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Exponential Expansiveness and Variational Integral Equations

Bogdan Sasu

Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara, Parvan Blv. No. 4,
300223-Timişoara, Romania
E-mail: bsasu@math.uvt.ro

Abstract

We associate with a linear skew-product flow $\pi = (\Phi, \sigma)$ a variational integral equation and we characterize the exponential expansiveness of π in terms of the solvability of the associated equation. We prove that a linear skew-product flow on $X \times \Theta$ is uniformly exponentially expansive if and only if the pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is uniformly exactly admissible for π, where $V(\mathbb{R}_+, X)$ denotes one of the spaces $C_0(\mathbb{R}_+, X)$ or $C_b(\mathbb{R}_+, X)$.

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1. Introduction

Let Θ be a metric space and let σ be a flow on Θ. We consider the homogeneous equation

$$(A) \qquad \dot{x}(t) = A(\sigma(\theta, t))x(t), \quad t \geq 0$$

and the nonhomogeneous equation

$$(A, v) \qquad \dot{x}(t) = A(\sigma(\theta, t))x(t) + v(t), \quad t \geq 0$$

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where $A(\theta)$ are linear operators on a Banach space X. If there exists a linear skew-product flow $\pi = (\Phi, \sigma)$ associated with (A), then for every $\theta \in \Theta$ the mild solution of (A, v) is given by:

$$f(t) = \Phi(\sigma(\theta, s), t-s)f(s) + \int_s^t \Phi(\sigma(\theta, \tau), t-\tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0.$$

The existence of the linear skew-product flow associated with the variational differential equation (A) is conditioned by specific conditions regarding the family of linear operators $\{A(\theta)\}_{\theta\in\Theta}$. Thus in the last few years a special attention was devoted directly to the general case of linear skew-product flows, providing characterizations for the asymptotic properties of cocycles over flows and consequently for the solutions of variational differential equations. In this sense we refer the reader to [1, 4, 5, 10, 11] and the references therein.

In recent years, beside stability and dichotomy (see [1–4, 10]) a special attention was devoted to the study of expansiveness of evolution equations (see [5–9, 11]). The autonomous case was treated in [8], the case of evolution families on the half-line was considered in [6, 7] and the variational case was studied in [5, 9] and [11]. In the variational case, we may associate with a linear skew-product flow $\pi = (\Phi, \sigma)$ at every point $\theta \in \Theta$ the integral equation

$$(E_\theta) \qquad f(t) = \Phi(\sigma(\theta, s), t-s)f(s) + \int_s^t \Phi(\sigma(\theta, \tau), t-\tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0$$

with $v \in I(\mathbb{R}_+, X)$ — the input space and $f \in O(\mathbb{R}_+, X)$ — the output space. Then, the exponential expansiveness can be expressed in terms of the admissibility of the pair $(O(\mathbb{R}_+, X), I(\mathbb{R}_+, X))$ for π, which means that for every $(\theta, v) \in \Theta \times I(\mathbb{R}_+, X)$, the equation (E_θ) has a unique solution $f \in O(\mathbb{R}_+, X)$. In [5], we proved that the uniform complete admissibility of the pair $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ is equivalent with the uniform exponential expansiveness of π. The main result in [9] states that the uniform complete admissibility of the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is a sufficient condition for the uniform exponential expansiveness of π.

The aim of the present paper is to obtain new characterizations for uniform exponential expansiveness of linear skew-product flows. We will introduce a general concept of admissibility, which optimizes and generalizes the admissibility concepts in [5] and [9]. We will consider $C_c(\mathbb{R}_+, X)$ as input space and using constructive steps we will establish the connections between the exact admissibility of the pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ and the expansiveness of a linear skew-product flow. Our main result improves and extends the expansiveness results in [5–9].

2. Main Results

Let X be a real or a complex Banach space. The norm on X and on $\mathcal{L}(X)$ — the Banach algebra of all bounded linear operators on X will be denoted by $||\cdot||$. Let I be the identity operator on X.

Let (Θ, d) be a metric space and $\mathcal{E} := X \times \Theta$.

Definition 2.1. A mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ if $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$ and $\sigma(\theta, t+s) = \sigma(\sigma(\theta, t), s)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}^2$.

Definition 2.2. A pair $\pi = (\Phi, \sigma)$ is called *linear skew-product flow* on $\mathcal{E}$ if σ is a flow on Θ and $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ has the following properties:

(i) $\Phi(\theta, 0) = I$, for all $\theta \in \Theta$ and $\Phi(\theta, t+s) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$ (*the cocycle identity*), for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$;

(ii) there are $M, \omega > 0$ such that $||\Phi(\theta, t)|| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iii) for every $x \in X$ the mapping $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous.

Definition 2.3. A linear skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly exponentially expansive* if for every $(\theta, t) \in \Theta \times \mathbb{R}_+$ the operator $\Phi(\theta, t)$ is invertible and there are two constants $K, \nu > 0$ such that $||\Phi(\theta, t)x|| \geq Ke^{\nu t}||x||$, for all $(x, \theta, t) \in \mathcal{E} \times \mathbb{R}_+$.

Let $C_b(\mathbb{R}_+, X)$ be the linear space of all continuous functions $u : \mathbb{R}_+ \to X$ with $\sup\limits_{t \geq 0} ||u(t)|| < \infty$ and let $C_0(\mathbb{R}_+, X) := \{u \in C_b(\mathbb{R}_+, X) : \lim\limits_{t\to\infty} u(t) = 0\}$. With respect to the norm

$$|||u||| := \sup_{t \geq 0} ||u(t)||$$

$C_b(\mathbb{R}_+, X)$ and $C_0(\mathbb{R}_+, X)$ are Banach spaces.

For every $p \in [1, \infty)$, we consider $L^p(\mathbb{R}_+, X)$ the space of all Bochner measurable functions $f : \mathbb{R}_+ \to X$ with $\displaystyle\int_0^\infty ||f(\tau)||^p \, d\tau < \infty$, which is a Banach space with respect to the norm

$$||f||_p := \left(\int_0^\infty ||f(\tau)||^p \, d\tau \right)^{1/p}.$$

We denote by $C_c(\mathbb{R}_+, X)$ the linear space of all continuous functions $v : \mathbb{R}_+ \to X$ with compact support.

Let $V(\mathbb{R}_+, X) \in \{C_b(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X)\}$.

Definition 2.4. The pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is said to be *exactly admissible* for $\pi = (\Phi, \sigma)$ if for every $\theta \in \Theta$ and every $v \in C_c(\mathbb{R}_+, X)$ there exists a unique function $f \in V(\mathbb{R}_+, X)$ such that the pair (f, v) satisfies the integral equation

$$(E_\theta) \qquad f(t) = \Phi(\sigma(\theta, s), t-s)f(s) + \int_s^t \Phi(\sigma(\theta, \tau), t-\tau)v(\tau) \, d\tau, \quad \forall t \geq s \geq 0.$$

Theorem 2.5. If the pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is exactly admissible for $\pi = (\Phi, \sigma)$, then $\Phi(\theta, t)$ is invertible, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

Proof. Let $(\theta, r) \in \Theta \times (0, \infty)$.

Injectivity. Let $x \in \mathrm{Ker}\ \Phi(\theta, r)$. We define $f : \mathbb{R}_+ \to X, f(t) = \Phi(\theta, t)x$. Then $f \in V(\mathbb{R}_+, X)$ and an easy computation shows that the pair $(f, 0)$ satisfies the equation (E_θ). From the exact admissibility, it follows that $f \equiv 0$. In particular, $x = f(0) = 0$, so $\Phi(\theta, r)$ is injective.

Surjectivity. Let $x \in X$ and let $\alpha : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with compact support such that $\mathrm{supp}\,\alpha \subset (r, r+1)$ and $\displaystyle\int_r^{r+1} \alpha(\tau)\, d\tau = 1$. We consider

$$v : \mathbb{R}_+ \to X, \quad v(t) = -\alpha(t)\Phi(\sigma(\theta, r), t - r)x$$

and then $v \in C_c(\mathbb{R}_+, X)$. Let $f \in V(\mathbb{R}_+, X)$ be such that the pair (f, v) satisfies the equation (E_θ). This implies that

$$f(t) = \Phi(\sigma(\theta, s), t - s)f(s) - \left(\int_s^t \alpha(\tau)\, d\tau\right) \Phi(\sigma(\theta, r), t - r)x \tag{2.1}$$

for all $t \geq s \geq 0$. From (2.1) we have that $f(r) = \Phi(\theta, r)f(0)$. We define

$$g : \mathbb{R}_+ \to X, \quad g(t) = f(t + r) - \left(\int_{t+r}^\infty \alpha(\tau)\, d\tau\right) \Phi(\sigma(\theta, r), t)x.$$

Then $g \in V(\mathbb{R}_+, X)$ and using (2.1) we deduce that the pair $(g, 0)$ satisfies the equation $(E_{\sigma(\theta,r)})$. From the exact admissibility, we deduce that $g = 0$. In particular, $g(0) = 0$, so

$$f(r) = \left(\int_r^\infty \alpha(\tau)\, d\tau\right) x = x.$$

It follows that $x = f(r) = \Phi(\theta, r)f(0) \in \mathrm{Im}\ \Phi(\theta, r)$, so $\Phi(\theta, r)$ is surjective and the proof is complete. ■

Remark 2.6. If the pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is exactly admissible for $\pi = (\Phi, \sigma)$, then for every $\theta \in \Theta$, we may consider the linear operator $L_\theta : C_c(\mathbb{R}_+, X) \to V(\mathbb{R}_+, X)$, $L_\theta(v) = f$, where $f \in V(\mathbb{R}_+, X)$ has the property that the pair (f, v) satisfies the equation (E_θ).

Definition 2.7. The pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is said to be *uniformly exactly admissible* for π if there are $p \in (1, \infty)$ and $\lambda > 0$ such that $|||L_\theta(v)||| \leq \lambda\, ||v||_p$, for all $(\theta, v) \in \Theta \times C_c(\mathbb{R}_+, X)$.

Theorem 2.8. If the pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is uniformly exactly admissible for $\pi = (\Phi, \sigma)$, then there is $\delta > 0$ such that

$$||\Phi(\theta, t)x|| \geq \delta\, ||x||, \quad \forall (x, \theta) \in \mathcal{E}, \forall t \geq 0.$$

Proof. Let $\lambda > 0$ and $p \in (1, \infty)$ be given by Definition 2.7. Let $M, \omega > 0$ be given by Definition 2.2.

Let $(x,\theta)\in\mathcal{E}$ and let $t_0\geq 0$. We consider a function $\alpha:\mathbb{R}_+\to[0,2]$ with compact support such that $\operatorname{supp}\alpha\subset(t_0,t_0+1)$ and $\displaystyle\int_{t_0}^{t_0+1}\alpha(\tau)\,d\tau=1$. We define

$$v:\mathbb{R}_+\to X,\quad v(t)=-\alpha(t)\Phi(\theta,t)x$$

$$f:\mathbb{R}_+\to X,\quad f(t)=\int_t^\infty\alpha(\tau)\,d\tau\ \Phi(\theta,t)x.$$

We have that $f,v\in C_c(\mathbb{R}_+,X)$ and an easy computation shows that the pair (f,v) satisfies the equation (E_θ). This shows that $f=L_\theta(v)$, so $|||f|||\leq\lambda\,||v||_p$. In particular, we deduce that

$$||x||=||f(0)||\leq|||f|||\leq\lambda\,||v||_p. \tag{2.2}$$

Since $||\Phi(\theta,t)x||\leq Me^\omega||\Phi(\theta,t_0)x||$, for all $t\in(t_0,t_0+1)$, we have that

$$||v(t)||=\alpha(t)\,||\Phi(\theta,t)x||\leq\alpha(t)Me^\omega||\Phi(\theta,t_0)x||,\quad\forall t\geq 0$$

which implies that

$$||v||_p\leq 2Me^\omega||\Phi(\theta,t_0)x||. \tag{2.3}$$

Setting $\delta=1/(2\lambda Me^\omega)$, from (2.2) and (2.3) it follows that $||\Phi(\theta,t_0)\,x||\geq\delta\,||x||$. Taking into account that δ does not depend on θ,t_0 or x the proof is complete. ■

The main result of this paper is the following.

Theorem 2.9. A linear skew-product flow $\pi=(\Phi,\sigma)$ is uniformly exponentially expansive if and only if the pair $(V(\mathbb{R}_+,X),C_c(\mathbb{R}_+,X))$ is uniformly exactly admissible for π.

Proof. Necessity. Let $K,\nu>0$ be such that $||\Phi(\theta,t)x||\geq Ke^{\nu t}||x||$, for all $(x,\theta)\in\mathcal{E}$ and all $t\geq 0$. Let $\theta\in\Theta$ and $v\in C_c(\mathbb{R}_+,X)$. We define

$$f:\mathbb{R}_+\to X,\quad f(t)=-\int_t^\infty\Phi(\sigma(\theta,t),\tau-t)^{-1}v(\tau)\,d\tau.$$

Since $v\in C_c(\mathbb{R}_+,X)$ we have that $f\in C_c(\mathbb{R}_+,X)$. In particular, $f\in V(\mathbb{R}_+,X)$. An easy computation shows that the pair (f,v) satisfies the equation (E_θ).

Let $\tilde{f}\in V(\mathbb{R}_+,X)$ be such that the pair $(\tilde{f},v)$ satisfies the equation (E_θ). Setting $g:=\tilde{f}-f$ we have that $g(t)=\Phi(\sigma(\theta,s),t-s)g(s)$, for all $t\geq s\geq 0$.

Let $s\geq 0$. From

$$||g(s)||\leq\frac{1}{K}\,e^{-\nu(t-s)}||g(t)||\leq\frac{1}{K}\,e^{-\nu(t-s)}|||g|||,\quad\forall t\geq s$$

it follows that $g(s)=0$. This shows that $g=0$, so f is uniquely determined. Then, we have that the pair $(V(\mathbb{R}_+,X),C_c(\mathbb{R}_+,X))$ is exactly admissible for π.

Let $p \in (1,\infty)$. Setting $p' = p/(p-1)$ and using Hölder's inequality we obtain that

$$\begin{aligned} ||f(t)|| &\leq \frac{1}{K} \int_t^{\infty} e^{-\nu(\tau-t)} ||v(\tau)||\, d\tau \\ &\leq \frac{1}{K} \left(\int_0^{\infty} e^{-\nu p' s}\, ds \right)^{1/p'} ||v||_p = \frac{1}{K(\nu p')^{1/p'}}\, ||v||_p, \quad \forall t \geq 0. \end{aligned}$$

This implies that

$$|||L_\theta(v)||| \leq \frac{1}{K(\nu p')^{1/p'}}\, ||v||_p, \quad \forall v \in C_c(\mathbb{R}_+, X), \forall \theta \in \Theta$$

so the pair $(V(\mathbb{R}_+, X), C_c(\mathbb{R}_+, X))$ is uniformly exactly admissible for π.

Sufficiency. From Theorem 2.5 we have that $\Phi(\theta,t)$ is invertible, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$. According to Theorem 2.8 there is $\delta > 0$ such that

$$||\Phi(\theta,t)x|| \geq \delta\, ||x||, \quad \forall (x,\theta) \in \mathcal{E}, \forall t \geq 0. \tag{2.4}$$

Let $\lambda > 0$, $p \in (1,\infty)$ be given by Definition 2.7. We set $k := \left(\int_0^{\infty} \frac{1}{(t+1)^p}\, dt \right)^{1/p}$ and let $h > 0$ be such that

$$\int_0^h \frac{1}{s+1}\, ds \geq \frac{\lambda k e}{\delta}. \tag{2.5}$$

Let $\alpha : \mathbb{R}_+ \to [0,1]$ be a continuous function with $\alpha(t) = 1$, for all $t \in [0,h]$ and $\alpha(t) = 0$, for all $t \geq h+1$.

Let $(x,\theta) \in \mathcal{E}$. We consider the functions

$$\begin{aligned} v &: \mathbb{R}_+ \to X, \quad v(t) = -\frac{\alpha(t)}{t+1}\, \Phi(\theta,t)x \\ f &: \mathbb{R}_+ \to X, \quad f(t) = \int_t^{\infty} \frac{\alpha(\tau)}{\tau+1}\, d\tau\, \Phi(\theta,t)x. \end{aligned}$$

We have that $f \in V(\mathbb{R}_+, X), v \in C_c(\mathbb{R}_+, X)$ and an easy computation shows that the pair (f,v) satisfies the equation (E_θ). It follows that $f = L_\theta(v)$, so

$$|||f||| \leq \lambda\, ||v||_p. \tag{2.6}$$

Using (2.4) we have that

$$||v(t)|| \leq \frac{\chi_{[0,h+1)}(t)}{t+1}\, ||\Phi(\theta,t)x|| \leq \frac{1}{\delta(t+1)}\, ||\Phi(\theta,h+1)x||, \quad \forall t \geq 0$$

which implies that $||v||_p \leq (k/\delta)\ ||\Phi(\theta, h+1)x||$. Then, from (2.6) we deduce that

$$\left(\int_0^h \frac{1}{\tau+1}\,d\tau\right)\ ||x|| \leq \left(\int_0^\infty \frac{\alpha(\tau)}{\tau+1}\,d\tau\right)\ ||x|| = ||f(0)|| \\ \leq |||f||| \leq \frac{\lambda k}{\delta}\ ||\Phi(\theta, h+1)x||. \qquad (2.7)$$

From (2.5) and (2.7) we obtain that $||\Phi(\theta, h+1)x|| \geq e\ ||x||$. We set $l = h+1$ and note that l does not depend on x or θ. It follows that

$$||\Phi(\theta, l)x|| \geq e\ ||x||, \quad \forall (x, \theta) \in \mathcal{E}. \qquad (2.8)$$

Let $(x, \theta) \in \mathcal{E}$ and $t \geq 0$. Then, there are $j \in \mathbb{N}$ and $r \in [0, l)$ such that $t = jl + r$. Using (2.4) and (2.8) we have that

$$||\Phi(\theta, t)x|| \geq \delta\ ||\Phi(\theta, jl)x|| \geq \delta e^j\ ||x||\ \geq Ke^{\nu t}||x||$$

where $\nu = \dfrac{1}{l}$ and $K = \dfrac{\delta}{e}$. In conclusion, π is uniformly exponentially expansive. ■

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