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On Existence of Positive Solutions for Linear Difference Equations with Several Delays

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Abstract

We study the existence of positive solutions for a scalar linear difference equation with several delays:

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n) x(h_l(n)), \;\; h_l(n) \le n, \;\; n > n_0.$$

Nonoscillation criteria, comparison theorems and some explicit nonoscillation results are presented. Some known nonoscillation tests for equations with constant delays and with one variable delay are obtained as special cases.

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1. Introduction

Recently many publications on oscillation of linear difference equations appeared (see, for example, [7, 12]), including monograph [1]. Nonoscillation of difference equations is less studied compared to sufficient oscillation conditions.

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Let us discuss some results on the existence of a positive solution for linear nonautonomous difference equations with several delays. First recall some known results. For the equation

$$x(n+1) - x(n) = -p(n)x(n-k) \tag{1.1}$$

Erbe and Zhang [3] proved: if

$$0 \le p(n) \le \frac{k^k}{(k+1)^{k+1}},$$

then Eq. (1.1) has a nonoscillatory solution.

Ladas in 1990 conjectured [5] that the condition

$$p(n) \ge 0, \ \frac{1}{k} \sum_{i=n-k}^{n-1} p(i) \le \frac{k^k}{(k+1)^{k+1}}$$

implies nonoscillation of Eq. (1.1). However this conjecture is not true and a counterexample was given by Yu, Zhang, Wang [10].

Tang and Yu [8] proved: if

$$p(n) \ge 0, \ \ \sum_{i=n-k}^{n} p(i) \le \frac{1}{e},$$

then Eq. (1.1) has a nonoscillatory solution. As a corollary Tang and Yu obtained the following "corrected Ladas conjecture": if

$$p(n) \ge 0, \ \frac{1}{k} \sum_{i=n-k}^{n} p(i) \le \frac{k^k}{(k+1)^{k+1}},$$

then Eq. (1.1) has a nonoscillatory solution.

Zhang and Tian [11] studied the equation with one variable delay

$$x(n+1) - x(n) = -p(n)x(h(n)), \ h(n) \le n, \ \ \lim_{n\to\infty} h(n) = \infty. \tag{1.2}$$

They proved that if

$$\sum_{i=h(n)}^{n-1} p^+(i) \le \frac{1}{4},$$

then Eq. (1.2) has a nonoscillatory solution. The constant $\frac{1}{4}$ is the best possible one since for the equation

$$x(n+1) - x(n) = -px(n-1), \ \ p > 0$$

the condition $p \le \dfrac{1}{4}$ is necessary and sufficient for nonoscillation. One of the purposes of the present paper is to extend the result of Zhang and Tian to difference equations with several variable delays.

Interesting results on nonoscillation were obtained by Philos, Purnaras [6].

In the present paper we consider a scalar linear difference equation with several delays. After some preliminaries (Section 2) we present nonoscillation criteria (Section 3), comparison theorems (Section 4) and explicit nonoscillation and oscillation results (Section 5). Section 6 contains some numerical examples. For difference equations with variable delays we extend several results, which are well known for delay differential equations. The main tool in this investigation is the solution representation formula and properties of the fundamental function.

2. Preliminaries

Consider a scalar linear difference equation with several delays:

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n)x(h_l(n)) + f(n), \;\; n \ge n_0, \tag{2.1}$$

$$x(n) = \varphi(n), \;\; n \le n_0, \tag{2.2}$$

where $h_l(n)$ are integers satisfying $h_l(n) \le n$, $\lim_{n\to\infty} h_l(n) = \infty, \; l = 1, 2, \dots, m$.

Further we will extensively apply the solution representation formula and properties of the fundamental function. We start with the definition of this function.

Definition 2.1. The solution $X(n,k)$ of the problem

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n)x(h_l(n)), \; n \ge k, \quad x(n) = 0, \; n < k, \;\; x(k) = 1$$

is called **the fundamental function** of Eq. (2.1).

The following lemma is a corollary of the Elaydi [2] solution representation formula.

Lemma 2.2. For the solution $x(n)$ of problem (2.1), (2.2) we have the following representation:

$$\begin{aligned} x(n) = X(n,n_0)x(n_0) &+ \sum_{k=n_0}^{n-1} X(n,k+1)f(k) \\ &- \sum_{k=n_0}^{n-1} X(n,k+1)\sum_{l=1}^{m} a_l(k)\varphi(h_l(k)), \end{aligned} \tag{2.3}$$

where $\varphi(h_l(k)) = 0, h_l(k) \ge n_0$.

As an application of Lemma 2.2, let us present the following example.

Example 2.3. Consider the linear difference equation

$$x(n+1) - x(n) + u(n)x(n) = 0,$$

its fundamental function can be easily computed as

$$X(n,k) = \prod_{j=k}^{n-1} [1 - u(j)].$$

Thus the general solution of the nonhomogeneous equation

$$x(n+1) - x(n) + u(n)x(n) = z(n),\ n \geq n_0,\ \ x(n_0) = x_0$$

can be presented as

$$x(n) = x_0 \prod_{j=n_0}^{n-1} [1 - u(j)] + \sum_{k=n_0}^{n-1} z(k) \prod_{j=k+1}^{n-1} [1 - u(j)]. \tag{2.4}$$

Here and in future we assume any product which does not involve any factors is equal to one and any sum which does not include any terms is equal to zero.

3. Existence of Positive Solutions

Consider now the homogeneous equation and corresponding inequalities:

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n)x(h_l(n)),\ \ n \geq 0, \tag{3.1}$$

$$y(n+1) - y(n) \leq -\sum_{l=1}^{m} a_l(n)y(h_l(n)),\ \ n \geq 0, \tag{3.2}$$

$$z(n+1) - z(n) \geq -\sum_{l=1}^{m} a_l(n)z(h_l(n)),\ \ n \geq 0. \tag{3.3}$$

The following result contains nonoscillation criteria.

Theorem 3.1. Suppose $a_l(n) \geq 0$, $l = 1, 2, \dots, m$. Then the following conditions are equivalent:

1) Eq. (3.1) has an eventually positive solution.

2) Inequality (3.2) has an eventually positive solution or (3.3) has an eventually negative solution.

3) There exists a sequence $\{u(n)\},\ n \geq 0$, and a number $n_0 \geq 0$, such that $0 \leq u(n) < 1$ and

$$u(n) \geq \sum_{l=1}^{m} a_l(n) \prod_{k=h_l(n)}^{n-1} [1-u(k)]^{-1},\ n \geq n_0. \tag{3.4}$$

4) The fundamental function $X(n,k)$ is eventually positive: there exists $n_0 \geq 0$ such that $X(n,k) > 0,\ n \geq k \geq n_0$.

If (3.4) holds for $n \geq n_0$, then $X(n,k) > 0,\ n \geq k \geq n_0$.

Proof. Let us prove the implications $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1)$.

$1) \Rightarrow 2)$ is obvious since any solution of (3.1) is also a solution of inequality (3.2).

$2) \Rightarrow 3)$. Suppose $\{x(n)\}$ is a solution of (3.2) which is positive beginning with some n_1. Then the sequence is nonincreasing

$$x(n+1) \leq x(n) - \sum_{l=1}^{m} a_l(n) x(h_l(n)) < x(n),$$

beginning with such n_0 that $h_l(n) > n_1$, $n > n_0$, for any l. Thus for

$$u(n) = \frac{x(n) - x(n+1)}{x(n)}$$

we have $0 \leq u(n) < 1$.

Consider an auxiliary equation

$$x(n+1) - x(n) + u(n)x(n) = 0, \quad \text{or} \qquad x(n+1) = [1-u(n)]x(n),\ n \geq n_0.$$

Thus $x(n) = x(n_0) \prod_{j=n_0}^{n-1} (1-u(j))$. Substituting this into (3.2) we have

$$x(n_0) \prod_{j=n_0}^{n} (1-u(j)) - x(n_0) \prod_{j=n_0}^{n-1} (1-u(j)) + \sum_{l=1}^{m} a_l(n) x(n_0) \prod_{j=n_0}^{h_l(n)-1} (1-u(j)) \leq 0.$$

After dividing by a positive factor of $x(n_0) \prod_{j=n_0}^{n-1} (1-u(j))$ we have for $n \geq n_0$

$$1 - u(n) - 1 + \sum_{l=1}^{m} a_l(n) \prod_{k=h_l(n)}^{n-1} [1-u(k)]^{-1} \leq 0,$$

which is equivalent to (3.4). Here, if (3.4) holds for some positive $u(k)$, then (3.4) is also satisfied with the initial conditions $u(k)=0$, $k<n_0$. This is equivalent to the existence of a positive solution of (3.2), with the initial conditions $x(k)=x(n_0)$, $k<n_0$.

$3) \Rightarrow 4)$. Consider the initial value problem (2.1), (2.2) with the zero initial conditions: $x(n)=0$, $n\le n_0$. We will apply an auxiliary equation

$$x(n+1)-x(n)+u(n)x(n)=z(n), \tag{3.5}$$

which has a solution (see (2.4)) $x(n)=\sum_{k=n_0}^{n-1} z(k)\prod_{j=k+1}^{n-1}[1-u(j)]$, where $u(n)$ are as in (3.4). Substituting this into (2.1) and applying $x(n+1)-x(n)=z(n)-u(n)x(n)$ we obtain

$$\begin{aligned} z(n) &- \sum_{k=n_0}^{n-1} u(n)z(k)\prod_{j=k+1}^{n-1}[1-u(j)] \\ &+\sum_{l=1}^{m} a_l(n)\sum_{k=n_0}^{h_l(n)-1} z(k)\prod_{j=k+1}^{h_l(n)-1}[1-u(j)] = f(n). \end{aligned}$$

Hence

$$\begin{aligned} z(n) &= \sum_{k=n_0}^{n-1} u(n)z(k)\prod_{j=k+1}^{n-1}[1-u(j)] \\ &\quad -\sum_{l=1}^{m} a_l(n)\sum_{k=n_0}^{h_l(n)-1} z(k)\prod_{j=k+1}^{h_l(n)-1}[1-u(j)]+f(n) \\ &= \sum_{k=n_0}^{n-1} u(n)z(k)\prod_{j=k+1}^{n-1}[1-u(j)] - \sum_{k=n_0}^{n-1} z(k)\sum_{l=1}^{m} a_l(n)\prod_{j=k+1}^{h_l(n)-1}[1-u(j)] \\ &\quad +\sum_{l=1}^{m} a_l(n)\sum_{k=h_l(n)}^{n-1} z(k)\prod_{j=k+1}^{h_l(n)-1}[1-u(j)]+f(n) \\ &= \sum_{k=n_0}^{n-1} z(k)\left\{u(n)\prod_{j=k+1}^{n-1}[1-u(j)]-\sum_{l=1}^{m} a_l(n)\prod_{j=k+1}^{h_l(n)-1}[1-u(j)]\right\}+g(n) \\ &= \sum_{k=n_0}^{n-1} z(k)\prod_{j=k+1}^{n-1}[1-u(j)]\left\{u(n)-\sum_{l=1}^{m} a_l(n)\prod_{h_l(n)}^{n-1}(1-u(j))^{-1}\right\}+g(n), \end{aligned}$$

where $g(n) = \sum_{l=1}^{m} a_l(n) \sum_{k=h_l(n)}^{n-1} z(k) \prod_{j=k+1}^{h_l(n)-1} [1-u(j)] + f(n)$. Here by (3.4) the expression in the braces is nonnegative. If $f(n)$ is nonnegative for any n, then $g(n)$ is nonnegative and by induction $z(n)$ is also nonnegative.

On one hand, since $x(n)$ has the representation (2.3) and $0 \le u(j) < 1$, then $\{x(n)\} \ge 0$ as far as $\{f(n)\} \ge 0$ and $x(n_0) \ge 0$. On the other hand, by (2.3) the solution of (2.1), (2.2), with the zero initial conditions, can be written as

$$x(n) = \sum_{k=n_0}^{n-1} X(n, k+1) f(k).$$

We can assume $f(k) = 1, f(i) = 0, i \neq k$ for any k, which implies $X(n, k+1) \ge 0$. Further, representation (2.3) yields that if $z(k) \ge 0$ for any k and $z(n) > 0$, then $x(l) > 0, l \ge n+1$. Thus $f(k) > 0$ implies $z(k) > 0$, which yields that $X(n, k+1) > 0$, $n \ge k+1$, consequently, the fundamental function $X(n, k)$ is positive for $n \ge k \ge n_0$.

$4) \Rightarrow 1)$. A sequence $X(n, n_0)$ is a positive solution of (3.1). ■

Assuming constant $u(n) = 1 - \mu$ we obtain as a corollary the following sufficient condition for the existence of an eventually positive solutions which is a part of [4, Theorem 7.8.1] and partially extends this theorem to the case of variable delays.

Corollary 3.2. Suppose there exists $\mu \in (0, 1]$ such that for some $n_0 \ge 0$ and $n \ge n_0$ the following inequality holds

$$1 - \mu \ge \sum_{l=1}^{m} a_l(n) \mu^{h_l(n)-n}.$$

Then (3.1) has an eventually positive solution and for fundamental function of this equation we have $X(n, k) > 0, \ n \ge k \ge n_0$.

The following is a corollary of the proof of the implication $2) \Rightarrow 3)$ and the remark in the end of the proof.

Corollary 3.3. If $u(n)$ is a solution of (3.4), then

$$x(n) = x(n_0) \prod_{k=n_0}^{n-1} [1 - u(k)], \ n > n_0, \ \ x(n) = x(n_0), \ n \le n_0,$$

is a solution of inequality (3.2).

Inequalities similar to (3.4) appear in most papers on nonoscillation as implicit sufficient conditions for the existence of a positive solution. A discussion on these conditions can be found in the paper by Zhang and Tian [11] for equations with one delay. Difference equations with several variable delays were studied by Zhou [12]. As a necessary and sufficient condition for the positiveness of the fundamental function, condition 3) of Theorem 3.1 probably appears for the first time.

4. Comparison Theorems

As the first application of the nonoscillation criteria given in Theorem 3.1 we will obtain several comparison results. To this end consider together with Eq. (3.1) the following one

$$x(n+1) - x(n) = -\sum_{l=1}^{m} b_l(n) x(g_l(n)), \ n \geq 0, \tag{4.1}$$

where $g_l(n) \leq n$. Denote by $Y(n,k)$ the fundamental function of Eq. (4.1).

Theorem 4.1. Suppose $a_l(n) \geq b_l(n) \geq 0, \ g_l(n) \geq h_l(n)$ for sufficiently large n.

If Eq. (3.1) has an eventually positive solution, then Eq. (4.1) has an eventually positive solution and its fundamental function $Y(n,k)$ is eventually positive.

If all solutions of Eq. (4.1) are oscillatory, then all solutions of Eq. (3.1) are oscillatory.

Proof. By Theorem 3.1 there exists a positive solution $u(n)$ of inequality (3.4) for some $n_0 \geq 0$ and $n \geq n_0$. Since $[1-u(k)]^{-1} > 1$, $g_l(n) \geq h_l(n)$ and $a_l(n) \geq b_l(n) \geq 0$, then

$$u(n) \geq \sum_{l=1}^{m} a_l(n) \prod_{k=h_l(n)}^{n-1} [1-u(k)]^{-1} \geq \sum_{l=1}^{m} b_l(n) \prod_{k=g_l(n)}^{n-1} [1-u(k)]^{-1}.$$

Applying Theorem 3.1 once again, we deduce that the fundamental function $Y(n,k)$ of (4.1) is eventually positive; moreover, it is positive for $n \geq k \geq n_0$.

The second part of the theorem is a corollary of the first one. ■

Comparison results are very popular in literature. Our result probably is given in the most general form: several variable delays are included and both coefficients and delays are compared. Similar results were published by Yan and Qian [9], but this paper involves some wrong statements.

In Theorem 4.1 we assumed $b_l(n) \geq 0$. For some further statements, let us avoid this assumption. To this end consider together with Eq. (3.1) the following one

$$x(n+1) - x(n) = -\sum_{l=1}^{m} b_l(n) x(h_l(n)), \ n \geq 0. \tag{4.2}$$

Denote as before by $X(n,k), Y(n,k)$ the fundamental functions of (3.1) and (4.2), respectively.

Theorem 4.2. Suppose $a_l(n) \geq 0, \ a_l(n) \geq b_l(n), \ X(n,k) > 0, n \geq k \geq n_0 \geq 0$. Then $Y(n,k) \geq X(n,k) > 0, n \geq k \geq n_0 \geq 0$.

Proof. Without loss of generality, assume $n_0 = 0$. Denote $x(n) = X(n,0), \ y(n) = Y(n,0)$. We have

$$y(n+1) - y(n) = -\sum_{l=1}^{m} a_l(n) y(h_l(n)) + \sum_{l=1}^{m} [a_l(n) - b_l(n)] y(h_l(n)).$$

Thus by solution representation formula (2.3)

$$y(n) = x(n) + \sum_{k=0}^{n-1} X(n,k+1) \sum_{l=1}^{m} [a_l(k) - b_l(k)] y(h_l(k)).$$

We have $y(n) = 0, n < 0, y(0) = 1$, $x(n) > 0$, $n \geq 0$, $X(n,k+1) > 0$, $n \geq 0$, $a_l(k) - b_l(k) \geq 0$. Hence by induction $y(n) > 0, n > 0$. Then $y(n) = Y(n,0) \geq x(n) = X(n,0)$. Similarly $Y(n,k) \geq X(n,k) > 0$, $n \geq k \geq 0$. ■

Corollary 4.3. Suppose the inequality

$$y(n+1) - y(n) \leq -\sum_{l=1}^{m} a_l^+(n) y(h_l(n)),\ n \geq 0,$$

has an eventually positive solution, where $a^+ = \max\{0, a\}$. Then Eq. (3.1) has an eventually positive solution.

Proof. By Theorem 3.1 the fundamental function of the equation

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l^+(n) x(h_l(n))$$

is eventually positive. By Theorem 4.2 the fundamental function of (3.1) is also eventually positive. ■

Compare now solutions of two difference equations. To this end consider together with Eq. (2.1) the following comparison equation with the same initial conditions:

$$x(n+1) - x(n) = -\sum_{l=1}^{m} b_l(n) x(h_l(n)) + r(n),\ \ n \geq n_0, \tag{4.3}$$

$$x(n) = \varphi(n),\ \ n \leq n_0. \tag{4.4}$$

Theorem 4.4. Suppose $a_l(n) \geq b_l(n), n \geq n_0,\ r(n) \geq f(n), X(n,k) > 0, n \geq k \geq n_0$. If $x(n) > 0$, then $y(t) \geq x(n)$, where $x(n)$ and $y(n)$ are solutions of (2.1), (2.2) and (4.3), (4.4), respectively.

Proof. By Theorem 4.2 we have $Y(n,k) \geq X(n,k) > 0,\ n \geq k \geq n_0$. Let us rewrite (2.1) in the form

$$x(n+1) - x(n) = -\sum_{l=1}^{m} b_l(n) x(h_l(n)) - \sum_{l=1}^{m} [a_l(n) - b_l(n)] x(h_l(n)) + f(n).$$

Applying representation (2.3), positiveness of x and inequalities assumed, we obtain

$$\begin{aligned}
x(n) &= Y(n,k)x(n_0) + \sum_{k=n_0}^{n-1} Y(n,k+1)f(k) - \sum_{k=n_0}^{n-1} Y(n,k+1)\sum_{l=1}^{m} b_l(k)\varphi(h_l(k)) \\
&\quad - \sum_{k=n_0}^{n-1} Y(n,k+1)\sum_{l=1}^{m}[a_l(k)-b_l(k)]x(h_l(k)) \\
&\le Y(n,k)y(n_0) + \sum_{k=n_0}^{n-1} Y(n,k+1)r(k) \\
&\quad - \sum_{k=n_0}^{n-1} Y(n,k+1)\sum_{l=1}^{m} b_l(k)\varphi(h_l(k)) = y(n),
\end{aligned}$$

which completes the proof. ∎

Corollary 4.5. Suppose $a_l(n) \ge 0, X(n,k) > 0, n \ge k \ge n_0$. If $y(n) > 0,\ n > n_0$, and $y(n) = z(n) = x(n),\ n \le n_0$, then $y(n) \le x(n) \le z(n),\ n > n_0$, where $y(n)$ and $z(n)$ are solution of difference inequalities (3.2) and (3.3), respectively, and $x(n)$ is a solution of difference equation (3.1).

Corollary 4.6. Suppose $X_0(n,k) > 0, n \ge k \ge n_0$, is the fundamental function of the equation

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l^+(n)x(h_l(n))$$

and $x_0(n) > 0,\ n \ge n_0$, is a solution of this equation. Suppose $x(n) = x_0(n),\ n \le n_0$, where $x(n)$ is a solution of (3.1). Then $x(n) \ge x_0(n),\ n > n_0$.

5. Explicit Nonoscillation and Oscillation Conditions

To obtain explicit conditions for positiveness of the fundamental function we apply Theorem 3.1. By this theorem, if inequality (3.4) has a nonnegative solution for $n \ge n_0$, then $X(n,k) > 0,\ n \ge k \ge n_0$. In the following theorem we also use some ideas from [11], where the authors obtain nonoscillation conditions for Eq. (3.1) with $m = 1$.

Theorem 5.1. Suppose for some $n_0 \ge 0$

$$\sup_{n\ge n_0}\sum_{l=1}^{m} a_l^+(n) < \frac{1}{2}, \quad \sup_{n\ge n_0}\sum_{l=1}^{m}\sum_{k=\max\{n_0,\min_l h_l(n)\}}^{n-1} a_l^+(k) \le \frac{1}{4}. \tag{5.1}$$

Then for Eq. (3.1) we have $X(n,k) > 0,\ n \ge n_0$.

Proof. By Corollary 4.3 it is sufficient to prove the theorem for $a_l(n) \geq 0$ only. Inequality (3.4) is a corollary of the following one

$$u(n) \geq a(n) \prod_{k=h(n)}^{n-1} [1-u(k)]^{-1}, \ n \geq n_0, \tag{5.2}$$

where

$$a(n) := \sum_{l=1}^{m} a_l(n), \ \ h(n) := \min_l h_l(n).$$

The first inequality in (5.1) implies $0 \leq a(n) < \dfrac{1}{2}$.

We will show that $u(n) = \begin{cases} 2a(n), & n \geq n_0, \\ 0, & n < n_0, \end{cases}$ is a solution of (5.2), such that $0 \leq u(n) < 1$. It is equivalent to the inequality

$$\prod_{k=h(n)}^{n-1} [1-2a(n)] \geq \frac{1}{2}.$$

We have

$$\prod_{k=h(n)}^{n-1} (1-2a(n)) \geq 1 - 2\sum_{k=h(n)}^{n-1} a(n) \geq 1 - 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

Hence the sequence $\{u(n)\}$ is a solution of inequality (3.4). Thus by Theorem 3.1 the fundamental solution of Eq. (3.1) is positive. ∎

The following theorem is an analogue of the well-known result for delay differential equations ([4, Theorem 3.3.1]). To the best of our knowledge such results for linear difference equations have not been published yet.

Theorem 5.2. Suppose $a_l(n) \geq 0$ and for some $n_0 \geq 0$ (5.1) holds. If $x(n_0) > 0$, $0 \leq \varphi(n) \leq x(n_0)$, then for solution $x(n)$ of (3.1), (2.2) we have $x(n) > 0, \ n > n_0$.

Proof. By the proof of Theorem 5.1 the sequence

$$u(n) = \begin{cases} 2\displaystyle\sum_{l=1}^{m} a_l(n), & n \geq n_0, \\ 0, & n < n_0, \end{cases}$$

is a solution of inequality (3.4) such that $0 \leq u(n) < 1$. By Corollary 3.3

$$y(n) = \begin{cases} x(n_0) \displaystyle\prod_{l=n_0}^{n-1} [1-u(l)], & n \geq n_0, \\ x(n_0), & n < n_0, \end{cases}$$

is a solution of (3.2) with initial conditions $y(n) = x(n_0), n \leq n_0$.

Solution representation formula (2.3) implies

$$x(n) = X(n,n_0)x(n_0) - \sum_{k=n_0}^{n-1} X(n,k+1) \sum_{l=1}^{m} a_l(k)\varphi(h_l(k)),$$

where $\varphi(h_l(k)) = 0, h_l(k) \geq n_0$. By Corollary 4.5 the sequence $y(n)$ does not exceed the solution of (3.1), with $\varphi(n) = x(n_0)$, $n < n_0$, thus

$$y(n) = X(n,n_0)x(n_0) - \sum_{k=n_0}^{n-1} X(n,k+1) \sum_{l=1}^{m} a_l(k)x(n_0)\xi(h_l(k)) - g(n),$$

where $\xi(h_l(k)) = 0, h_l(k) \geq n_0$, $\xi(h_l(k)) = 1$, $h_l(k) \leq n_0$, and $g(n)$ is a nonnegative function.

Since $0 \leq \varphi(n) \leq x(n_0)$, we have $x(n) \geq y(n) > 0, n \geq n_0$. ■

The following result demonstrates that condition (5.1) cannot be replaced by a more general condition $X(n,k) > 0, n \geq k \geq n_0$.

Example 5.3. The solution of the equation

$$x(n+1) = x(n) - 10^{2-4n}x(n-1) \tag{5.3}$$

with positive increasing initial conditions $x(-1) = 0.1, x(0) = 1$ is not positive:

$$x(1) = x(0) - 100x(-1) = 1 - 10 = -9.$$

The fundamental function $X(n,k) > 0$ for $n \geq k \geq 0$. For example, the positiveness of $X(n,0)$ is obtained by the immediate computation $X(0,0) = X(1,0) = 1$,

$$X(2,0) = 1 - 0.01 = 0.99,\ X(3,0) = 0.99 - 10^{-6}, \dots, 1 - \frac{0.01}{1 - 0.0001} < X(n,0) < 1.$$

Condition (5.1) does not holds for $n \geq n_0$, where $n_0 = 0$ or $n_0 = 1$. If $n \geq n_0 = 2$, then this condition holds for Eq. (5.3). If we take as before $x(0) = 0.1, x(1) = 1$, then the solution of Eq. (5.3) is positive for $n \geq 2$.

However, with some additional restriction of the positiveness of a specific solution the condition $X(n,k) > 0, n \geq k \geq n_0$ is sufficient for the positiveness of all solutions, with a positive initial value and an initial function, which does not exceed this value.

Theorem 5.4. Suppose $a_l(n) \geq 0$ for $n \geq n_0$, $X(n,k) > 0, n \geq k \geq n_0$, $x(n_0) > 0$ and the solution of the initial value problem (3.1), (2.2), with $\varphi(n) = x(n_0)$ is positive. If in (2.2) $0 \leq \varphi(n) \leq x(n_0)$, then for the solution $x(n)$ of (3.1), (2.2) we have $x(n) > 0,\ n > n_0$.

Proof. Let $y(n)$ be a solution of the initial value problem (3.1), (2.2) with $\varphi(n) = x(n_0)$. Then (2.3) implies

$$\begin{aligned} 0 < y(n) &= X(n, n_0)x(n_0) - \sum_{k=n_0}^{n-1} X(n, k+1) \sum_{l=1}^{m} a_l(k) x(n_0) \xi(h_l(k)) \\ &\leq X(n, n_0)x(n_0) - \sum_{k=n_0}^{n-1} a_l(k)\varphi(h_l(k)) = x(n), \end{aligned}$$

where $\xi(h_l(k)) = 0, h_l(k) \geq n_0$, $\xi(h_l(k)) = 1, h_l(k) \leq n_0$. Thus $x(n) > 0$. ■

Remark 5.5. Eq. (3.1) is linear, so the condition that the solution with the initial function $\varphi(n) = x(n_0)$ can be changed by any constant initial function, say, $\varphi(n) = x(n_0) = 1$.

Remark 5.6. [4, Theorem 7.8.1] presents sufficient conditions for the positiveness of a solution with a given initial function for the equation with constant delays. However the conditions imposed on initial values are rather restrictive: $\varphi(n+1) \geq \mu\varphi(n)$, where μ is as in Corollary 3.2. The following example presents the case of an equation with constant delays, where [4, Theorem 7.8.1] fails to establish positiveness of the solution but our results work.

Example 5.7. The solution of the equation

$$x(n+1) - x(n) = -\frac{1}{32}x(n-4) - \frac{1}{32}x(n-2), \ n \geq 0,$$

with the initial conditions $x(-4) = 1$, $x(-3) = 0$, $x(-2) = 1$, $x(-1) = 0$, $x(0) = 1$ is positive by Theorem 5.2. [4, Theorem 7.8.1] cannot be applied since inequality $0 \geq \mu \cdot 1$ cannot be satisfied for a positive μ.

Remark 5.8. It is to be noted that, unlike delay differential equations, under the condition $\lim\limits_{n\to\infty} h_l(n) = \infty$ the set of initial functions of a difference equation is finite dimensional. Thus, taking any basis of this set, such that solutions are positive, we obtain that any initial function which is a linear combination of basis functions with positive coefficients, leads to a positive solution. For example, taking basis functions as $\varphi(n) = 1$, $j \leq n \leq 0$, we get the following result.

Suppose the solution $y_j(n)$ of (3.1), (2.2), with $\varphi(n) = 0$, $n < j$, $\varphi(n) = 1$, $j \leq n \leq 0$, is positive for any $j \geq H$. Then the solution of (3.1), (2.2) is positive for any nonnegative nondecreasing sequence $\varphi(n)$ and $\varphi(0) = x(0) > 0$.

In particular cases it is possible to check the positiveness of all basis solutions, as the following example illustrates.

Example 5.9. Consider the equation

$$x(n+1) - x(n) = -\frac{1}{n+k}\, x(n-1), \ n \geq 0, \tag{5.4}$$

where $k > 0$. For $k = 1$ and the initial function $x(-1) = x(0) = 1$ we have $x(1) = 0$, $x(2) = -0.5$, solutions of this equation are not necessarily positive for a nonnegative increasing initial function, for example, $x(-1) = 0.9,\ x(1) = 1,\ x(2) = 1 - 0.9 = 0.1,\ x(3) = 0.1 - 0.5 = -0.4$. For $k \geq 4$ the solution with $x(-1) = 0$, $x(0) = 1$ is positive (see [11]). It is easy to check that the solution with $x(-1) = x(0) = 1$ is also positive. So is any solution with $x(-1) \leq x(0)$, $x(0) > 0$.

By analyzing the proof of Theorem 5.2 we have the following comparison result.

Theorem 5.10. Suppose $a_l(n) \geq 0$ and for some $n_0 \geq 0$ inequality (3.4) has a solution $0 \leq u(n) < 1$. If $x(n)$ and $y(n)$ are two solutions of (3.1) such that $0 < x(n_0) = y(n_0)$, $0 \leq x(n) \leq y(n),\ n < n_0$, $y(n) > 0,\ n \geq n_0$, then $x(n) \geq y(n),\ n \geq n_0$.

Proof. Repeats the proof of Theorem 5.2. ■

A shortcoming of inequality (5.1) is in the application of the "worst" delay only. The following sufficient condition employs all delays.

Theorem 5.11. Suppose there exist $\lambda_1 \geq 1, \dots, \lambda_l \geq 1$ such that for $n \geq n_0$

$$\sum_{j=1}^{m} \lambda_j a_j^+(n) < 1,\ \sum_{j=1}^{m} \lambda_j \sum_{k=h_l(n)}^{n-1} a_j^+(k) \leq 1 - \frac{1}{\lambda_l},\ l = 1, \dots, m. \tag{5.5}$$

Then for Eq. (3.1) we have $X(n,k) > 0,\ n \geq k \geq n_0$.

Proof. By Corollary 4.3 it is sufficient to consider $a_l(n) \geq 0$. We will show that the function $u(n) = \sum_{l=1}^{m} \lambda_l a_l(n)$ is a solution of inequality (3.4), i.e.,

$$\sum_{l=1}^{m} \lambda_l a_l(n) \geq \sum_{l=1}^{m} a_l(n) \prod_{k=h_l(n)}^{n-1} \left[1 - \sum_{j=1}^{m} \lambda_j a_j(k) \right]^{-1}.$$

This inequality holds if

$$\lambda_l \geq \prod_{k=h_l(n)}^{n-1} \left[1 - \sum_{j=1}^{m} \lambda_j a_j(k) \right]^{-1},\ \ l = 1, \dots, m,$$

which can be written as

$$\prod_{k=h_l(n)}^{n-1} \left[1 - \sum_{j=1}^{m} \lambda_j a_j(k) \right] \geq \frac{1}{\lambda_l},\ \ l = 1, \dots, m.$$

The latter inequality, if the first inequality in (5.5) is valid, is a consequence of the following one

$$1 - \sum_{j=1}^{m} \lambda_j \sum_{k=h_l(n)}^{n-1} a_j(k) \geq \frac{1}{\lambda_l},\ \ l = 1, \dots, m,$$

which is equivalent to the second inequality in (5.5). ■

Remark 5.12. If $h_1(n) \equiv n$, then $\lambda_1 = 1$ and the system in Theorem 5.11 contains $l-1$ inequalities only.

Consider the equation with two delays:

$$x(n+1) - x(n) = -a(n)x(h(n)) - b(n)x(g(n)). \tag{5.6}$$

Corollary 5.13. Suppose there exist $\lambda_1 \geq 1, \lambda_2 \geq 1$ such that for $n \geq n_0$

$$\lambda_1 a^+(n) + \lambda_2 b^+(n) < 1,$$

$$\lambda_1 \sum_{k=h(n)}^{n-1} a^+(k) + \lambda_2 \sum_{k=h(n)}^{n-1} b^+(k) \leq 1 - \frac{1}{\lambda_1},$$

$$\lambda_1 \sum_{k=g(n)}^{n-1} a^+(k) + \lambda_2 \sum_{k=g(n)}^{n-1} b^+(k) \leq 1 - \frac{1}{\lambda_2}.$$

Then the fundamental solution of Eq. (5.6) is positive for $n \geq k \geq n_0$.

Corollary 5.14. Suppose $h(n) \equiv n$ and there exists $\lambda \geq 1$ such that for $n \geq n_0$

$$a^+(n) + \lambda b^+(n) < 1, \quad \sum_{k=g(n)}^{n-1} a^+(k) + \lambda \sum_{k=g(n)}^{n-1} b^+(k) \leq 1 - \frac{1}{\lambda}.$$

Then the fundamental solution of Eq. (5.6) is positive for $n \geq k \geq n_0$.

Let us note that Theorem 5.11 and its corollaries can be applied to equations with unbounded delays.

Now let us present an explicit oscillation result.

Lemma 5.15. [11] Suppose

$$\limsup_{n\to\infty} p(n) > 0, \tag{5.7}$$

$$\liminf_{n\to\infty} \sum_{i=h(n)}^{n-1} p(i) > \frac{1}{e}. \tag{5.8}$$

Then all solutions of Eq. (1.2) are oscillatory.

Theorem 5.16. Suppose $a_l(n) \geq 0$, $\limsup_{n\to\infty} \sum_{l=1}^{m} a_l(n) > 0$, and

$$\liminf_{n\to\infty} \sum_{l=1}^{m} \sum_{k=\max_l h_l(n)}^{n-1} a_l(k) > \frac{1}{e}. \tag{5.9}$$

Then all solutions of Eq. (3.1) are oscillatory.

Proof. Lemma 5.15 implies that all solutions of the equation

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n) x(\max_l h_l(n)) \tag{5.10}$$

are oscillatory. But Eq. (5.10) also has form (4.1), with $b_l = a_l$, $g_l = \max_l h_l(n)$. By Theorem 4.1 all solutions of Eq. (3.1) are oscillatory. ■

6. Numerical Examples

Example 6.1. Let us illustrate comparison Theorem 4.4 and Corollary 4.5. We consider the equation

$$x(n+1) - x(n) = f(n) - 0.02x(n-2) - 0.06x(n-3), \tag{6.1}$$

which by Theorem 5.1 has a positive fundamental function. Here the initial function and the initial value are equal to one, for solutions y, x and z of (6.1) we have $f(n) \equiv -0.01, 0, 0.01$, respectively. See Fig. 1 for illustration.

Example 6.2. Now let us analyze the sharpness of the condition (5.1) for the equation with two delays

$$x(n+1) - x(n) = -ax(n-2) - bx(n-3). \tag{6.2}$$

Consider the solution with $x(0) = \varphi(n) \equiv 1$. According to (5.1) the solution should be positive for $3(a+b) \leq 0.25$, or $b < 0.25/3 - a$. We also compare the result to the following estimate

$$\sup_{n \geq n_0} \sum_{l=1}^{m} \sum_{k=h_l(n)}^{\max\{n-1, h_l(n)\}} a_l(k) \leq \frac{1}{4}, \tag{6.3}$$

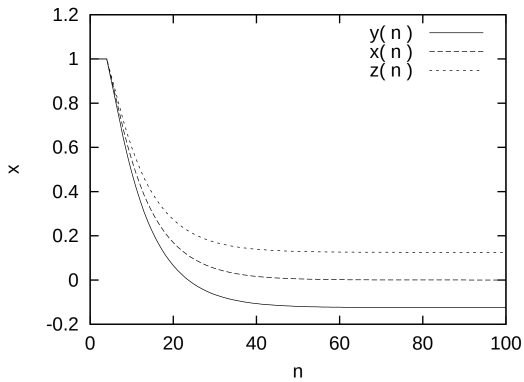

Figure 1: Solutions $y \leq x \leq z$ of (6.1) for $f(n) \equiv -0.01, 0, 0.01$, respectively. Here the initial function and the initial value are equal to one.

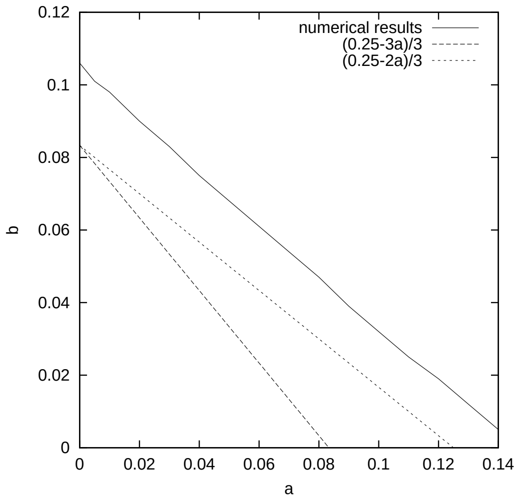

Figure 2: Oscillation bounds for Eq. (6.2): numerical results, sufficient condition (5.1) and the estimate (6.3). Non-oscillatory domains are below the lines. Here for numerical simulations we assumed $x(0) = \varphi(n) \equiv 1$.

or $2a + 3b \leq 0.25$, which is more general than (5.1) for any linear difference equation including delay terms.

Let us also present a similar graph for oscillation conditions, based on numerical results, sufficient oscillation condition (5.9) and the estimate

$$\sup_{n \geq n_0} \sum_{l=1}^{m} \sum_{k=h_l(n)}^{n-1} a_l(k) > \frac{1}{e}. \tag{6.4}$$

Example 6.3. Let us illustrate the sharpness of Corollary 5.14 for the equation

$$x(n+1) - x(n) = -ax(n) - bx(n-\tau),\ a, b \geq 0, \tau \geq 0. \tag{6.5}$$

There should be $\lambda \geq 1$, such that

$$a + \lambda b < 1,\ \tau a + \tau\lambda b \leq 1 - \frac{1}{\lambda}. \tag{6.6}$$

Since $\lambda > 0$, the second inequality in (6.6) is equivalent to

$$f(\lambda) = (\tau b)\lambda^2 - (1 - \tau a)\lambda + 1 \leq 0. \tag{6.7}$$

We have $f(1) = \tau(a+b) > 0$, so (6.7) has a solution $\lambda \geq 1$ only if the x-coordinate of the vertex of the parabola $f(x)$ exceeds one and the quadratic inequality (6.7) has real solutions:

$$\frac{1 - \tau a}{2\tau b} > 1, \quad (1 - \tau a)^2 > 4\tau b,$$

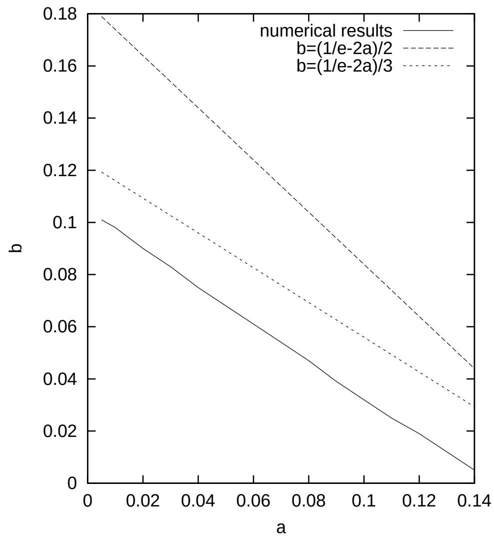

Figure 3: Oscillation bounds for Eq. (6.2): numerical results, sufficient condition (5.9) and the estimate (6.4). Non-oscillatory domains are below the lines. Here for numerical simulations we assumed $x(0) = \varphi(n) \equiv 1$.

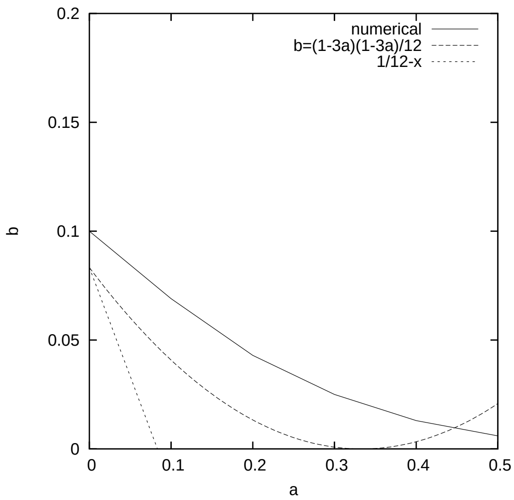

Figure 4: Oscillation bounds for Eq. (6.5): numerical results, second sufficient condition (6.8) and (5.1) for $\tau = 3$.

which can be rewritten as

$$b < \frac{1-\tau a}{2\tau}, \quad b < \frac{(1-\tau a)^2}{4\tau}. \tag{6.8}$$

If the second inequality in (6.6) is satisfied, with $\lambda \geq 1$, then the first inequality is also valid. Thus (6.8) are sufficient conditions for nonoscillation. Note that the second inequality in (6.8) implies the first one. So, if the second inequality (6.8) holds, then Eq. (6.6) has a nonoscillatory solution.

Fig. 4 presents bounds (6.8) for a particular case $\tau = 3$, which is $b < (1-3a)^2/12$, together with numerically established bounds of oscillation. For comparison we present a stricter sufficient condition $3(a+b) < 1/4$ (5.1) of Theorem 5.1.

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