)y_1(t - \tau_1(t)) + a_{12}(t)y_2(t - \tau_2(t))\right], \\
\dot{y}_2(t) &= y_2(t)\left[r_2(t) + a_{21}(t)y_1(t - \tau_1(t)) - a_{22}(t)y_2(t - \tau_2(t))\right].
\end{align*}
\]
(1.3)

Now we generalize the system to
\[
\begin{align*}
\dot{y}_1(t) &= y_1(t)\left[r_1(t) - a_{11}(t)y_1(t - \tau_1(t)) + a_{12}(t)y_2(t - \tau_2(t))\right], \\
\dot{y}_2(t) &= y_2(t)\left[r_2(t) + a_{21}(t)y_1(t - \sigma_1(t)) - a_{22}(t)y_2(t - \sigma_2(t))\right],
\end{align*}
\]
(1.4)

where \(a_{ij}(t), i, j = 1, 2, \tau_i(t), \sigma_i(t), i = 1, \ldots, n\) are positive periodic functions with period \(\omega > 0\) and \(\int_0^\omega r_i(t)dt > 0, i = 1, 2\). The assumption of periodicity of the parameters \(a_{ij}(t)\) is a way of incorporating the periodicity of the environment (e.g., seasonal effects of weather condition, food supplies, temperature, mating habits, harvesting etc.). The growth functions \(r_i(t)\) are unnecessary to remain positive, since the environment fluctuates randomly, in bad condition, \(r_i(t)\) may be negative.

Obviously, system (1.3) is a special case of system (1.4). Unlike system (1.3), \(\sigma_i(t) \neq \tau_i(t)\). Thus the existing argument in Yang etc. [8] for obtaining bounds of solutions to the operator equation \(Lx = \lambda Nx\) are not applicable to our case and some new arguments are employed for the first time.

The initial conditions for system (1.4) take the form of
\[
\begin{align*}
\dot{y}_i(s) &= \psi_i(s), \quad \tau = \max \sup_{t \in [0, \omega]} \{\tau_i(t), \sigma_i(t), i = 1, 2\} \\
\dot{\psi}_i(t) &\in C([-\tau, 0], R_+), \psi_i(0) > 0, \ i = 1, 2.
\end{align*}
\]
(1.5)

Throughout this paper, we will use the following notation:
\[
\mathcal{f} = \frac{1}{\omega} \int_0^\omega f(t)dt, \quad f^s = \min_{t \in [0, \omega]} f(t), \quad f^s = \max_{t \in [0, \omega]} f(t),
\]
where \(f(t)\) is an \(\omega\)-periodic function.

### 2. Existence of Positive Periodic Solutions

In order to obtain the existence of positive periodic solutions of (1.4), for convenience, we shall summarize in the following a few concepts and results from [10] that will be prerequisite for this section.
Let $X, Y$ be real Banach spaces, let $L : \text{Dom}L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping $L$ is called a Fredholm mapping of index zero if $\dim \ker L = \text{codim} \text{Im} L < +\infty$ and $\text{Im} L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$, and $Q : Y \to Y$ such that $\text{Im} P = \ker L$, $\ker Q = \text{Im} L = \text{Im}(I - Q)$. It follows that $L|\text{dom}L \cap \ker P : (I - P)X \to \text{Im} L$ is invertible. We denote the inverse of that map by $K_p$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\text{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \text{Im} Q \to \ker L$.

**Lemma 2.1.** [10] Let $\Omega \subset X$ be an open bounded set. Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\overline{\Omega}$. Assume

(a) for each $\lambda \in (0, 1)$, $x \in \partial \Omega \cap \text{Dom} L$, $Lx \neq \lambda Nx$;
(b) for each $x \in \partial \Omega \cap \ker L$, $QN x \neq 0$;
(c) deg\{JQN, $\Omega \cap \ker L$, 0\} $\neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \text{Dom} L$.

**Lemma 2.2.** Assume that $f, g$ are continuous nonnegative functions defined on the interval $[a, b]$. Then there exists $\xi \in [a, b]$ such that
\[
\int_a^b f(t)g(t)dt = f(\xi)\int_a^b g(t)dt.
\]

**Theorem 2.3.** Let
\[
\begin{align*}
b_{11}' &= \min_{t \in [0, \omega]} \frac{a_{11}(t)}{1 - \tau_1(t)}, & b_{12}' &= \max_{t \in [0, \omega]} \frac{a_{12}(t)}{1 - \tau_2(t)}, \\
b_{22}' &= \min_{t \in [0, \omega]} \frac{a_{22}(t)}{1 - \sigma_2(t)}, & b_{21}' &= \max_{t \in [0, \omega]} \frac{a_{21}(t)}{1 - \sigma_1(t)}.
\end{align*}
\]
If $b_{11}'b_{22}' > b_{21}'b_{12}'$ and $\tau_i(t) < 1$, $\sigma_i(t) < 1$, ($i = 1, 2$) hold, then system (1.4) has at least one positive $\omega$-periodic solution.

**Proof.** We make the change of variables
\[
x_i(t) = \ln y_i(t), \quad i = 1, 2.
\]
Then (1.4) is rewritten as
\[
\begin{align*}
\dot{x}_1(t) &= r_1(t) - a_{11}(t) \exp\{x_1(t - \tau_1(t))\} + a_{12}(t) \exp\{x_2(t - \tau_2(t))\}, \\
\dot{x}_2(t) &= r_2(t) + a_{21}(t) \exp\{x_1(t - \sigma_1(t))\} - a_{22}(t) \exp\{x_2(t - \sigma_2(t))\}.
\end{align*}
\]
Take
\[ X = Y = \{ x = (x_1, x_2)^T \in C(\mathbb{R}, \mathbb{R}^2) : x(t + \omega) = x(t) \} \]
and define
\[ \| x \| = \max_{t \in [0, \omega]} |x_1(t)| + \max_{t \in [0, \omega]} |x_2(t)|, \quad x = (x_1, x_2)^T \in X \text{ or } Y \]
(here \( | \cdot | \) denotes the Euclidean norm). Then \( X \) and \( Y \) are Banach spaces with the norm \( \| \cdot \| \). For any \( x = (x_1, x_2)^T \in X \), because of the periodicity, we can easily check that
\[ r_1(t) - a_{11}(t) \exp\{x_1(t - \tau_1(t))\} + a_{12}(t) \exp\{x_2(t - \tau_2(t))\} := \Delta_1(x, t) \in C(\mathbb{R}, \mathbb{R}), \]
\[ r_2(t) + a_{21}(t) \exp\{x_1(t - \sigma_1(t))\} - a_{22}(t) \exp\{x_2(t - \sigma_2(t))\} := \Delta_2(x, t) \in C(\mathbb{R}, \mathbb{R}), \]
are \( \omega \)-periodic. Set
\[ L : \text{Dom}L \cap X, \quad L(x_1(t), x_2(t)) = \left( \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt} \right), \]
where \( \text{Dom}L = \{(x_1(t), x_2(t)) \in C^1(\mathbb{R}, \mathbb{R}^2)\} \) and \( N : X \to X, \)
\[ N\left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} \Delta_1(x, t) \\ \Delta_2(x, t) \end{array} \right). \]
Define
\[ P\left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = Q\left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\omega} \int_0^\omega x_1(t)dt \\ 1 \end{array} \right), \quad \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in X = Y. \]
It is not difficult to show that
\[ \text{Ker}L = \{ x \in X, \ x = C_0, \ C_0 \in \mathbb{R}^2 \}, \]
\[ \text{Im}L = \left\{ y \in Y, \ \int_0^\omega y(t)dt = 0 \right\} \text{ is closed in } Y, \]
\[ \dim \text{Ker}L = \text{codim} \text{Im}L = 2, \]
and \( P \) and \( Q \) are continuous projections such that
\[ \text{Im}P = \text{Ker}L, \quad \text{Ker}Q = \text{Im}L = \text{Im}(I - Q). \]
It follows that \( L \) is a Fredholm mapping of index zero. Furthermore, the inverse \( K_p \) of \( L_p \) exists and has the form \( K_p : \text{Im}L \to \text{Dom}L \cap \text{Ker}P \)
\[ K_p(y) = \int_0^t y(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t y(s)dsdt. \]
Then \( QN : X \to Y \) and \( K_p(I - Q)N : X \to X \) read

\[
QN x = \begin{pmatrix}
\frac{1}{\omega} \int_0^\omega \Delta_1(x, t) dt \\
\frac{1}{\omega} \int_0^\omega \Delta_2(x, t) dt
\end{pmatrix}
\]

\[
K_p(I - Q)N x = \int_0^t Nx(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t Nx(s) ds dt - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega Nx(s) ds.
\]

Clearly, \( QN \) and \( K_p(I - Q)N \) are continuous. By using the Arzelà–Ascoli Theorem, it is not difficult to prove that \( K_p(I - Q)N(\Omega) \) is compact for any open bounded set \( \Omega \subset X \). Moreover, \( QN(\Omega) \) is bounded. Therefore, \( N \) is \( L \)-compact on \( \Omega \) for any open bounded set \( \Omega \subset X \).

Now we reach the position to search for an appropriate open bounded subset \( \Omega \) for the application of Lemma 2.1.

Corresponding to the operator equation \( Lx = \lambda Nx, \lambda \in (0, 1) \), we have

\[
\begin{cases}
\dot{x}_1(t) = \lambda [r_1(t) - a_{11}(t) \exp\{x_1(t - \tau_1(t))\} + a_{12}(t) \exp\{x_2(t - \tau_2(t))\}], \\
\dot{x}_2(t) = \lambda [r_2(t) + a_{21}(t) \exp\{x_1(t - \sigma_1(t))\} - a_{22}(t) \exp\{x_2(t - \sigma_2(t))\}].
\end{cases}
\]

(2.3)

Suppose \( x = (x_1, x_2)^T \in X \) is a solution of (2.3) for a certain \( \lambda \in (0, 1) \). Integrating (2.3) over the interval \([0, \omega]\), we obtain

\[
\begin{cases}
\int_0^\omega [r_1(t) - a_{11}(t) \exp\{x_1(t - \tau_1(t))\} + a_{12}(t) \exp\{x_2(t - \tau_2(t))\}] dt = 0, \\
\int_0^\omega [r_2(t) + a_{21}(t) \exp\{x_1(t - \sigma_1(t))\} - a_{22}(t) \exp\{x_2(t - \sigma_2(t))\}] dt = 0.
\end{cases}
\]

Hence,

\[
\tilde{\tau}_1 \omega + \int_0^\omega a_{12}(t) \exp\{x_2(t - \tau_2(t))\} dt = \int_0^\omega a_{11}(t) \exp\{x_1(t - \tau_1(t))\} dt,
\]

(2.4)

\[
\tilde{\tau}_2 \omega + \int_0^\omega a_{21}(t) \exp\{x_1(t - \sigma_1(t))\} dt = \int_0^\omega a_{22}(t) \exp\{x_2(t - \sigma_2(t))\} dt.
\]

(2.5)

Let \( s = t - \tau_1(t) \). We note that \( \tilde{\tau}_1(t) < 1 \), which implies \( \frac{ds}{dt} = 1 - \tilde{\tau}_1(t) > 0 \), \( ds = (1 - \tilde{\tau}_1(t)) dt \). Therefore, the function \( s = t - \tau_1(t) \) has the continuous inverse function \( t = \tau_1^*(s) \), \( s \in [-\tau_1(0), \omega - \tau_1(\omega)] \). So we have

\[
\int_0^\omega a_{11}(t) \exp\{x_1(t - \tau_1(t))\} dt = \int_{-\tau_1(0)}^{\omega - \tau_1(\omega)} \frac{a_{11}(\tau_1^*(s))}{1 - \tilde{\tau}_1(\tau_1^*(s))} \exp\{x_1(s)\} ds.
\]
By Lemma 2.2, there exists \( \xi \in [\tau_1(0), \omega - \tau_1(\omega)] = [-\tau_1(0), \omega - \tau_1(0)] \) such that

\[
\int_{0}^{\omega} a_{11}(t) \exp\{x_1(t - \tau_1(t))\} \, dt = \frac{a_{11}(\tau_1(\xi))}{1 - \tau_1(\tau_1^{*}(\xi))} \int_{-\tau_1(0)}^{\omega} \exp\{x_1(s)\} \, ds
\]

\[
= \frac{a_{11}(\eta_1)}{1 - \tau_1(\eta_1)} \int_{0}^{\omega} \exp\{x_1(t)\} \, dt, \quad (\eta_1 = \tau_1^{*}(\xi)).
\]

For the equations (2.3) and (2.4), similar to the above discussion, there exist \( \eta_i \in [0, \omega], (i = 1, 2, 3, 4) \) such that

\[
\bar{r}_1 \omega + \frac{a_{12}(\eta_2)}{1 - \tau_1(\eta_2)} \int_{0}^{\omega} \exp\{x_2(t)\} \, dt = \frac{a_{11}(\eta_1)}{1 - \tau_1(\eta_1)} \int_{0}^{\omega} \exp\{x_1(t)\} \, dt, \quad (2.6)
\]

\[
\bar{r}_2 \omega + \frac{a_{21}(\eta_3)}{1 - \bar{\sigma}_1(\eta_3)} \int_{0}^{\omega} \exp\{x_1(t)\} \, dt = \frac{a_{22}(\eta_4)}{1 - \bar{\sigma}_2(\eta_4)} \int_{0}^{\omega} \exp\{x_2(t)\} \, dt. \quad (2.7)
\]

We denote

\[
b_{11} = \frac{a_{11}(\eta_1)}{1 - \tau_1(\eta_1)}, \quad b_{12} = \frac{a_{12}(\eta_2)}{1 - \tau_1(\eta_2)}, \quad b_{21} = \frac{a_{21}(\eta_3)}{1 - \bar{\sigma}_1(\eta_3)}, \quad b_{22} = \frac{a_{22}(\eta_4)}{1 - \bar{\sigma}_2(\eta_4)}.\]

It follows from (2.6)–(2.7) that

\[
\bar{r}_1 \omega + b_{12} \int_{0}^{\omega} \exp\{x_2(t)\} \, dt = b_{11} \int_{0}^{\omega} \exp\{x_1(t)\} \, dt, \quad (2.8)
\]

\[
\bar{r}_2 \omega + b_{21} \int_{0}^{\omega} \exp\{x_1(t)\} \, dt = b_{22} \int_{0}^{\omega} \exp\{x_2(t)\} \, dt. \quad (2.9)
\]

Under the assumption of Theorem 2.3, it is not difficult to derive that

\[
\int_{0}^{\omega} \exp\{x_1(t)\} \, dt = \frac{b_{12} \bar{r}_2 + b_{22} \bar{r}_1}{b_{11} b_{22} - b_{21} b_{12}} \omega \leq \frac{b_{12}^{\mu} \bar{r}_2 + b_{22}^{\mu} \bar{r}_1}{b_{11}^{\mu} b_{22} - b_{21}^{\mu} b_{12}^{\mu}} \omega \quad (2.10)
\]

and

\[
\int_{0}^{\omega} \exp\{x_2(t)\} \, dt = \frac{b_{21} \bar{r}_1 + b_{11} \bar{r}_2}{b_{11} b_{22} - b_{21} b_{12}} \omega \leq \frac{b_{21}^{\mu} \bar{r}_1 + b_{11}^{\mu} \bar{r}_2}{b_{11}^{\mu} b_{22} - b_{21}^{\mu} b_{12}^{\mu}} \omega. \quad (2.11)
\]

Therefore, there exists \( t_i \in [0, \omega], i = 1, 2 \) such that

\[
x_1(t_1) \leq \ln \frac{b_{12}^{\mu} \bar{r}_2 + b_{22}^{\mu} \bar{r}_1}{b_{11}^{\mu} b_{22} - b_{21}^{\mu} b_{12}^{\mu}}, \quad x_2(t_2) \leq \ln \frac{b_{21}^{\mu} \bar{r}_1 + b_{11}^{\mu} \bar{r}_2}{b_{11}^{\mu} b_{22} - b_{21}^{\mu} b_{12}^{\mu}}.
\]

Obviously, there exists positive constants \( M_i > 0, i = 1, 2 \) such that

\[
|x_i(t_i)| \leq M_i, \quad i = 1, 2. \quad (2.12)
\]
On the other hand, it follows from (2.3), (2.4) and (2.10) that
\[
\int_0^\omega |\dot{x}_1(t)| dt = \lambda \int_0^\omega \left| r_1(t) - a_{11}(t) \exp\{x_1(t - \tau_1(t))\} + a_{12}(t) \exp\{x_2(t - \tau_2(t))\}\right| dt,
\]
\[
< \bar{r}_1 \omega + \int_0^\omega a_{12}(t) \exp\{x_2(t - \tau_2(t))\} dt + \int_0^\omega a_{11}(t) \exp\{x_1(t - \tau_1(t))\} dt
\]
\[
= 2 \int_0^\omega a_{11}(t) \exp\{x_1(t - \tau_1(t))\} dt
\]
\[
\leq 2b_{11}^\mu \int_0^\omega \exp\{x_1(t)\} dt \leq 2b_{11}^\mu \frac{b_{12}^\mu \bar{r}_1 + b_{11}^\mu \bar{r}_2}{b_{11}^\mu b_{22}^\mu - b_{21}^\mu b_{12}^\mu} \omega := H_1.
\]
Similarly, it follows from (2.3), (2.5) and (2.11) that
\[
\int_0^\omega |\dot{x}_2(t)| dt < 2b_{11}^\mu \frac{b_{12}^\mu \bar{r}_1 + b_{11}^\mu \bar{r}_2}{b_{11}^\mu b_{22}^\mu - b_{21}^\mu b_{12}^\mu} \omega := H_2.
\]
It follows from (2.12)–(2.14) that
\[
|x_i(t)| \leq |x_i(t_i)| + \int_0^\omega |\dot{x}_i(t)| dt < M_i + H_i.
\]
Clearly, \(M_i, H_i, i = 1, 2\) are independent of the choice of \(\lambda\). We note that
\[
\bar{a}_{11} = \frac{1}{\omega} \int_0^\omega a_{11}(t) dt = \frac{1}{\omega} \int_{\omega-\tau_1(0)}^{\omega} \frac{a_{11}(\tau_1^+(s))}{1 - \tau_1(\tau_1^+(s))} ds
\]
\[
= \frac{a_{11}(\tau_1^+(s_1))}{1 - \tau_1(\tau_1^+(s_1))}, \quad s_1 \in [-\tau_1(0), \omega - \tau_1(0)],
\]
where \(t = \tau_1^+(s)\) is the inverse function of \(s = t - \tau_1(t)\). Similar to the discussion of (2.15),
\[
\bar{a}_{12} = \frac{a_{12}(\tau_2^+(s_2))}{1 - \tau_2(\tau_2^+(s_2))}, \quad \bar{a}_{21} = \frac{a_{21}(\sigma_1^+(s_3))}{1 - \sigma_1(\sigma_1^+(s_3))}, \quad \bar{a}_{22} = \frac{a_{22}(\sigma_2^+(s_4))}{1 - \sigma_2(\sigma_2^+(s_4))}.
\]
Thus, under the assumptions in Theorem 2.3, it follows from (2.15)–(2.16) that
\[
\bar{a}_{11}\bar{a}_{22} - \bar{a}_{12}\bar{a}_{21} > b_{11}^\mu b_{22}^\mu - b_{12}^\mu b_{21}^\mu > 0.
\]
Now it is easy to show that the system of algebraic equations
\[
\bar{r}_i - \bar{a}_{ii}v_i + \bar{a}_{ij}v_j = 0, \quad i, j = 1, 2.
\]
Yonghui Xia has a unique solution \((v^*_1, v^*_2)^T \in \mathbb{R}^2_+\) with \(x^*_i > 0\). Take \(H = \max\{M_i + H_i + C, M_i + H_i + C\}\), where \(C > 0\) is taken sufficiently large such that \(\|\ln v^*_1, \ln v^*_2\|^T < C\). Define \(\Omega = \{x(t) \in X : \|x\| < H\}\). It is clear that \(\Omega\) satisfies condition (a) of Lemma 2.1. Let \(x \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap \mathbb{R}^2_+\), \(x\) be a constant vector in \(\mathbb{R}^2_+\) with \(\|x\| = H\). Then

\[QN x = \begin{bmatrix} \bar{r}_i - \bar{a}_{ii} \exp\{x_i\} + \bar{a}_{ij} \exp\{x_j\} = 0 \end{bmatrix}_{2 \times 1} \neq 0.\]

Furthermore, let \(J : \text{Im} Q \rightarrow \text{Ker} L\). In view of assumptions in Theorem 2.3 and (2.15)–(2.17), it is easy to see that

\[\deg \{JQ N, \Omega \cap \text{Ker} L, 0\} = \text{sgn det} \begin{bmatrix} -\bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & -\bar{a}_{22} \end{bmatrix} \geq \text{sgn}(b_{11}^\mu b_{22} - b_{12}^\mu b_{21}^\mu) \neq 0,\]

where \(\deg(\cdot)\) is the Brouwer degree and the \(J\) is the identity mapping since \(\text{Im} Q = \text{Ker} L\).

By now, we have shown that \(\Omega\) verifies all requirements of Lemma 2.1. Then it follows that \(Lx = Nx\) has at least one solution in \(\text{Dom} L \cap \Omega\). By (2.1), we derive that (1.4) has at least one positive \(\omega\)-periodic solution. The proof is completed. \(\blacksquare\)

**Remark 2.4.** Our results and the method used in the proof are much different from those in the known literature. We remark that we generalize system (1.3) in [8] to system (1.4), where \(\tau_i(t) \neq \sigma_i(t)\), i.e., all the delays are different. It is in this aspect that we generalize the results in [8].

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