Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 1 Number 1 (2006), pp. 49–57

http://www.ripublication.com/adsa.htm

On Some Stochastic Fractional Integro-Differential Equations

Mahmoud M. El-Borai
Faculty of Science, Alexandria University, Alexandria, Egypt
E-mail: m_m_elborai@yahoo.com

Abstract

Some classes of stochastic fractional integro-partial differential equations are investigated. Mild solutions of the nonlocal Cauchy problem for the considered classes are studied. The Leray–Schauder principle is used to establish the existence of stochastic solutions. The uniqueness of the solution of the considered problem is also studied under suitable conditions.

AMS subject classification: 34K30, 34K05, 26A33, 35A05.
Keywords: Nonlocal Cauchy problem, stochastic integro-partial differential equations, fractional order.

1. Introduction

Many physical models are represented by semi-linear stochastic fractional integro-partial differential systems of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = L(x,D)u(x,t) + f(u(x,t)) + \int_0^t g(u(x,s))dW(s), \tag{1.1}$$

with the nonlocal condition

$$u(x,0) = \varphi(x) + \sum_{i=1}^{p} c_i u(x,t_i), \tag{1.2}$$

where $0 < \alpha \leq 1$, $0 \leq t_1 < t_2 < \cdots < t_p$, x is an element of the n-dimensional Euclidean space $\mathbb{R}^n$, $D = (D_1, \ldots, D_n)$, $D_i = \dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, $W(t)$ is a standard

Received May 1, 2006; Accepted July 2, 2006

Brownian motion defined over the filtered probability space (Ω, F, F_t, P), $u \in \mathbb{R}^k$ and $\{F_t : 0 \leq t \leq T\}$ is a right-continuous, increasing family of sub σ-algebras of F.

Let H be the space of all k-dimensional vectors of functions defined on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \sum_{i=1}^{k} u_i^2(x)dx < \infty, \ u = (u_1, \ldots, u_k).$$

A scalar product (u, v) in H is defined by

$$(u, v)_H = \sum_{i=1}^{k} \int_{\mathbb{R}^n} u_i(x)v_i(x)dx.$$

If $S = \{V(x, t, \omega) : \Omega \to H | 0 \leq t \leq T\}$ is a stochastic process, then we shall write for simplicity $V(x, t)$ and $V_t : [0, T] \to H$ in place of S.

The collection of all strongly measurable H-valued random variables denoted by $L^2(\Omega, H)$ is a Banach space equipped with the norm

$$\|V_t\|_{L^2(\Omega,H)} = [E \, \|V_t(\omega)\|_H^2]^{\frac{1}{2}}, \quad E(g) = \int_{\Omega} g(\omega)dP,$$

where $E(g)$ is the expectation of g. It is assumed that

$$L(x, D) = L_0(x, D) + L_1(x, D),$$

where

$$L_0(x, D) = \sum_{|q|=2m} A_q(x)D^q, \qquad L_1(x, D) = \sum_{|q|<2m} A_q(x)D^q,$$

$D^q = D_1^{q_1} \cdots D_n^{q_n}$, $q = (q_1, \ldots, q_n)$ is a multi index, $|q| = q_1 + \cdots + q_n$, and $\{A_q(x) : |q| \leq 2m\}$ is a family of deterministic square matrices of order k. Following Petrovsky it is assumed that

$$\det\{(-1)^m L_0(x, \sigma) - \lambda I\} = 0$$

has roots which satisfy the inequality $\mathrm{Re}\lambda < -\delta$, $\delta > 0$ for all $x \in \mathbb{R}^n$, $t \geq 0$ and for any real vector $\sigma, \sigma_1^2 + \cdots + \sigma_n^2 = 1$, [2]. If B is a matrix of order $m \times n$, then we introduce $|B|$ by

$$|B| = \sum_{i,j} |b_{ij}|.$$

It is assumed that the coefficients of $L(x, D)$ are bounded on $\mathbb{R}^n$ and satisfy the Hölder condition (with exponent $\gamma \in (0, 1]$). It is well known that there exists a fundamental matrix solution $Z(x, y, t)$, which satisfies the system

$$\frac{\partial Z(x, y, t)}{\partial t} = L(x, D)Z(x, y, t), \quad t > 0, x, y \in \mathbb{R}^n.$$

This fundamental matrix satisfies the inequality

$$|D^q Z(x,y,t)| \leq K_1 t^{-\rho_1} \exp(-K_2\rho_2), \tag{1.3}$$
$$|q| \leq 2m, \rho_1 = -\frac{n+|q|}{2m},\ \rho_2 = \sum_{i=1}^{n} |x_i - y_i|^{\lambda} t^{-\frac{1}{2m-1}},$$
$$\lambda = \frac{2m}{2m-1},\ (K_1 > 0 \text{ and } K_2 > 0 \text{ are constants}).$$

The set $\{v_t \in C([0,T]; L^2(\Omega, H)) : v_t \text{ is } F_t \text{ adapted}\}$ is denoted by $C([0,T];H)$, where $C([0,T];H)$ denotes the space of k-dimensional vectors of continuous functions from $[0,T]$ into H equipped with norm

$$\|v\|_C = \sup_{t\in[0,T]} [E\,\|v_t\|_H^2]^{\frac{1}{2}}.$$

The purpose of this paper is to study the mild solutions of the nonlocal Cauchy problem (1.1), (1.2), assuming that φ is an F_0-measurable H-valued stochastic process. The results in this note may be regarded as a generalization of some recent results developed in [3–11, 14, 16, 17].

In Section 2, we shall find the mild solutions of the nonlocal Cauchy problem (1.1), (1.2).

The nonlocal Cauchy problem (1.1), (1.2) has applications in many fields such as viscoelasticity and electromagnetic theory, [1, 12, 13, 15, 18].

2. Stochastic Integral Equation

A slightly modified version of the nonlocal Cauchy problem (1.1), (1.2) is considered. Using the definitions of the fractional derivatives and integrals, it is suitable to rewrite the considered problem in the form

$$\begin{aligned} u(x,t) = u(x,0) &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} L(x,D) u(x,\theta) d\theta \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(u(x,\theta)) d\theta \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\theta (t-\theta)^{\alpha-1} g(u(x,s)) dW(s) d\theta. \end{aligned} \tag{2.1}$$

Let $Z(t)$ be the operator defined on H, for every $t > 0$, by

$$Z(t)\varphi = \int_{\mathbb{R}^n} Z(x,y,t)\varphi(y) dy.$$

According to condition (1.3), there is a positive constant M such that

$$\|Z(t)\|_H \leq M. \tag{2.2}$$

Let us consider the integrals of the operator-valued functions

$$\psi(t) = \int_0^\infty \xi_\alpha(\theta) Z(t^\alpha \theta) d\theta,$$

$$\psi^*(t) = \alpha \int_0^\infty \theta t^{\alpha-1} \xi_\alpha(\theta) Z(t^\alpha \theta) d\theta,$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $[0, \infty]$ (see [3]). It is supposed that

$$cM < 1, \quad \text{where } c = \sum_{i=1}^{p} |c_i|. \tag{2.3}$$

Theorem 2.1. If $u \in C([0,T]; H)$ is an F_t-adapted stochastic process that satisfies equation (2.1) a.s. $[p]$, then u satisfies the equation (a.s. $[p]$)

$$\begin{aligned} u(x,t) &= (\psi(t)\Lambda^{-1}\varphi)(x) \\ &\quad + \psi(t)\Lambda^{-1} \sum_{i=1}^{p} c_i \int_0^{t_i} (\psi^*(t_i - \eta) f(u))(x,\eta) d\eta \\ &\quad + \psi(t)\Lambda^{-1} \sum_{i=1}^{p} c_i \int_0^{t_i} \int_0^{\eta} (\psi^*(t_i - \eta) g(u))(x,s) dW(s) d\eta \\ &\quad + \int_0^t (\psi^*(t-\eta) f(u))(x,\eta) d\eta \\ &\quad + \int_0^t \int_0^\eta (\psi^*(t-\eta) g(u))(x,s) dW(s) d\eta, \end{aligned} \tag{2.4}$$

where $\Lambda = I - \sum_{i=1}^{p} c_i \psi(t_i)$, I is the identity operator.

Proof. Using (2.2) and (2.3), we find that the inverse operator Λ^{-1} exists. Using the results in [3], the solution of equation (2.1) can be written in the form (a.s. $[p]$)

$$\begin{aligned} u(x,t) &= \int_0^\infty \int_{\mathbb{R}^n} Z(x,y,t^\alpha\theta)\xi_\alpha(\theta) u(y,0) dy d\theta \\ &\quad + \alpha \int_0^t \int_0^\infty \int_{\mathbb{R}^n} \theta(t-\eta)^{\alpha-1} \xi_\alpha(\theta) Z(x,y,(t-\eta)^\alpha\theta) f(u(y,\eta)) dy d\theta d\eta \\ &\quad + \alpha \int_0^t \int_0^\infty \int_{\mathbb{R}^n} \theta(t-\eta)^{\alpha-1} \xi_\alpha(\theta) Z(x,y,(t-\eta)^\alpha\theta) \\ &\quad \times \int_0^\eta g(u(y,s)) dW(s) dy d\theta d\eta. \end{aligned}$$

It is easy to see that (a.s. $[p]$)

$$\sum_{i=1}^{p} c_i u(x,t_i) = \left(\Lambda^{-1}\sum_{i=1}^{p} c_i\psi(t_i)\varphi\right)(x) + \Lambda^{-1}\sum_{i=1}^{p}\left(c_i\int_0^{t_i}\psi^*(t_i-\eta)f(u)\right)(x,y)d\eta + \Lambda^{-1}\sum_{i=1}^{p}\left(c_i\int_0^{t_i}\int_0^{\eta}\psi^*(t_i-\eta)g(u)\right)(x,s)dW(s)d\eta. \tag{2.5}$$

Using (1.2) and (2.5), one gets (a.s. $[p]$)

$$(\psi(t)\varphi)(x) + \left(\psi(t)\sum_{i=1}^{p} c_i u\right)(x_i,t_i) = (\psi(t)\Lambda^{-1}\varphi)(x) + \psi(t)\Lambda^{-1}\sum_{i=1}^{p} c_i\int_0^{t_i}(\psi^*(t_i-\eta)f(u))(x,\eta)d\eta + \psi(t)\Lambda^{-1}\sum_{i=1}^{p} c_i\int_0^{t_i}\int_0^{\eta}(\psi^*(t_i-\eta)g(u))(x,s)dW(s)d\eta.$$

Hence the required result follows. ■

Now we define a mild solution of equation (2.1) to be an F_t- adapted stochastic process $u \in ([0,T];H)$ which satisfies equation (2.4), a.s. $[p]$.

Theorem 2.2. If $\|f\|_C + \|g\|_C \le K$ for all $u \in S_\gamma = \{u \in C([0,T];H)\ \|u\|_C \le \gamma\}$ and if

$$ME(\|\Lambda^{-1}\varphi\|_H) + KM^2T^\alpha\|\Lambda^{-1}\|_H + KMT^\alpha + MKT^{1+\alpha} + KM^2T^{1+\alpha} \le \gamma, \tag{2.6}$$

where K and γ are positive constants, then there exists a mild solution of equation (2.1), a.s. $[p]$.

Proof. Let Q be a map defined on S_γ by

$$(Qu)(x,t) = (\psi(t)\Lambda^{-1}\varphi)(x) + \psi(t)\Lambda^{-1}\sum_{i-1}^{p} c_i\int_0^{t_i}(\psi^*(t_i-\eta)f(u)(x,\eta)d\eta + \psi(t)\Lambda^{-1}\sum_{i-1}^{p} c_i\int_0^{t_i}\int_0^{\eta}(\psi^*(t_i-\eta)g(u)(x,s)dW(s)d\eta + \int_0^{t}(\psi^*(t-\eta)f(u))(x,\eta)d\eta + \int_0^{t}\int_0^{\eta}(\psi^*(t-\eta)g(u)(x,s)dW(s)d\eta.$$

Using (2.6) and noting that $\int_0^\infty \theta\xi_\alpha(\theta)d\theta = 1$, one gets,

$$E(\|Qu_t\|_H) \leq \gamma,$$

where

$$\|v_t\|_H^2 = \int_{\mathbb{R}^n} \sum_{i=1}^k v_i^2(x,t)dx.$$

Thus Q maps $C([0,T];H)$ into itself. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&E\left\|Qu_{t_1} - Qu_{t_2}\right\|_H \\
&\quad \leq [E(\left\|\Lambda^{-1}\varphi\right\|_H) + 2cKT^\alpha \left\|\Lambda^{-1}\right\|_H] \left\|\psi(t_1) - \psi(t_2)\right\|_H \\
&\qquad + 2K \int_{t_1}^{t_2} \left\|\psi^*(t-\eta)\right\|_H d\eta + 2M \int_0^{t_1} \left\|\psi^*(t_1-\eta) - \psi^*(t_2-\eta)\right\|_H d\eta. \quad (2.7)
\end{aligned}$$

It can be proved that

$$\left\|[\psi(t_2) - \psi(t_1)]w\right\|_H \leq \alpha M[\log t_2 - \log t_1] \left\|w\right\|_H. \quad (2.8)$$

Consequently $\left\|\psi(t_2) - \psi(t_1)\right\|_H$ tends to zero as s tends to t. Similarly

$$\lim_{t_1 \to t_2} \int_0^{t_1} \left\|\psi^*(t_2-\eta) - \psi^*(t_1-\eta)\right\|_H d\eta = 0, \quad (2.9)$$

$$\lim_{t_1 \to t_2} \int_{t_1}^{t_2} \left\|\psi^*(t-\eta)\right\|_H d\eta = 0, \qquad \text{(see [7])}. \quad (2.10)$$

The right-hand side of inequality (2.7) is independent of u and by using (2.8), (2.9) and (2.10), one gets that $[\|Qu_{t_1} - Qu_{t_2}\|_H]$ tends to zero as $t_1 \to t_2$. Now it is clear that by Arzela–Ascoli's theorem, $\{(Qu)(x,t) : u \in C([0,T];H)\}$ is precompact. Hence by Leray–Schauder's fixed point theorem, Q has a fixed point in $C([0,T];H)$ and any fixed point of Q represents a mild solution of equation (2.1), a.s. [p]. This completes the proof of the theorem. ■

Theorem 2.3. Suppose the Lipschitz condition

$$E[\|f(u) - f(v)\|_H] + E[\|g(u) - g(v)\|_H] \leq K_1 E(\|u - v\|_H)$$

is satisfied, where $K_1 > 0$ is a constant. If Equation (2.1) has a mild solution $u \in C([0,T];H)$, then that mild solution is unique on $[0,T]$.

Proof. For $u, v, \in C([0,T];H)$ we infer from (2.4) that

$$\begin{aligned} E\left\|u_t - v_t\right\|_H^2 \le \frac{K_2 T^\alpha}{\alpha} & \int_0^t (t-\eta)^{\alpha-1} E\left\|u_\eta - v_\eta\right\|_H^2 d\eta \\ & + \frac{K_3 T^\alpha}{\alpha} \sum_{i=1}^p |c_i| \int_0^{t_i} (t_i-\eta)^{\alpha-1} E\left\|u_\eta - v_\eta\right\|_H^2 d\eta \\ & + K_2 \int_0^t (t-s)^\alpha E\left\|u_s - v_s\right\|_H^2 ds \\ & + K_3 \sum_{i=1}^p |c_i| \int_0^{t_i} (t_i-s)^\alpha E\left\|u_s - v_s\right\|_H^2 ds. \end{aligned} \tag{2.11}$$

Let $\rho(u,v) = \sup_{t\in[0,T]} (e^{-\lambda t} E\left\|u_t - v_t\right\|_H^2)$, where $\lambda > 1$. It is easy to see that

$$\begin{aligned} & \int_0^{t_i} (t_i-\eta)^{\alpha-1} E\left\|u_\eta - v_\eta\right\|_H^2 d\eta \\ & \quad \le \left[\lambda^{1-\alpha} \int_0^{t_i - \frac{1}{\lambda}} d\eta\right] \rho(u,v) \\ & \quad \le e^{\lambda t_i} \left(1 + \frac{1}{\alpha}\right) \left(\frac{1}{\lambda}\right)^\alpha \rho(u,v). \end{aligned} \tag{2.12}$$

Let us consider the following two cases. The first one is $t \ge t_p$ and the second is $t \le t_p$. For the first case $t \ge t_p$, we deduce from (2.11) and (2.12) that

$$e^{-\lambda t} E\left\|u_t - v_t\right\|_H^2 \le K_4 \left[\left(1 + \frac{1}{\lambda}\right)^\alpha + \frac{1}{\lambda}\right] \rho(u,v).$$

Thus $\rho(u,v) = 0$, for sufficiently large λ, ($K_4 > 0$ is a constant). For the second case $t \le t_p$, one gets

$$E\left\|u_t - v_t\right\|_H^2 \le K_5 t_p^\alpha \sup_{t\in[0,T]} E\left\|u_t - v_t\right\|_H^2, \ (K_5 > 0 \text{ is a constant}).$$

Now if t_p is sufficiently small such that $K_5 t_p^\alpha < 1$, we get

$$\sup E\left\|u_t - v_t\right\|_H^2 = 0.$$

This completes the proof of the theorem. ■

Acknowledgement

The author is very grateful to the anonymous referees and to Professor Martin Bohner for their careful reading of the paper and their valuable comments.

References

[1] Ronald L. Bagley and Peter J. Torvik. Fractional calculus in the transient analysis of viscoelastically damped structures. *AIAA J.*, 23:918–925, 1985.

[2] S. D. Èĭdel′man. On fundamental solutions of parabolic systems. *Mat. Sb. N.S.*, 38(80):51–92, 1956.

[3] Mahmoud M. El-Borai. Some probability densities and fundamental solutions of fractional evolution equations. *Chaos Solitons Fractals*, 14(3):433–440, 2002.

[4] Mahmoud M. El-Borai. The fundamental solutions for fractional evolution equations of parabolic type. *J. Appl. Math. Stoch. Anal.*, (3):197–211, 2004.

[5] Mahmoud M. El-Borai. Semigroups and some nonlinear fractional differential equations. *Appl. Math. Comput.*, 149(3):823–831, 2004.

[6] Mahmoud M. El-Borai. On some fractional differential equations in the Hilbert space. *Discrete Contin. Dyn. Syst.*, (suppl.):233–240, 2005.

[7] Mahmoud M. El-Borai. On some fractional evolution equations with nonlocal conditions. *Int. J. Pure Appl. Math.*, 24(3):405–413, 2005.

[8] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, Osama L. Mostafa, and Hamdy M. Ahmed. Volterra equations with fractional stochastic integrals. *Math. Probl. Eng.*, (5):453–468, 2004.

[9] Mahmoud M. El-Borai, Khairia El-said El-Nadi, Osama L. Mostafa, and Hamdy M. Ahmed. Numerical methods for some nonlinear stochastic differential equations. *Appl. Math. Comput.*, 168(1):65–75, 2005.

[10] Khairia El-Said El-Nadi. On some stochastic parabolic differential equations in a Hilbert space. *J. Appl. Math. Stoch. Anal.*, (2):167–173, 2005.

[11] Khairia El-Said El-Nadi. On some stochastic differential equations and fractional Brownian motion. *Int. J. Pure Appl. Math.*, 24(3):415–423, 2005.

[12] Mikael Enelund and B.Lennart Josefson. Time-domain finite element analysis of viscoelastic structures with fractional derivatives constitutive relations. *AIAA J.*, 35(10):1630–1637, 1997.

[13] Christian Friedrich. Linear viscoelastic behavior of branched polybutadiens: a fractional calculus approach. *Acta Polym.*, 46:385–390, 1995.

[14] T. E. Govindan. Autonomous semilinear stochastic Volterra integro-differential equations in Hilbert spaces. *Dynam. Systems Appl.*, 3(1):51–74, 1994.

[15] T. E. Govindan and M. C. Joshi. Stability and optimal control of stochastic functional-differential equations with memory. *Numer. Funct. Anal. Optim.*, 13(3-4):249–265, 1992.

[16] Akira Ichikawa. Stability of semilinear stochastic evolution equations. *J. Math. Anal. Appl.*, 90(1):12–44, 1982.

[17] David N. Keck and Mark A. McKibben. Functional integro-differential stochastic evolution equations in Hilbert space. *J. Appl. Math. Stochastic Anal.*, 16(2):141–161, 2003.

[18] Kazimierz Sobczyk. *Stochastic differential equations*, volume 40 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.