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Multiplicity Results for a Dirichlet Boundary Value Problem in the Higher Dimensional Case

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Abstract

In this paper, we establish the existence of three weak solutions to a Dirichlet boundary value problem involving the p-Laplacian. The approach is based on variational methods and critical points.

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1. Introduction

In this work, we study the boundary value problem

$$\begin{cases} -\triangle_p u + \lambda f(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a nonempty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a negative continuous function. Let us recall that a weak solution of problem (1.1) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x))dx + \lambda \int_\Omega f(u(x))v(x)dx = 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

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In recent years, many authors have studied multiple solutions from several points of view and with different approaches and we refer to [1–4] and the references therein for more details.

For instance, in [1], using variational methods, the authors ensure the existence of a sequence of arbitrarily small positive solutions for problem

$$\begin{cases} \triangle_p u + \lambda f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

when the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has a suitable oscillating behaviour at zero. Also, in [2] the authors studied problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

by using a multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on f.

In particular, in [3], the author proves multiplicity results for the problem (1.3) which for each $\lambda \in [0, +\infty[$, admits at least three solutions in $W_0^{1,2}([0,1])$ where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

In the present paper, under novel assumptions, we are interested in ensuring the existence of at least three weak solutions for the problem (1.1). Our approach is based on a three critical points theorem proved in [6], recalled below for the reader's convenience (Theorem 1.1), and on a technical lemma that allow us to apply it. Theorem 2.2 which is our main result, under novel assumptions ensures the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.1) admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega)$ are less than q. As a consequence of Theorem 2.2, we obtain Theorem 2.3 which deals with the case $N = 1$, $p = 2$ and it ensures that, for any negative continuous function $f : \mathbb{R} \to \mathbb{R}$, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -u''(x) + \lambda f(u(x)) = 0 & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.4}$$

admits at least three solutions whose norms in $W_0^{1,2}([0,1])$ are less than q.

The aim of the present paper is to extend the main result of [4] to the problem (1.1).

Finally, we here recall for the reader's convenience the three critical points theorem of [6], [5, Proposition 3]:

Theorem 1.1. Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, \infty[\to \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

Proposition 1.2. Let X be a nonempty set and Φ, J two real functions on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that

$$\Phi(x_0) = J(x_0) = 0, \ \Phi(x_1) > r,$$

$$\sup_{x \in \Phi^{-1}(]-\infty, r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x \in \Phi^{-1}(]-\infty, r])} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))).$$

2. Main Results

Here and in the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$|| \, u \, || = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p},$$

and put

$$g(t) = \int_0^t f(\xi) d\xi$$

for each $t \in \mathbb{R}$.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $o < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega.$$

Put

$$k_1 = \frac{1}{r_2 - r_1}\left(\left(r_2^N - r_1^N\right)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/p} c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \tag{2.1}$$

and

$$k_2 = \frac{1}{r_2 - r_1}\left(\frac{r_2^N - r_1^N}{r_1^N}\right)^{1/p} c|\Omega|^{\frac{1}{N}-\frac{1}{p}}, \tag{2.2}$$

where Γ denotes the Gamma function, $c = c(N, p)$ is a positive constant and $|\Omega|$ is the measure of the set Ω.

Our main results fully depend on the following lemma.

Lemma 2.1. Assume that there exist two positive constants α and β with $k_1\alpha > \beta$ such that

$$|\Omega| k_2^p \frac{\min_{t\in[-\beta,\beta]} g(t)}{\beta^p} > \frac{g(\alpha)}{\alpha^p},$$

where k_1 is given in (2.1) and k_2 by (2.2). Then, there exist $r > 0$ and $w \in X$ such that $||w||^p > pr$ and

$$|\Omega| \min g(t) > pr\frac{\int_\Omega g(w(x))dx}{||w||^p},$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.

Proof. We put

$$w(x) = \begin{cases} 0 & \text{for } x \in \Omega \setminus S(x^0, r_2) \\ \dfrac{\alpha}{r_2 - r_1}\left[r_2 - \sqrt{\displaystyle\sum_{i=1}^N (x_i - x_i^0)^2}\right] & \text{for } x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ \alpha & \text{for } x \in S(x^0, r_1) \end{cases}$$

and $r = \dfrac{\beta^p}{pc^p|\Omega|^{\frac{p}{N}-1}}$. It is easy to see that $w \in X$ and, in particular, one has

$$||w||^p = (r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\left(\frac{\alpha}{r_2 - r_1}\right)^p.$$

Hence, taking into account that $k_1\alpha > \beta$, one has

$$pr < ||w||^p.$$

Moreover, owing to our assumptions and since $\displaystyle\int_{S(x^0,r_1)} g(\alpha)dx = r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\alpha)$,

we have

$$|\Omega| \min g(t) > \left(\frac{\beta}{k_2\alpha}\right)^p g(\alpha) = \frac{\frac{\beta^p}{c^p|\Omega|^{\frac{p}{N}-1}}}{\left(r_2^N - r_1^N\right)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\left(\frac{\alpha}{r_2-r_1}\right)^p} r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\alpha)$$

$$= pr\frac{\int_{S(x^0,r_1)} g(\alpha)dx}{||w||^p}$$

$$\geq pr\frac{\int_\Omega g(w(x))dx}{||w||^p},$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$. So, the proof is complete. ■

Now, we state our main result.

Theorem 2.2. Assume that there exist three positive constants α, β and s with $k_1\alpha > \beta$, $s < p$ and a negative constant η such that

(i) $|\Omega|\left(\frac{k_2}{\beta}\right)^p \min_{t\in[-\beta,\beta]} g(t) > \frac{g(\alpha)}{\alpha^p}$,

(ii) $g(t) \geq \eta(1 + |t|^s)$ for each $t \in \mathbb{R}$, where k_1 is given in (2.1) and k_2 by (2.2).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.1) admits at least three solutions in X whose norms are less than q.

Proof. For each $u \in X$, we put

$$\Phi(u) = \frac{||u||^p}{p},$$

$$\Psi(u) = \int_\Omega g(u(x))dx.$$

Of course, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x))dx,$$

$$\Psi'(u)(v) = \int_\Omega f(u(x))v(x)dx.$$

Hence, the weak solutions of (1.1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

Thanks to (ii), for each $\lambda > 0$ one has that

$$\lim_{||u||\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\sup_{x\in\Omega} |u(x)| \leq c|\Omega|^{\frac{1}{N}-\frac{1}{p}}||u||$$

for each $u \in X$, it follows that

$$\inf_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = \inf_{||u||^p\leq pr} \int_\Omega g(u(x))dx \geq |\Omega| \min g(t)$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}]$. Thanks to Lemma 2.1, there exist $r > 0$ and $w \in X$ such that

$$|\Omega| \min g(t) > pr\frac{\int_\Omega g(w(x))dx}{||w||^p},$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}]$. So

$$\inf_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) > r\frac{\Psi(w)}{\Phi(w)},$$

namely

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < \rho < r\frac{(-\Psi(w))}{\Phi(w)}$$

and define $h(\lambda) = \lambda\rho$ for every $\lambda \geq 0$. From Proposition 1.2, with $x_0 = 0$, $x_1 = w$, $J = -\Psi$ we obtain

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 1.1. ■

We now want to point out a simple consequence of Theorem 2.2 in the case where $N = 1$ and $p = 2$.

For simplicity, we fix $\Omega =]0, 1[$ and consider a negative continuous function $f : \mathbb{R} \to \mathbb{R}$. Moreover, put $g(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$.

Taking into account that, in this situation, $c = \frac{1}{2}$, $k_1 = \frac{1}{2}\sqrt{\frac{1}{r_2 - r_1}}$ and $k_2 = \frac{1}{2}\sqrt{\frac{1}{r_1(r_2 - r_1)}}$, we have the following result:

Theorem 2.3. Assume that there exist three positive constants α, β, s with $\frac{1}{2}\sqrt{\frac{1}{r_2 - r_1}}\alpha > \beta$, $s < 2$ and a negative constant η such that

(i′) $\frac{1}{4r_1(r_2 - r_1)}\frac{g(\beta)}{\beta^2} > \frac{g(\alpha)}{\alpha^2}$,

(ii′) $g(t) \geq \eta(1 + |t|^s)$ for each $t \in \mathbb{R}$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.4) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q.

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