Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 1 Number 1 (2006), pp. 103–112

http://www.ripublication.com/adsa.htm

The Asymptotic Behavior of Nonoscillatory Solutions of Some Nonlinear Dynamic Equations on Time Scales

Xiangyun Shi[1], **Xueyong Zhou**[1] **and Weiping Yan**[2]

[1]*Department of Mathematics, Xinyang Normal University, Xinyang 464000, Henan, P.R. China*
Corresponding author: E-mail: xiangyunshi@126.com
[2]*Department of Mathematics, Shanxi University, Taiyuan 030006, Shanxi, P.R. China*

Abstract

In this paper, the asymptotic behavior of nonoscillatory solutions of the nonlinear dynamic equation on time scales

$$\left[r(t)g(y^{\Delta}(t))\right]^{\Delta} - f(t, y(t)) = 0, \quad t \geq t_0$$

is considered under the condition

$$\Big| \int_{t_0}^{\infty} g^{-1}\Big(\frac{m_1}{r(t)}\Big)\Delta t\Big| = \infty \ \text{ for } m_1 \neq 0.$$

Three sufficient and necessary conditions are obtained, which include and improve M.R.S. Kulenović and Ć. Ljubović's recent results in the continuous case [5] and provide some new results in the discrete case, as well as other more general situations.

AMS subject classification: 39A10.
Keywords: Time scales, dynamic equation, asymptotic behavior, nonoscillatory solutions.

Received December 15, 2005; Accepted March 12, 2006

1. Introduction

The study of dynamic equations on time scales is an area of mathematics that recently has received a lot of attention. It has been created in order to unify the study of differential equations and difference equations, and we refer the reader to the paper [3] for a comprehensive treatment of the subject.

Much recent research has been given to the asymptotic properties of solutions of differential equation

$$\big[r(t)g(y'(t))\big]' - p(t)f(y(t)) = 0, \quad t \in \mathbb{R}.$$

We refer the reader to the papers [4, 5]. But it's discrete counterpart

$$\Delta\big[r(t)g(\Delta y(t))\big] - p(t)f(y(t)) = 0, \quad t \in \mathbb{N}$$

has few results. In this paper, we consider the dynamic equation

$$\big[r(t)g(y^{\Delta}(t))\big]^{\Delta} - f(t, y(t)) = 0, \quad t \geq t_0 \tag{1.1}$$

under the conditions

(A_1) $r \in C^1_{rd}\big[(t_0, \infty), (0, \infty)\big]$.

(A_2) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, f is continuously increasing with respect to the second variable and $yf(t, y) > 0$, for $y \neq 0$.

(A_3) $g \in C^1[\mathbb{R}, \mathbb{R}]$, g is a strictly increasing differentiable function on $\mathbb{R}$ and $yg(y) > 0$, for $y \neq 0$.

If $\mathbb{T} = \mathbb{R}$, $f(t, y(t)) = p(t)y(t)$, then equation (1.1) reduces to the differential equation in [5]. If $\mathbb{T} = \mathbb{N}$, the difference equation

$$\Delta\big[r(t)g(\Delta y(t))\big] - f(t, y(t)) = 0, \quad t \in \mathbb{N}$$

is another special case of (1.1). The aim of this paper, on the one hand, is to revisit the proofs of all theorems in [5] which use the Knaster–Tarski fixed-point theorem under decreasing mapping, on the other hand, is to extend the results that we have obtained to the discrete case as well as to more general situations.

The paper is organized as follows: In the next section we present some basic definitions concerning the calculus on time scales. In Section 3, we give three sufficient and necessary conditions for the asymptotic behavior of every nonoscillatory solution of (1.1). In the final section, we also apply our results to discrete systems by two examples.

2. Some Definitions on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} :\ s > t\} \text{ and } \rho(t) := \inf\{s \in \mathbb{T} :\ s < t\}.$$

The point $t \in \mathbb{T}$ is called *right-scattered*, *right-dense*, *left-scattered*, *left-dense* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, $\rho(t) = t$ holds, respectively. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t^*, then $\mathbb{T}^\kappa = \mathbb{T} - \{t^*\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \le b$, define the closed interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} :\ a \le t \le b\}.$$

Other open, half-open intervals in $\mathbb{T}$ can be similarly defined.

Definition 2.1. If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$, then the Δ-*derivative* of f at the point t is defined to be the number $f^\Delta(t)$ with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)\right| \le \epsilon\left|\sigma(t) - s\right|$$

for all $s \in U$. The function f is called Δ-*differentiable* on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 2.2. If $F^\Delta = f$ holds on $\mathbb{T}^\kappa$, then we define the integral of f by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \quad s, t \in \mathbb{T}^\kappa.$$

We refer to [1, 2, 6] for additional details concerning the calculus on time scales. By a solution of (1.1), we mean a nontrivial real valued function x satisfying equation (1.1) for $t \ge t_0$. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor negative; otherwise, it is nonoscillatory.

3. Main Results and Proof

The following result provides useful information on the global asymptotic behavior of nonoscillatory solutions of (1.1).

Lemma 3.1. Let (A_1)–(A_3) be satisfied. If

$$(A_4) \quad \left|\int_{t_0}^\infty g^{-1}\left(\frac{m_1}{r(t)}\right)\Delta t\right| = \infty \text{ for } m_1 \ne 0$$

holds, then every solution y of (1.1) eventually satisfies either $|y(t)| \le K_1$ or $|y(t)| \ge K_2 + \int_{T_0}^{t} g^{-1}\left(\frac{M_1}{r(s)}\right)\Delta s$, where K_1, K_2 and M_1 are positive constants. Furthermore, every positive (negative) nondecreasing (nonincreasing) solution tends to $+\infty(-\infty)$.

Proof. Without loss of generality, we can assume $y(t) > 0$ eventually, that is $y(t) > 0$ for $t \ge t_1 \ge t_0$. Equation (1.1) implies $\left[r(t)g(y^{\Delta}(t))\right]^{\Delta} > 0$ for $t \ge t_1$. Then there exists $t_2 > t_1$ such that $r(t)g(y^{\Delta}(t))$ has constant sign for $t \ge t_2$.

If $r(t)g(y^{\Delta}(t)) < 0$ for $t \ge t_2$, then (A_1) and (A_3) imply $y^{\Delta}(t) < 0$ for $t \ge t_2$, that is, $y(t)$ is an eventually positive decreasing function, so there exist constants $K_1 > 0$ and $t_3 \ge t_2$ such that $y(t) \le K_1$ for $t \ge t_3$. If $r(t)g(y^{\Delta}(t)) > 0$ for $t \ge t_2$, then $y^{\Delta}(t) > 0$ for $t \ge t_2$. Now, $r(t)g(y^{\Delta}(t))$ is an eventually positive increasing function, so there exist constants $M_1 > 0$, and $T_0 \ge t_2$ such that $r(t)g(y^{\Delta}(t)) \ge M_1$ holds for $t \ge T_0$. Then $y^{\Delta}(t) \ge g^{-1}\left(\frac{M_1}{r(t)}\right)$ holds for $t \ge T_0$. Integrating this inequality from T_0 to $t > T_0$ we obtain

$$y(t) \ge y(T_0) + \int_{T_0}^{t} g^{-1}\left(\frac{M_1}{r(s)}\right)\Delta s, \quad y(T_0) = K_2 > 0.$$

The proof is complete. ■

Now we are ready to present the main results of this paper.

Theorem 3.2. Assume that (A_1)–(A_4) hold. Then every, in absolute value, nondecreasing solution y of (1.1) satisfies $\lim_{t\to\infty} |y(t)| < \infty$, if and only if there exists a constant $C \ne 0$ such that

$$\left|\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(t)}\int_{t_0}^{t} f(s,C)\Delta s\right)\right|\Delta t < \infty. \tag{3.1}$$

Proof. Without loss of generality, we can suppose that $y(t) > 0$ eventually.

First assume that (3.1) holds. Then $T_0 \ge t_0$ and $C > 0$ can be chosen such that

$$\int_{T_0}^{\infty} g^{-1}\left(\frac{1}{r(t)}\int_{T_0}^{t} f(s,C)\Delta s\right)\Delta t \le \frac{C}{2}.$$

If we can prove that y is the solution of the dynamic equation

$$y(t) = \frac{C}{2} + \int_{T_0}^{t} g^{-1}\left(\frac{1}{r(s)}\int_{T_0}^{s} f(u,y(u))\Delta u\right)\Delta s,$$

then we can see that y is the desired solution of (1.1). Now we construct the sequence $\{x_m\}$:

$$x_0(t) = \frac{C}{2},$$
$$x_m(t) = \frac{C}{2} + \int_{T_0}^{t} g^{-1}\left(\frac{1}{r(s)}\int_{T_0}^{s} f(u,x_{m-1}(u))\Delta u\right)\Delta s, \quad m = 1, 2\cdots.$$

Thus

$$\frac{C}{2} \le x_0 \le x_1 \le x_2 \le \cdots \le x_{m-1} \le x_m \le C.$$

Then the limit of the sequence $\{x_m\}$ exists. We denote it by x^*, i.e.,

$$\lim_{m\to\infty} x_m(t) = x^*(t).$$

Further,

$$f(s, x_m) \le f(s, C),$$

$$\int_{T_0}^{\infty} g^{-1}\left(\frac{1}{r(t)}\int_{T_0}^{t} f(s, x_m)\Delta s\right)\Delta t \le \frac{C}{2}$$

hold. By Lebesgue's dominated convergence theorem, we have

$$\lim_{m\to\infty} x_m(t) = \frac{C}{2} + \int_{T_0}^{t} g^{-1}\left(\frac{1}{r(s)}\int_{T_0}^{s} f(u, \lim_{m\to\infty} x_{m-1}(u))\Delta u\right)\Delta s.$$

That is,

$$x^*(t) = \frac{C}{2} + \int_{T_0}^{t} g^{-1}\left(\frac{1}{r(s)}\int_{T_0}^{s} f(u, x^*(u))\Delta u\right)\Delta s.$$

So x^* satisfies (1.1). Furthermore, the limit of $x^*(t)$ as $t \to \infty$ exists.

Conversely, let $y > 0$ be a bounded nondecreasing solution of (1.1). Then

$$\lim_{t\to\infty} y(t) = m \in (0, \infty).$$

Let $T_0 \ge t_0$ be such that $y(t) \ge \dfrac{m}{2}$ for $t \ge T_0$. Integrating (1.1) from T_0 to t we have

$$\begin{aligned} r(T_0)g(y^\Delta(t)) &= r(T_0)g(y^\Delta(T_0)) + \int_{T_0}^{t} f(s, y(s))\Delta s \\ &\ge \int_{T_0}^{t} f(s, y(s))\Delta s \\ &\ge \int_{T_0}^{t} f\left(s, \frac{m}{2}\right)\Delta s, \end{aligned}$$

which yields

$$y^\Delta(t) \ge g^{-1}\left(\frac{1}{r(t)}\int_{T_0}^{t} f\left(s, \frac{m}{2}\right)\Delta s\right).$$

Integrating this inequality from T_0 to $t > T_0$, and letting $t \to \infty$, we obtain

$$\int_{T_0}^{\infty} g^{-1}\left(\frac{1}{r(t)}\int_{T_0}^{t} f\left(s, \frac{m}{2}\right)\Delta s\right)\Delta t < \infty.$$

The proof is complete. ■

The next result gives a characterization of another type of asymptotic solution.

Theorem 3.3. If (A_1)–(A_4) are satisfied,

$$g(y) = y^{\alpha}, \quad \text{for every } \; y \neq 0, \alpha = \frac{p}{q} \; (p, q \text{ are both odd}) \tag{3.2}$$

and $\alpha > 1$, then equation (1.1) has an, in absolute value, nondecreasing solution y, such that $|y(t)| \to MR(t, t_0)$, $t \to \infty$ for some constant $M > 1$, if and only if

$$\left| \int_{t_0}^{\infty} f(t, \pm mR(t, t_0)) \Delta t \right| < \infty \tag{3.3}$$

for some $m > 1$, where $R(t, t_0) = \displaystyle\int_{t_0}^{t} g^{-1} \left(\frac{1}{r(s)} \right) \Delta s$.

Proof. Without loss of generality, we assume $y(t) > 0$ eventually.

First assume (3.3) holds. Then we can find $m_1 > 0, T_0 \geq t_0$ such that

$$\int_{T_0}^{\infty} f(t, mR(t, T_0)) \Delta t \leq m_1,$$

and $(1 + m_1)^{1/\alpha} \leq m$.

If we can prove that y is the solution of the dynamic equation

$$y(t) = \int_{T_0}^{t} g^{-1} \left(\frac{1}{r(s)} + \frac{1}{r(s)} \int_{T_0}^{s} f(u, y(u)) \Delta u \right) \Delta s,$$

then we can see that the function y is a nondecreasing solution of (1.1).

Now we construct the sequence $\{x_m\}$:

$$\begin{aligned} x_0 &= R(t, T_0), \\ x_m &= \int_{T_0}^{t} g^{-1} \left(\frac{1}{r(s)} + \frac{1}{r(s)} \int_{T_0}^{s} f(u, x_{m-1}(u)) \Delta u \right) \Delta s. \end{aligned}$$

Then

$$R(t, T_0) \leq x_0 \leq x_1 \leq \cdots \leq x_{m-1} \leq x_m \leq mR(t, T_0).$$

Obviously, the limit of the sequence $\{x_m\}$ exists. We denote it by x^*. Applying Lebesgue's dominated convergence theorem, we have

$$x^*(t) = \int_{T_0}^{t} g^{-1} \left(\frac{1}{r(s)} + \frac{1}{r(s)} \int_{T_0}^{s} f(u, x^*(u)) \Delta u \right) \Delta s.$$

Then y satisfies (3.1) and when $t \to \infty$, $y(t) \to mR(t, t_0)$.

Conversely, if $y(t) > 0, t \geq t_1 \geq t_0$, let y be a nondecreasing solution with the property $\lim\limits_{t \to \infty} y(t) = MR(t, t_0)$. Then there exists $L > 0$ such that $MR(t, t_0) \geq y(t) \geq LR(t, t_0)$, and $r(t) g(y^{\Delta}(t)) > 0$.

We will show that $\lim_{t\to\infty} r(t)g(y^\Delta(t)) < \infty$. Obviously, the limit exists because $r(t)g(y^\Delta(t))$ is an eventually positive and increasing function. If $\lim_{t\to\infty} r(t)g(y^\Delta(t)) = \infty$, then there exists T_0 such that $r(t)g(y^\Delta(t)) > M$ for $t \geq T_0$. This implies $y^\Delta(t) \geq g^{-1}\left(\frac{M}{r(t)}\right)$ and $y(t) \geq M^{1/\alpha}R(t,t_0)$, and consequently,

$$\frac{y(t)}{R(t,t_0)} \geq M^{1/\alpha} > M,$$

which is a contradiction. Therefore, the limit of $r(t)g(y^\Delta(t))$ exists, say m^*, and we have

$$\begin{aligned}2m^* \geq r(t)g(y^\Delta(t)) &= r(T_0)g(y^\Delta(T_0)) + \int_{T_0}^{t} f(s,y(s))\Delta s\\ &\geq \int_{T_0}^{t} f(s,LR(s,s_0))\Delta s,\end{aligned}$$

which shows that (3.3) is satisfied. The proof is complete. ■

Remark 3.4. If $\mathbb{T} = \mathbb{R}$, then $y(t) = O(R(t,t_0))$, that is, $y(t)$ and $R(t,t_0)$ are infinity of the same order.

Next we characterize bounded solutions under the condition (A_4).

Theorem 3.5. Consider (1.1) under conditions (A_1)–(A_4) and (3.2). Let y be an, in absolute value, nonincreasing solution of (1.1). Then $\lim_{t\to\infty}|y(t)| = K \in (0,\infty)$ if and only if there is a constant $C \neq 0$ such that

$$\left|\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(t)}\int_{t}^{\infty} f(s,C)\Delta s\right)\Delta t\right| < \infty. \tag{3.4}$$

Proof. Without loss of generality, we can assume $y(t) > 0$ eventually. If (3.4) holds, then $T_0 \geq t_0$ and $C > 0$ can be chosen such that

$$\int_{T_0}^{\infty} g^{-1}\left(\frac{1}{r(t)}\int_{t}^{\infty} f(s,C)\Delta s\right)\Delta t < \frac{C}{2}.$$

Since g is an odd function, if y is the solution of dynamic equation

$$y(t) = \frac{C}{2} + \int_{t}^{\infty} g^{-1}\left(\frac{1}{r(s)}\int_{s}^{\infty} f(u,C)\Delta u\right)\Delta s,$$

then y is a nonincreasing solution of (1.1).

Now we construct the sequence $\{x_m(t)\}$:

$$\begin{aligned}x_0(t) &= \frac{C}{2},\\ x_m(t) &= \frac{C}{2} + \int_{t}^{\infty} g^{-1}\left(\frac{1}{r(s)}\int_{s}^{\infty} f(u,x_{m-1})\Delta u\right)\Delta s.\end{aligned}$$

Then
$$\frac{C}{2} = x_0(t) \le x_1(t) \le \cdots \le x_m(t) \le C.$$
Clearly, the limit of the sequence $\{x_m\}$ exists. We denote it by x^*. Applying Lebesgue's dominated convergence theorem, we have
$$x^*(t) = \frac{C}{2} + \int_t^\infty g^{-1}\left(\frac{1}{r(s)}\int_s^\infty f(u, x^*(u))\Delta u\right)\Delta s.$$
Then equation (1.1) has an, in absolute value, nonincreasing solution.

Conversely, let $y > 0$ be a bounded nonincreasing solution of (1.1) and $\lim_{t\to\infty} y(t) = K \in (0,\infty)$. Then $y^\Delta(t) < 0$ eventually and $y(t) \ge \dfrac{K}{2}$ for t large enough. Equation (1.1) implies $\lim_{t\to\infty} r(t)g(y^\Delta(t)) = L \in (-\infty, 0)$. Now, we will show $L = 0$. Otherwise, $L < 0$ and so eventually
$$r(t)g(y^\Delta(t)) \le L \Rightarrow y^\Delta(t) \le g^{-1}\left(\frac{L}{r(t)}\right) = L^{1/\alpha} g^{-1}\left(\frac{1}{r(t)}\right).$$
Integrating this relation from T_0 to t, we obtain
$$y(t) \le y(T_0) + L^{1/\alpha}\int_{T_0}^t g^{-1}\left(\frac{1}{r(s)}\right)\Delta s,$$
which by (A_4) implies that $\lim_{t\to\infty} y(t) = -\infty$. This is an immediate contradiction. Thus, $L = 0$. Integrating (1.1) from t to ∞, we get
$$-r(t)g(y^\Delta(t)) = \int_t^\infty f(s, y(s))\Delta s \ge \int_t^\infty f(s, K/2)\Delta s,$$
so that
$$-g(y^\Delta(t)) \ge g^{-1}\left(\frac{1}{r(t)}\int_t^\infty f(s, K/2)\Delta s\right).$$
Integrating this inequality from t to ∞ we obtain
$$y(t) \ge K + \int_t^\infty g^{-1}\left(\frac{1}{r(s)}\int_s^\infty f(u, \frac{K}{2})\Delta u\right)\Delta s.$$
Then (3.4) is satisfied. The proof is complete. ∎

In the present paper, we will apply the results of the above theorems to two discrete cases.

Example 3.6. If $\mathbb{T} = \mathbb{N}$, then (1.1) reduces to the difference equation
$$\Delta\big[r(t)(g(\Delta y(t)))\big] - f(t, y(t)) = 0, \quad t \in \mathbb{N}. \tag{3.5}$$

Let

$$r(t) \equiv 1, \;\; g(u) = u^{\alpha^*} = |u|^{\alpha}\mathrm{sgn}u, \;\; f(u_1, u_2) = (u_1 + u_2)^{\beta^*} = |u_1 + u_2|^{\beta}\mathrm{sgn}u_2. \quad (3.6)$$

Now (3.5) can be written as

$$\Delta(\Delta y(n))^{\alpha^*} - (n + y(n))^{\beta^*} = 0, \quad n \in \mathbb{N}. \tag{3.7}$$

In this special case, (3.1) becomes $\sum_{n=n_0}^{\infty}\left[\sum_{l=n_0}^{n-1}(l+C)^{\beta}\right]^{1/\alpha} < \infty$ for some $C > 0$. By Theorem 3.2, if $\beta < -\alpha - 1$, (3.7) has a nondecreasing bounded solution. In fact

$$\sum_{n=n_0}^{\infty}\left[\sum_{l=n_0}^{n-1}(l+C)^{\beta}\right]^{1/\alpha} \leq \sum_{n=n_0}^{\infty}\left[(n-n_0)(n-1+C)^{\beta}\right]^{1/\alpha}\sum_{n=n_0}^{\infty}(n-1+C)^{\frac{\beta+1}{\alpha}},$$

if $\beta < -\alpha - 1$, the progression $\sum_{n=n_0}^{\infty}(n+C)^{\frac{\beta+1}{\alpha}}$ is convergent, so the progression $\sum_{n=n_0}^{\infty}\left[\sum_{l=n_0}^{n-1}(l+C)^{\beta}\right]^{1/\alpha}$ is also convergent.

Example 3.7. If $\mathbb{T} = h\mathbb{N}$, $h > 0$, and (3.6) hold, then (1.1) can be written as

$$\left[(y^{\Delta}(t))^{\alpha^*}\right]^{\Delta} - (t + y(t))^{\beta^*} = 0, \quad t \in h\mathbb{N}. \tag{3.8}$$

In this case, condition (3.1) becomes

$$h\sum_{n=n_0/h}^{\infty}\left[h\sum_{l=n_0/h}^{n-1}(hl+C)^{\beta}\right]^{1/\alpha} < \infty,$$

where $\alpha = p/q$ (p and q are both odd). By Theorem 3.2, if $\beta < -\alpha - 1$, then equation (3.8) has a nondecreasing bounded positive solution. In fact,

$$h\sum_{n=n_0/h}^{\infty}\left[h\sum_{l=n_0/h}^{n-1}(hl+C)^{\beta}\right]^{1/\alpha} < h\sum_{n=n_0/h}^{\infty}\left[(h-n_0)(hn-h+C)^{\beta}\right]^{1/\alpha}.$$

Thus, if $\beta < -\alpha - 1$, the progression is convergent.

References

[1] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson. Dynamic equations on time scales: a survey. *J. Comput. Appl. Math.*, 141(1-2):1–26, 2002.

[2] Martin Bohner and Allan Peterson. *Dynamic equations on time scales*. Birkhäuser Boston Inc., Boston, MA, 2001.

[3] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[4] M. R. S. Kulenović. Asymptotic behavior of solutions of certain differential equations and inequalities of second order. *Rad. Mat.*, 1(2):275–295, 1985.

[5] M. R. S. Kulenović and Ć. Ljubović. The asymptotic behavior of nonoscillatory solutions of some nonlinear differential equations. *Nonlinear Anal.*, 42(5, Ser. A: Theory Methods):821–833, 2000.

[6] B. G. Zhang and Xinghua Deng. Oscillation of delay differential equations on time scales. *Math. Comput. Modelling*, 36(11-13):1307–1318, 2002.