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On Exponential Stability of Variational Difference Equations

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Abstract

We prove that a general system of variational difference equations is uniformly exponentially stable if and only if certain associated sets are of the second category. We also deduce necessary and sufficient conditions for uniform exponential stability of systems with uniformly bounded coefficients. We apply our results for the study of exponential stability of linear skew-product flows, generalizing some stability theorems recently obtained in [A. L. Sasu, *Math. Ineq. Appl.* 7 (2004), 535-541].

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1. Introduction

In recent years, important attempts have been made to study asymptotic properties of evolution equations in infinite-dimensional spaces. Significant progress has been made in this direction pointing out that an impressive list of the classical problems can be treated in the unified setting of linear skew-product flows (see [3, 7, 11–13, 15, 17, 18]. Among the techniques used in the study of qualitative properties, the theory of function spaces became a valuable tool, providing new applicability areas (see [1,2,4–7,9,10,12–16,18]). The idea of characterizing the exponential stability of semigroups in terms of Banach function spaces goes back to the paper of Neerven (see [9]). The main result in [9] states that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if there is a Banach function space B with $\lim\limits_{t\to\infty} F_B(t) = \infty$ such that for every $x \in X$, the mapping $t \mapsto ||T(t)x||$ lies in B. Later, the author generalized his theorem using category type

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arguments in [10]. Neerven's results in [9] were generalized at the non-autonomous case (see [2, 6]) and also at the variational case (see [7]). A new approach was given in [18], considering a general class of Banach sequence spaces $(B, |\cdot|_B)$ with the properties

$$\inf_{n\in\mathbb{N}} |\chi_{\{n\}}|_B > 0 \quad \text{and} \quad \sup_{n\in\mathbb{N}} |\chi_{\{0,\dots,n\}}|_B = \infty. \tag{1.1}$$

The main result in [18] states that a system of variational difference equations is uniformly exponentially stable if and only if there exists a Banach sequence space B with the properties (1.1) such that the set of all vectors with the corresponding orbits uniformly contained in B, is of the second category.

In what follows we will continue the study begun in [18]. We associate with a system (A) of variational difference equations on a Banach space X general sets of vectors in X with certain properties. We prove that (A) is uniformly exponentially stable if and only if the associated sets are of the second category. After that, we consider the case of systems of variational difference equations with uniformly bounded coefficients and we give necessary and sufficient conditions for their uniform exponential stability. Finally, we apply our main results in order to deduce characterizations for uniform exponential stability of linear skew-product flows, generalizing the main results in [7, 12, 13].

2. Preliminaries

In this section we recall some basic properties of Banach sequence spaces. Let $\mathcal{S}$ be the linear space of all sequences $s : \mathbb{N} \to \mathbb{R}$. For every set $A \subset \mathbb{N}$ we denote by χ_A the characteristic function of A. We recall that a linear subspace B of $\mathcal{S}$ is a normed linear space if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ such that

(i) $|s|_B = 0$ if and only if $s = 0$;

(ii) if $|s| \le |u|$, then $|s|_B \le |u|_B$;

(iii) $|\alpha s|_B = |\alpha|\, |s|_B$, for all $(\alpha, s) \in \mathbb{R} \times B$;

(iv) $|s + u|_B \le |s|_B + |u|_B$, for all $s, u \in B$.

Moreover if $(B, |\cdot|_B)$ is complete, then B is called a *Banach sequence space*.

Example 2.1. [Orlicz sequence spaces] Let $\varphi : \mathbb{R}_+ \to [0, \infty]$ be a non-decreasing, left continuous function which is not identically 0 or ∞ on $(0, \infty)$. The function

$$Y_\varphi(t) = \int_0^t \varphi(s)\, ds$$

is called *the Young function* associated with φ. For every $s \in \mathcal{S}$ we consider

$$M_\varphi(s) := \sum_{n=0}^{\infty} Y_\varphi(|s(n)|).$$

The set O_φ of all sequences with the property that there exists $k > 0$ such that $M_\varphi(ks) < \infty$ is easily checked to be a linear space. With respect to the norm $|s|_\varphi := \inf\{k > 0 : M_\varphi(s/k) \le 1\}$, O_φ is a Banach sequence space, called *Orlicz sequence space*.

Remark 2.2. If $p \in [1, \infty)$ and $\varphi_p : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_p(t) = pt^{p-1}$, then $O_{\varphi_p} = \ell^p(\mathbb{N}, \mathbb{R})$.

Remark 2.3. If

$$\psi : \mathbb{R}_+ \to [0, \infty], \quad \psi(t) = \begin{cases} 0 & , \quad t \in [0, 1] \\ \infty & , \quad t > 1 \end{cases}$$

then $O_\psi = \ell^\infty(\mathbb{N}, \mathbb{R})$.

Remark 2.4. If $\varphi(0) = 0$ and $\varphi(t) \in (0, \infty)$, for all $t > 0$, then O_φ has the properties

$$\inf_{n\in\mathbb{N}} |\chi_{\{n\}}|_\varphi > 0 \quad \text{and} \quad \sup_{n\in\mathbb{N}} |\chi_{\{0,\dots,n\}}|_\varphi = \infty.$$

For proof details we refer to [6, Proposition 2.1].

3. Exponential Stability for Variational Difference Equations

Let X be a real or complex Banach space. The norm on X and on $\mathcal{L}(X)$, the Banach algebra of all bounded linear operators on X, will be denoted by $||\cdot||$. Let (Θ, d) be a metric space and let $J \in \{\mathbb{N}, \mathbb{Z}\}$.

We recall that a *discrete flow* on Θ is a mapping $\sigma : \Theta \times J \to \Theta$ with $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, m + n) = \sigma(\sigma(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times J^2$.

Let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the linear system of variational difference equations

$$(A) \qquad x(\theta)(n + 1) = A(\sigma(\theta, n))x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}.$$

The *discrete cocycle* associated with the system (A) is

$$\Phi : \Theta \times \mathbb{N} \to \mathcal{L}(X), \quad \Phi(\theta, n) = \begin{cases} A(\sigma(\theta, n-1)) \dots A(\theta) & , \quad n \in \mathbb{N}^* \\ I & , \quad n = 0 \end{cases}$$

where I denotes the identity operator on X.

Remark 3.1. We observe that $\Phi(\theta, m+n) = \Phi(\sigma(\theta, n), m)\Phi(\theta, n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{N}^2$.

Definition 3.2. We say that the system (A) is *uniformly exponentially stable* if the discrete cocycle associated with (A) is uniformly exponentially stable, i.e., there are two constants $K, \nu > 0$ such that $||\Phi(\theta, n)|| \le Ke^{-\nu n}$, for all $(\theta, n) \in \Theta \times \mathbb{N}$.

Let $\mathcal{F}$ be the set of all continuous non-decreasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and $f(t) > 0$, for all $t > 0$.

Theorem 3.3. Let $f \in \mathcal{F}$ and let $p \in \mathbb{N}^*$. If Φ is the discrete cocycle associated with (A), then the set

$$F = \left\{ x \in X : \sup_{\theta \in \Theta} \sum_{j=0}^{\infty} f(||\Phi(\theta, j)x||) \leq p \right\}$$

is closed.

Proof. For every $(\theta, h) \in \Theta \times \mathbb{N}$, let

$$F_{\theta,h} = \left\{ x \in X : \sum_{j=0}^{h} f(||\Phi(\theta, j)x||) \leq p \right\}.$$

We prove that $F_{\theta,h}$ is closed, for all $(\theta, h) \in \Theta \times \mathbb{N}$.

Let $(\theta, h) \in \Theta \times \mathbb{N}$ and $x \in \overline{F}_{\theta,h}$. Then there exists a sequence $(x_n) \subset F_{\theta,h}$ such that $x_n \to x$. Let $a := \sup_{n \in \mathbb{N}} ||x_n||$ and $b := \max\{||\Phi(\theta, j)|| : j \in \{0, \ldots, h\}\}$. Since f is continuous on $[0, ab]$, it is uniformly continuous on $[0, ab]$.

Let $\varepsilon > 0$. There is $\delta > 0$ such that for every $s, s' \in [0, ab]$ with $|s - s'| < \delta$ we have

$$|f(s) - f(s')| < \frac{\varepsilon}{h+1}. \tag{3.1}$$

Let $l \in \mathbb{N}$ be such that $||x_l - x|| < \delta/b$. Then for every $j \in \{0, \ldots, h\}$, $||\Phi(\theta, j)x_l||, ||\Phi(\theta, j)x|| \in [0, ab]$ and $|\ ||\Phi(\theta, j)x_l|| - ||\Phi(\theta, j)x||\ | < \delta$. Using (3.1) it follows that

$$\sum_{j=0}^{h} f(||\Phi(\theta, j)x||) \leq \sum_{j=0}^{h} f(||\Phi(\theta, j)x_l||) + \varepsilon. \tag{3.2}$$

Since $x_l \in F_{\theta,h}$ from (3.2) we obtain that

$$\sum_{j=0}^{h} f(||\Phi(\theta, j)x||) \leq p + \varepsilon.$$

Since ε was arbitrary, we deduce that $x \in F_{\theta,h}$, so $F_{\theta,h}$ is closed. Taking into account that

$$F = \bigcap_{\theta \in \Theta} \bigcap_{h \in \mathbb{N}} F_{\theta,h},$$

the proof is complete. ∎

The first main result of this section is:

Theorem 3.4. The system (A) is uniformly exponentially stable if and only if there is $f \in \mathcal{F}$ with $\lim\limits_{t\to\infty} f(t) = \infty$ such that the set

$$\mathcal{L} = \left\{ x \in X : \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} f(||\Phi(\theta, j)x||) < \infty \right\}$$

is of the second category.

Proof. *Necessity.* Consider $f(t) = t$, for all $t \geq 0$ and an easy computation shows that $\mathcal{L} = X$.

Sufficiency. For every $p \in \mathbb{N}^*$, let

$$F_p = \left\{ x \in X : \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} f(||\Phi(\theta, j)x||) \leq p \right\}.$$

Then, we have that $\mathcal{L} = \bigcup\limits_{p=1}^{\infty} F_p$. Moreover, from Theorem 3.3 we obtain that F_p is closed, for every $p \in \mathbb{N}^*$. Since $\mathcal{L}$ is a set of the second category, it follows that there is $q \in \mathbb{N}^*$ such that the interior of the set F_q is not empty. This yields that there are $x_0 \in X$ and $r > 0$ such that $D(x_0, r) = \{x \in X : ||x - x_0|| \leq r\} \subset F_q$. It follows that

$$\sup_{\theta\in\Theta} \sum_{j=0}^{\infty} f(||\Phi(\theta, j)x||) \leq q, \quad \forall x \in D(x_0, r). \tag{3.3}$$

Since $\lim\limits_{t\to\infty} f(t) = \infty$, there is $\gamma > 0$ such that $f(t) > q$, for all $t \geq \gamma$. Then, from (3.3) we deduce that

$$||\Phi(\theta, j)x|| \leq \gamma, \quad \forall x \in D(x_0, r), \forall(\theta, j) \in \Theta \times \mathbb{N}.$$

Let $x \in X \setminus \{0\}$ and $(\theta, j) \in \Theta \times \mathbb{N}$. Using the above relation we have that

$$\left\| \Phi(\theta, j)\frac{r}{||x||}\, x|| \leq ||\Phi(\theta, j)\left(x_0 + \frac{r}{||x||}\, x\right)\right\| + ||\Phi(\theta, j)x_0)|| \leq 2\gamma$$

which shows that

$$||\Phi(\theta, j)x|| \leq \frac{2\gamma}{r}\, ||x||, \quad \forall(x, \theta) \in X \times \Theta, \forall j \in \mathbb{N}.$$

Let O_f be the Orlicz space associated with f and let Y_f be the associated Young function. For every $(x, \theta) \in X \times \Theta$, let

$$s_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad s_{x,\theta}(n) = ||\Phi(\theta, n)x||.$$

Let $\lambda = \max\{1, (2\gamma q/r)\cdot(||x_0|| + r)\}$. If $x \in D(x_0, r)$ and $\theta \in \Theta$, then

$$\begin{aligned} Y_f\left(\frac{1}{\lambda}\, s_{x,\theta}(n)\right) &\leq \frac{1}{\lambda}\, ||\Phi(\theta, n)x||\, f\left(\frac{1}{\lambda}\, s_{x,\theta}(n)\right) \\ &\leq \frac{1}{q}\, f(||\Phi(\theta, n)x||), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.4}$$

Thus from (3.3) and (3.4) we deduce that $M_f(s_{x,\theta}/\lambda) \leq 1$, which shows that $s_{x,\theta} \in O_f$ and $|s_{x,\theta}|_f \leq \lambda$, for all $(x, \theta) \in D(x_0, r) \times \Theta$. Now, we consider the set

$$\mathcal{S} := \left\{x \in X :\ \sup_{\theta\in\Theta} |s_{x,\theta}|_f < \infty\right\}$$

and we have that $D(x_0, r) \subset \mathcal{S}$. So $\mathcal{S}$ is a set of the second category. By applying [Theorem 2.1] [18] for $B = O_f$ and using Remark 2.4, we obtain that the system (A) is uniformly exponentially stable. ■

Theorem 3.5. Let $\varphi : \mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\varphi(n, \cdot) \in \mathcal{F}$ and $\lim_{t\to\infty} \varphi(n, t) = \infty$, for all $n \in \mathbb{N}$. If the set

$$\mathcal{Q} = \left\{x \in X :\ \exists\alpha(x) \in \mathbb{N} \text{ such that } \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} \varphi(\alpha(x), ||\Phi(\theta, j)x||) < \infty\right\}$$

is of the second category, then the system (A) is uniformly exponentially stable.

Proof. For every $n \in \mathbb{N}$, let $\varphi_n = \varphi(n, \cdot)$ and let

$$\mathcal{Q}_n := \left\{x \in X :\ \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} \varphi_n(||\Phi(\theta, j)x||) < \infty\right\}.$$

Then $\mathcal{Q} = \bigcup_{n\in\mathbb{N}} \mathcal{Q}_n$. According to our hypothesis, we deduce that there is $h \in \mathbb{N}$ such that $\mathcal{Q}_h$ is a set of the second category. By applying Theorem 3.4 for $f = \varphi_h$ we obtain the conclusion. ■

In the second part of this section, we will focus on systems (A) with uniformly bounded coefficients, i.e.,

$$\sup_{\theta\in\Theta} ||A(\theta)|| < \infty.$$

Theorem 3.6. A system (A) with uniformly bounded coefficients is uniformly exponentially stable if and only if there is a function $g \in \mathcal{F}$ and an unbounded sequence (k_n) with $\sup_{n\in\mathbb{N}} |k_{n+1} - k_n| < \infty$ such that the set

$$\mathcal{B} := \left\{x \in X :\ \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} g(||\Phi(\theta, k_j)x||) < \infty\right\}$$

is of the second category.

Proof. *Necessity.* We consider $g(t) = t$, for all $t \geq 0$ and $k_n = n$, for all $n \in \mathbb{N}$. Then $\mathcal{B} = X$.

Sufficiency. Without loss of generality we may assume that (k_n) is increasing. Using similar arguments as in the proof of Theorem 3.3, we obtain that for every $n \in \mathbb{N}^*$, the set

$$\mathcal{B}_n = \left\{ x \in X : \sup_{\theta \in \Theta} \sum_{j=0}^{\infty} g(||\Phi(\theta, k_j)x||) \leq n \right\}$$

is closed. Since $\mathcal{B} = \bigcup_{n \in \mathbb{N}^*} \mathcal{B}_n$, it follows that there is $m \in \mathbb{N}^*$ such that the interior of $\mathcal{B}_m$ is not empty. Thus, there is $x_0 \in X$ and $r > 0$ such that

$$\sup_{\theta \in \Theta} \sum_{j=0}^{\infty} g(||\Phi(\theta, k_j)x||) \leq m, \quad \forall x \in D(x_0, r). \tag{3.5}$$

Let $L = \max\{1, \sup_{\theta \in \Theta} ||A(\theta)||\}$. Let $h \in \mathbb{N}^*$ be such that $m < hg(1)$. If $p = \sup_{n \in \mathbb{N}}(k_{n+1} - k_n)$, we set $\gamma = L^{ph}$.

Let $(y, \theta) \in D\left(0, \frac{r}{\gamma}\right) \times \Theta$. If $n \geq h$, from

$$\left\| \Phi(\theta, k_n)\left(\frac{x_0}{\gamma} + y\right) \right\| \leq ||\Phi(\theta, k_j)(x_0 + \gamma y)||, \quad \forall j \in \{n - h + 1, \ldots, n\}$$

using (3.5) it follows that

$$h\ g\left(\left\| \Phi(\theta, k_n)\left(\frac{x_0}{\gamma} + y\right) \right\| \right) \leq \sum_{j=n-h+1}^{n} g(||\Phi(\theta, k_j)(x_0 + \gamma y)||) \leq m.$$

Since g is non-decreasing and $m < h\ g(1)$, we deduce that

$$\left\| \Phi(\theta, k_n)\left(\frac{x_0}{\gamma} + y\right) \right\| \leq 1, \quad \forall y \in D\left(0, \frac{r}{\gamma}\right), \forall n \geq h. \tag{3.6}$$

Let $x \in X \setminus \{0\}$. Then, using (3.6) we obtain that

$$\left\| \Phi(\theta, k_n)\frac{rx}{\gamma||x||} \right\| \leq \left\| \Phi(\theta, k_n)\left(\frac{x_0}{\gamma} + \frac{rx}{\gamma||x||}\right) \right\| + \left\| \Phi(\theta, k_n)\frac{x_0}{\gamma} \right\| \leq 2.$$

Setting $M_1 = 2\gamma/r$ it follows that

$$||\Phi(\theta, k_n)x|| \leq M_1\ ||x||, \quad \forall (x, \theta) \in X \times \Theta, \forall n \geq h. \tag{3.7}$$

For $n \in \{0, \ldots, h - 1\}$ we have that $||\Phi(\theta, k_n)|| \leq L^{k_0 + p(h-1)}$. Setting $M = \max\{M_1, L^{k_0+p(h-1)}\}$, using (3.7) we deduce that

$$||\Phi(\theta, k_n)|| \leq M, \quad \forall (\theta, n) \in \Theta \times \mathbb{N}. \tag{3.8}$$

For every $(x, \theta) \in X \times \Theta$ let

$$u_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad u_{x,\theta}(n) = ||\Phi(\theta, k_n)x||.$$

Let O_g be the Orlicz space associated with g. Using relations (3.5) and (3.8) and employing similar arguments to those in the proof of Theorem 3.4 we obtain that for every $(x, \theta) \in D(x_0, r) \times \Theta$, $u_{x,\theta} \in O_g$ and $\sup\limits_{\theta \in \Theta} |u_{x,\theta}|_g < \infty$, for all $x \in D(x_0, r)$. By applying [18, Corollary 2.1] for $B = O_g$, we deduce that the system (A) is uniformly exponentially stable. ■

Theorem 3.7. Let (A) be a system with uniformly bounded coefficients. Let $\Psi : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\psi(n, \cdot) \in \mathcal{F}$, for all $n \in \mathbb{N}$. For every $p \in \mathbb{N}$ let $(k_j^p)_{j \in \mathbb{N}}$ be an unbounded sequence with $\sup\limits_{j \in \mathbb{N}} |k_{j+1}^p - k_j^p| < \infty$. If

$$\mathcal{H} = \Bigg\{ x \in X : \ \exists \alpha(x), p(x) \in \mathbb{N} \text{ such that}$$

$$\sup_{\theta \in \Theta} \sum_{j=0}^{\infty} \psi(\alpha(x), ||\Phi(\theta, k_j^{p(x)})x||) < \infty \Bigg\}$$

is a set of the second category, then (A) is uniformly exponentially stable.

Proof. For every $m, n \in \mathbb{N}$, let

$$\mathcal{H}_{n,m} = \left\{ x \in X : \ \sup_{\theta \in \Theta} \sum_{j=0}^{\infty} \Psi(n, ||\Phi(\theta, k_j^m)x||) < \infty \right\}.$$

Then $\mathcal{H} = \bigcup\limits_{n,m} \mathcal{H}_{n,m}$. Since $\mathcal{H}$ is of the second category, there are n_0, m_0 such that $\mathcal{H}_{n_0,m_0}$ is of the second category. By applying Theorem 3.6 for $g = \Psi(n_0, \cdot)$ and $k_j = k_j^{m_0}$, for all $j \in \mathbb{N}$, we deduce that the system (A) is uniformly exponentially stable. ■

4. Applications for Uniform Exponential Stability of Linear Skew-Product Flows

Let X be a real or complex Banach space and let (Θ, d) be a metric space.

Let $J \in \{\mathbb{R}_+, \mathbb{R}\}$. A *flow* on Θ is a mapping $\sigma : \Theta \times J \to \Theta$ with $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$ and $\sigma(\theta, t + s) = \sigma(\sigma(\theta, t), s)$, for all $(\theta, t, s) \in \Theta \times J^2$.

Let σ be a flow on Θ. A pair $\pi = (\Phi, \sigma)$ is called a *linear skew-product flow* on $\mathcal{E} = X \times \Theta$ if the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ has the properties: $\Phi(\theta, 0) = I$, $\Phi(\theta, t + s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbb{R}_+^2$, and there are $M, \omega > 0$ such that $||\Phi(\theta, t)|| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$. Moreover, if σ is continuous and for every $x \in X$, the mapping $t \mapsto \Phi(\theta, t)x$ is continuous, then we say that π is *strongly continuous*.

Definition 4.1. A linear skew-product flow $\pi = (\Phi, \sigma)$ is *uniformly exponentially stable* if there are $K, \nu > 0$ such that $||\Phi(\theta, t)|| \leq Ke^{-\nu t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

Remark 4.2. Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $X \times \Theta$. For every $\theta \in \Theta$ let $A(\theta) = \Phi(\theta, 1)$. Then, considering the system of variational difference equations

$$(A_\pi) \qquad x(\theta)(n+1) = A(\sigma(\theta, n))x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}$$

we have that the discrete cocycle associated with the system (A_π) is $\Phi_{A_\pi}(\theta, n) = \Phi(\theta, n)$, for all $(\theta, n) \in \Theta \times \mathbb{N}$. So π is uniformly exponentially stable if and only if the system (A_π) is uniformly exponentially stable.

Theorem 4.3. Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $X \times \Theta$. Then π is uniformly exponentially stable if and only if there is a function $g \in \mathcal{F}$ and an unbounded sequence $(t_n) \subset (0, \infty)$ with $\sup\limits_{n \in \mathbb{N}} |t_{n+1} - t_n| < \infty$ such that the set

$$\mathcal{R} = \left\{ x \in X : \sup_{\theta \in \Theta} \sum_{j=0}^{\infty} g(||\Phi(\theta, t_j)x||) < \infty \right\}$$

is of the second category.

Proof. *Necessity* is immediate.

Sufficiency. Let $k_n = [t_n] + 1$, for all $n \in \mathbb{N}$. By applying Theorem 3.6 and Remark 4.2, we obtain the conclusion. ■

Remark 4.4. A different proof of Theorem 4.3 was given in [12, Theorem 3.1].

Theorem 4.5. Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $X \times \Theta$. Then π is uniformly exponentially stable if and only if there is a function $F : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ with $F(n, \cdot) \in \mathcal{F}$, for all $n \in \mathbb{N}$ such that the set

$$\mathcal{W} = \left\{ x \in X : \ \exists \alpha(x) \in \mathbb{N} \text{ such that } \sup_{\theta \in \Theta} \int_0^\infty F(\alpha(x), ||\Phi(\theta, t)x||)\, dt < \infty \right\}$$

is a set of the second category.

Proof. *Necessity* is immediate, taking $F(n, t) = t$, for all $(n, t) \in \mathbb{N} \times \mathbb{R}_+$.

Sufficiency. Let $M, \omega > 0$ be given. We define

$$\Psi : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+, \quad \Psi(n, t) = F\left(n, \frac{t}{Me^\omega}\right).$$

We observe that

$$\sum_{k=0}^{\infty} \Psi(n, ||\Phi(\theta, k+1)x||) \leq \int_0^{\infty} F(n, ||\Phi(\theta, t)x||)\, dt, \quad \forall (n, \theta) \in \mathbb{N} \times \Theta.$$

According to our hypothesis, we deduce that the set

$$\mathcal{S} = \Big\{ x \in X : \ \exists \alpha(x) \in \mathbb{N} \text{ such that}$$

$$\sup_{\theta \in \Theta} \sum_{j=0}^{\infty} \Psi(\alpha(x), ||\Phi(\theta, j+1)x||) < \infty \Big\}$$

is a set of the second category. By applying Theorem 3.7 and Remark 4.2, we obtain that π is uniformly exponentially stable. ■

Remark 4.6. The above theorem generalizes a stability result proved in [13, Theorem 3.2].

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