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Stability of Neural Networks with Time Varying Delays in the Presence of Impulses

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Abstract

We develop a new approach to the stability analysis of Hopfield-type neural networks with time varying delays in the presence of impulses. With the new approach, we improve and generalize some previous works of other researchers. We study stability of equilibrium points of impulsive systems which are either a generalization of those existing or new.

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1. Introduction

A neural network is a network that performs computational tasks such as associative memory, pattern recognition, optimization, model identification, signal processing, etc. on a given pattern via interaction between a number of interconnected units characterized by simple functions. From the mathematical point of view, an artificial neural network corresponds to a nonlinear transformation of some inputs into certain outputs. Many types of neural networks have been proposed and studied in the literature and the Hopfield-type network has become an important one due to its potential for applications in various fields of daily life. The model proposed by Hopfield, also known as Hopfield's graded response neural network is based on analog circuit consisting of capacitors, resistors and amplifiers. Among the most popular models in the literature of artificial neural networks (see, e.g., [1–3, 6, 8, 11–15]) is the continuous time model described by a system of ordinary differential equations:

$$\frac{dx_i}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + c_i, \quad t > 0, \tag{1.1}$$

where $x_i(t)$ corresponds to the membrane potential of the unit i at time t; $f_j(\cdot)$ denotes a measure of response or activation to its incoming potentials; b_{ij} denotes the synaptic connection weight of the unit j on the unit i; the constants c_i correspond to the external bias or input from outside the network to the unit i; the coefficient a_i is the rate with which the unit self-regulates or resets its potential when isolated from other units and inputs. We refer for more detail to [1, 2, 6, 8, 11–15] and the references cited therein.

Dynamical systems are often broadly classified into two categories: continuous time systems or discrete time systems. Recently there has been introduced a somewhat new category of dynamical systems which is neither purely continuous time nor purely discrete time ones; these are called dynamical systems with impulses (see for instance [1, 2, 8] and references therein). Stability conditions for various types of stability of neural networks problems such as complete stability, asymptotic stability, absolute stability and exponential stability have been studied extensively. One should underline the fact that stability properties of a neural network basically depend on the intended problems. For example in the solution of optimization problems, the neural network must be designed to have only one equilibrium point and this equilibrium point is globally stable. See more details in [2, 8] and references given therein.

The differences between functional and neural networks and the advantages of using functional networks instead of standard neural networks can be represented as follows: Functional networks are a generalization of the standard neural networks in the sense that the weights are now replaced by neural functions, which can exhibit, in general, a multivariate character. In addition, when working with functional networks we are able to connect different neuron outputs at convenience. Furthermore, different neurons can be associated with neural functions from different families of functions. As a result of these properties, the functional networks allow more flexibility than the standard neural networks [11]. Differences can be summarized in the following way:

1. In neural networks each neuron returns an output $y = f\left(\sum \omega_{ik} x_k\right)$ that depends only on the value $\sum \omega_{ik} x_k$, where $x_1, x_2, \ldots, x_n$ are the received inputs. Therefore, their neural functions have only one argument. In contrast, neural functions in functional networks can have several arguments.

2. In neural networks, the neural functions are univariate: a neuron can show different outputs but all of them represent the same values. In functional networks, the neural functions can be multivariate.

3. In a given functional network the neural functions can be different, while in neural networks they are identical.

4. In neural networks there are weights, which must be learned. These weights do not appear in functional networks, where neural functions are learned instead.

5. In neural networks the neuron outputs are different, while in functional networks neuron outputs can be coincident. This fact leads to a set of functional equations.

All these features show that the functional networks exhibit more interesting possibilities than standard neural networks. Recently [12] the stability of the Hopfield-type neural networks with time varying delays was studied describing the state equations of the form

$$\frac{d}{dt} u_i(t) = -\frac{1}{R_i} u_i(t) + \sum_{j=1}^{n} \omega_{ij} f_j(u_j(t - \tau_{ij}(t))) + I_i, \quad i = \overline{1, n}, \tag{1.2}$$

where R_i are time constants, ω_{ij} are the connection strengths, f_i are the input-output transfer functions, τ_{ij} are the time varying transmission delays, and I_i are the signals from outside. Defining the dynamical characteristic of the network by the dynamics of the system of ordinary differential equations is one of the most popular and typical neural network models. Some other models, such as the continuous bi-directional associative memory networks, can be deduced from a special form of system (1.1) (see for more details [3, 8, 9, 11–15] and references therein.)

In the sequel we consider the system (1.2) subjected to certain impulsive state displacements at fixed moments of time:

$$\begin{cases} \dfrac{d}{dt} u_i(t) = -\dfrac{1}{R_i} u_i(t) + \displaystyle\sum_{j=1}^{n} \omega_{ij} f_j(u_j(t - \tau_{ij}(t))) + I_i, \\ t > t_0, \ t \neq t_k, \quad i = \overline{1, n}, \\ u(t_0+) = u_0 \in \mathbb{R}^n, \\ u_i(t_k+) - u_i(t_k-) = J_{ik}(u_i(t_k-)), \quad i = \overline{1, n}, \quad k = 1, 2, 3, \ldots, \\ t_0 < t_1 < t_2 < \cdots < t_k \to \infty \ \text{ as } \ k \to \infty. \end{cases} \tag{1.3}$$

By a solution of (1.3) we mean $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathbb{R}^n$, in which $u(\cdot)$ is piecewise continuous on $[t_0, \alpha)$ for some $\alpha > t_0$ such that $u(t_k+)$ and $u(t_k-)$ exist and $u(\cdot)$ is differentiable on the intervals of the form $(t_{k-1}, t_k) \subset (t_0, \alpha)$ and satisfies (1.3); we assume that $u(t)$ is left continuous with $u(t_k) = u(t_k-)$; the functions $J_{ik}(\cdot) : \mathbb{R} \to \mathbb{R}$ are assumed to be Lipschitz continuous. Throughout the present paper, we will use the following assumptions.

H1. Each transfer function f_i is monotonically increasing, and satisfies the Lipschitz condition $|f_i(u_i) - f_i(v_i)| \leq M_i|u_i - v_i|$, for some $M_i > 0$ and for all real numbers u_i, v_i.

H2. The delays $\tau_{ij}(t)$ are bounded, that is, there exists a constant b such that $0 \leq \tau_{ij} \leq b$ for all $t \neq t_k$ and $i, j = \overline{1, n}$.

H3. The number $i(t_0, t) = \max\{k \in \mathbb{Z}^+ : \ t_k < t\}$ of moments of impulse effect between t_0 and t satisfies

$$\limsup_{t \to +\infty} \frac{i(t_0, t)}{t} = p < +\infty$$

and the impulsive operators J_{ik} satisfy

$$|J_{ik}(u_i) - J_{ik}(v_i)| \leq c|u_i - v_i|, \quad i = \overline{1, n}, \quad k \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\},$$

for some positive constant c and any real numbers u_i, v_i.

In neural network applications, the transfer function f_i is generally chosen as a sigmoidal function. That is, $\lim\limits_{u_i \to \mp\infty} f_i(u_i) = \mp 1$ and $f_i'(0) \geq f_i'(u_i) > 0$ for all real numbers u_i. Thus the activation functions $f_i(\cdot)$ have been assumed to be continuously differentiable, monotonic and bounded. However, in some applications, one is required to use unbounded and non-monotonic activation functions. It has been shown that the capacity of an associative memory network can be significantly improved if the sigmoidal functions are replaced by non-monotonic activation functions.

This paper is organized as follows. In Section 2, we introduce a more general time delay impulsive system (2.1), (2.2) and present the necessary notations and concepts of the stability analysis of this system. In Section 3 we give the proofs of the main theorems on existence and stability of equilibrium points of the impulsive system (2.1), (2.2) and the Hopfield-type neural network with time varying delays in the presence of impulses (1.3).

2. Notations and Preliminaries

Here we follow [12] adapting the approach expounded therein to impulsive systems. In order to study the stability analysis of the general time delay system we rewrite system

(1.3) as

$$\frac{du(t)}{dt} = F(u(t)) + G(u_\tau(t)), \quad t > t_0,\ t \neq t_k, \tag{2.1}$$

$$\Delta u(t_k) = J_k(u(t_k-)) = u(t_k+) - u(t_k-), \quad k = 1, 2, 3, \ldots, \tag{2.2}$$

where $J_k(u) = (J_{1k}(u_1), \ldots, J_{nk}(u_n))^T$ and the operators J_{ik} satisfy the condition **H3**, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ is the state vector of the neural network, F and G both are mappings from an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, and $G(u_\tau(t))$ is defined as $G(u) = (G_1(u), G_2(u), \ldots, G_n(u))^T$ and

$$G_i(u_\tau(t)) = G_i(u_1(t - \tau_{i1}(t)), u_2(t - \tau_{i2}(t)), \ldots, u_n(t - \tau_{in}(t))).$$

Let $\mathbb{R}^n$ be the n-dimensional real vector space with vector norm $\|\cdot\|$. If $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$, then commonly used vector norms in $\mathbb{R}^n$ are $\|v\|_1,\ \|v\|_2,\ \|v\|_\infty$, where

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|, \qquad \|v\|_2 = \left\{\sum_{i=1}^{n} |v_i|^2\right\}^{1/2}, \qquad \|v\|_\infty = \max_{i=\overline{1,n}} |v_i|.$$

We note that $\|v\|_1$ and $\|v\|_2$ are special cases of a more general norm $\|v\|_p$, where

$$\|v\|_p = \left\{\sum_{i=1}^{n} |v_i|^p\right\}^{1/p}, \quad p \geq 1, \quad v \in \mathbb{R}^n.$$

We also use the so-called matrix-deduced norm $\|\cdot\|_P$ which is defined, given a nonsingular matrix P and a specific vector norm $\|\cdot\|$, by $\underline{\|x\|_p} = \|Px\|$. In particular, if the matrix $P = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $d_i \neq 0, \quad i = \overline{1,n}$, and the vector norm is $\|\cdot\|_1$, then the matrix-deduced norm is $\|x\|_{1,P} = \sum_{i=1}^{n} |d_i x_i|$. We recall the following matrix norms and matrix measures induced by respective vector norms.

If $A = (a_{ij})$ denotes an $n \times n$ matrix, then the norm $\|A\|$ of the matrix A induced by a vector norm $\|\cdot\|$ and the corresponding matrix measure $\mu(A)$ are defined respectively by

$$\|A\| = \sup_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|Av\|}{\|v\|} = \sup_{\|v\|=1} \|Av\| = \sup_{\|v\| \leq 1} \|Av\|, \quad \mu(A) = \lim_{\lambda \to 0^+} \frac{\|I + \lambda A\| - 1}{\lambda},$$

where I denotes the identity matrix. The matrix measure depends on the given vector norm of $\mathbb{R}^n$. For example, corresponding to the norms

$$\|A\|_1 = \max_{j=\overline{1,n}} \sum_{i=1}^{n} |a_{ij}| \quad \text{(column sum)}, \quad \|A\|_\infty = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}| \quad \text{(row sum)}$$

and $\|A\|_2 = \big(\lambda_{\max}(A^TA)\big)^{1/2}$, where $\lambda_{\max}(A^TA)$ is the maximum eigenvalue of the symmetric matrix A^TA, are the commonly used matrix measures of A, denoted by $\mu_1(A)$, $\mu_\infty(A)$ and $\mu_2(A)$, defined respectively by

$$\mu_1(A) = \max_{j=\overline{1,n}}\left\{a_{jj} + \sum_{i\neq j}^{n}|a_{ij}|\right\}, \quad \mu_\infty(A) = \max_{i=\overline{1,n}}\left\{a_{ii} + \sum_{j\neq i}^{n}|a_{ij}|\right\}$$

and $\mu_2(A) = \dfrac{1}{2}\lambda_{\max}(A^T + A)$. One can easily check that, for all diagonal matrices $P = \operatorname{diag}(d_1, d_2, \dots, d_n)$ $(d_i \neq 0,\ i = \overline{1,n})$, the inequality

$$\mu_\infty(A) \le \big(\min_{i=\overline{1,n}}(d_i)\big)^{-1}\mu_\infty(PA) \tag{2.3}$$

holds. For more details about the matrix norm and matrix measure see, e.g., [10].

The importance of the matrix measure in characterizing stability of a linear system with time delays can be shown by the fact that if $\mu(A)+\|B\| < 0$, then the linear system

$$\frac{d}{dt}x(t) = Ax(t) + Bx(t-\tau), \quad t \ge 0,$$

is exponentially stable. In order to introduce a similar quantity for nonlinear systems, one can observe that if A is a given $n \times n$ matrix, and $\mathbb{R}^n$ is endowed with the norm $\|\cdot\|_1$, then the corresponding matrix measure $\mu_1(A)$ of A can be also defined by

$$\mu_1(A) = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle Ax, \operatorname{sgn}(x)\rangle}{\|x\|_1},$$

where $\langle u, v\rangle$ represents the inner product of vectors $u, v \in \mathbb{R}^n$ and $\operatorname{sgn}(x)$ is the vector whose i-th component is the sign of the i-th component of x, that is, $\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \operatorname{sgn}(x_2), \dots, \operatorname{sgn}(x_n))^T$ and for any real t, $\operatorname{sgn}(t)$ is defined by

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$. A function $f:\ \Omega \to \mathbb{R}^n$ is said to be a *Lipschitz operator* on Ω whenever there exists a nonnegative constant M such that for any $x, y \in \Omega$,

$$\|f(x) - f(y)\| \le M\|x - y\|,$$

where M is called a *Lipschitz constant* of f.

The *minimal Lipschitz constant* (MLC) of f defined by

$$L(f) = \sup_{x,y\in\Omega,\ x\neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

is known as a semi-norm of the space of Lipschitz operators. Naturally the MLC of a Lipschitz operator varies with the norm adopted. Further on, the MLCs of the Lipschitz operator f corresponding to $\|\cdot\|_1$ and $\|\cdot\|_\infty$ will be denoted respectively by $L_1(f)$ and $L_\infty(f)$.

Definition 2.2. Let f be a Lipschitz operator on $\Omega \subset \mathbb{R}^n$ with the norm $\|\cdot\|_1$, then the constant

$$m(f) = \sup_{x,y\in\Omega,\, x\neq y} \frac{\langle f(x) - f(y), \operatorname{sgn}(x-y)\rangle}{\|x-y\|_1}$$

is called the *nonlinear Lipschitz measure* (NLM) of f on Ω.

From Definition 2.1 and Definition 2.2 one can immediately deduce that $m(f) \leq L(f)$ for any Lipschitz operator f. In addition, if f is a matrix, then because of the definition of $\mu_1(A)$ and $L(f)$ the NLM of f coincides exactly with its matrix measure.

Lemma 2.3. [12] Assume $\mathbb{R}^n$ is endowed with the norm $\|\cdot\|_1$, and f is a Lipschitz operator on $\Omega \subset \mathbb{R}^n$. If $m(f) < 0$, then

(i) f is one-to-one, that is, $f(x) \neq f(y)$ whenever $x \neq y$. Moreover, if $\Omega = \mathbb{R}^n$, the range of f denoted by $R(f)$ is the whole space $\mathbb{R}^n$, therefore f is a homeomorphism of $\mathbb{R}^n$.

(ii) The inverse function f^{-1} is a Lipschitz operator on $R(f)$ with

$$L(f^{-1}) \leq \frac{1}{-m(f)}.$$

We will also need the following simple assertion from calculus.

Lemma 2.4. [12] If $a > c > 0$, then, for each nonnegative real number b, the equation

$$\lambda - a + ce^{\lambda b} = 0$$

has a unique positive solution.

In fact, the left hand side of this equation is a strictly monotonic function for $\lambda \geq 0$, which is positive at $\lambda = a$ and negative at $\lambda = 0$.

Definition 2.5. The time-delay impulsive system (2.1), (2.2) is said to be *exponentially stable* on a neighbourhood Ω of an equilibrium point u^* if there are two positive constants α and M such that

$$\|u(t) - u^*\| \leq Me^{-\alpha(t-t_0)} \sup_{t_0 - b \leq s \leq t_0} \|u_0(s) - u^*\|, \quad t \geq t_0,$$

where $b = \sup\{\tau_{ij}(t) : \ i,j = \overline{1,n},\ t \in \mathbb{R}\}$ and $u(t)$ is the unique trajectory of the system initiated from $u_0(s) \in \Omega$ with $s \in (t_0 - b, t_0]$. If the equilibrium point of the system (2.1), (2.2) is unique and it is exponentially stable on the space $\mathbb{R}^n$, then system (2.1), (2.2) is said to be *exponentially global stable*.

3. Main Results

Theorem 3.1. Let $\Omega \subset \mathbb{R}^n$ be a neighbourhood of an equilibrium point u^* of the system (2.1), (2.2) and F, G be Lipschitz operators on Ω, $J_{ik},\ i = \overline{1,n},\ k \in \mathbb{Z}^+$, satisfy **H3** and $b = \sup\{\tau_{ij}(t) :\ i,j = \overline{1,n},\ t \in \mathbb{R}\} < \infty$. When $\mathbb{R}^n$ is endowed with the norm $\|\cdot\|_1$ and for some $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ with $a_i > 0$ we have $m(FA) + L_1(GA) < 0$, then by Lemma 2.4 the equation

$$\lambda \min_{i=\overline{1,n}} a_i + m(FA) + L_1(GA)e^{b\lambda} = 0$$

has a unique positive solution λ. If $p\ln(1+c) < \lambda$, then system (2.1), (2.2) is exponentially stable on Ω. More precisely, for any $\tilde{\lambda} \in (0, \lambda - p\ln(1+c))$ there exists a constant M such that

$$\|x(t) - y(t)\|_1 \le Me^{-\tilde{\lambda}(t-t_0)} \sup_{t_0-b\le s\le t_0} \|x_0(s) - y_0(s)\|_1 \quad \text{for all } t \ge t_0. \tag{3.1}$$

In the case of $\mathbb{R}^n$ endowed with the norm $\|\cdot\|_\infty$, F a matrix and if for some matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ with $a_i > 0$, we have $\mu_\infty(AF) + L_\infty(AG) < 0$, then by Lemma 2.4 the equation

$$\mu \min_{i=\overline{1,n}} a_i + \mu_\infty(AF) + L_\infty(AG)e^{b\mu} = 0$$

has a unique positive solution μ. If $p\ln(1+c) < \mu$, then system (2.1), (2.2) is exponentially stable on Ω. More precisely, for any $\tilde{\mu} \in (0, \mu - p\ln(1+c))$ there exists a constant M such that

$$\|x(t) - y(t)\|_\infty \le Me^{-\tilde{\mu}(t-t_0)} \sup_{t_0-b\le s\le t_0} \|x_0(s) - y_0(s)\|_\infty \quad \text{for all } t \ge t_0. \tag{3.2}$$

In both cases, $x(t)$ and $y(t)$ are the trajectories of system (2.1), (2.2) initiated respectively from $x_0(s)$ and $y_0(s)$ where $x_0(s), y_0(s) \in \Omega$ for all $s \in (t_0 - b, t_0]$.

Proof. Let us denote $u(t) = x(t) - y(t)$.

For any vector $w \in \mathbb{R}^n$ we have $\|w\|_1 = \langle w, \mathrm{sgn}(w)\rangle$ and $\|w\|_1 \ge \langle w, \mathrm{sgn}(z)\rangle$ for all $z \in \mathbb{R}^n$. Therefore for any $s \in \mathbb{R},\ s > 0$ we have

$$\frac{\|u(t)\|_1 - \|u(t-s)\|_1}{s} \le \frac{1}{s}\langle u(t) - u(t-s), \mathrm{sgn}(u(t))\rangle.$$

So, from system (2.1) for $t \ne t_k$ we have

$$\begin{aligned}
&\frac{d\|u(t)\|_1}{dt} \le \left\langle \frac{du(t)}{dt}, \mathrm{sgn}(u(t))\right\rangle \\
&= \langle F(x(t)) - F(y(t)), \mathrm{sgn}(A^{-1}u(t))\rangle + \langle G(x_\tau(t)) - G(y_\tau(t)), \mathrm{sgn}(u(t))\rangle \\
&\le m(FA)\|A^{-1}u(t)\|_1 + L_1(GA)\|A^{-1}x_\tau(t) - A^{-1}y_\tau(t)\|_1 \\
&\le \left\{ m(FA)\|u(t)\|_1 + L_1(GA) \sup_{t-b\le s\le t} \|u(s)\|_1 \right\} \left(\min_{i=\overline{1,n}} a_i \right)^{-1}.
\end{aligned}$$

Since $m(FA) + L_1(GA) < 0$ and $L_1(GA) \geq 0$, then by Halanay's inequality [7] and taking into account the presence of impulses, we have

$$\|u(t)\|_1 \leq e^{-\lambda(t-t_0)}(1+c)^{i(t_0,t)} \sup_{t_0-b\leq s\leq t_0} \|u(s)\|_1,$$

where λ is the unique positive solution of the equation

$$\lambda = -m(FA)\left(\min_{i=\overline{1,n}} a_i\right)^{-1} - L_1(GA)\left(\min_{i=\overline{1,n}} a_i\right)^{-1} e^{\lambda b}.$$

Let $\varepsilon > 0$ be such that $\lambda - (p+\varepsilon)\ln(1+c) > 0$. Then $i(t_0,t) \leq (p+\varepsilon)(t-t_0)$ for all t large enough and there exists a constant $M \geq 1$ such that $i(t,t_0) \leq (p+\varepsilon)(t-t_0) + \ln\, M/\ln(1+c)$ for all $t \geq t_0$. Then

$$(1+c)^{i(t_0,t)} \leq M \exp[(p+\varepsilon)\ln(1+c)(t-t_0)]$$

and the desired estimate (3.1) follows with $\tilde{\lambda} = \lambda - (p+\varepsilon)\ln(1+c)$.

When F is a matrix and $\mathbb{R}^n$ is endowed with the norm $\|\cdot\|_\infty$, we have from system (2.1)

$$u(t) = e^{F(t-s)}u(s) + \int_s^t e^{F(t-r)}\Big(G(x_\tau(r)) - G(y_\tau(r))\Big)\,dr$$

for all $t > s \geq t_0$. Using the well-known properties of the matrix measure $\mu_\infty(F)$ (see more details in [10]), $\|e^{Ft}\|_\infty \leq e^{\mu_\infty(F)t} \quad \forall t \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{\|u(t)\|_\infty - \|u(s)\|_\infty}{t-s} &\leq \frac{1}{t-s}\Big\{\big(\|e^{F(t-s)}\| - 1\big)\|u(s)\|_\infty \\
&+ \int_s^t \|e^{F(t-r)}\|\|G(x_\tau(r)) - G(y_\tau(r))\|_\infty\,dr\Big\} \\
&\leq \frac{1}{t-s}\Big\{\big(e^{\mu_\infty(F)(t-s)} - 1\big)\|u(s)\|_\infty \\
&+ \left(\min_{i=\overline{1,n}} a_i\right)^{-1} L_\infty(AG)\int_s^t e^{\mu_\infty(F)(t-r)}\|u_\tau(r)\|_\infty\,dr\Big\}.
\end{aligned}$$

For $s \to t$ and using the inequality (2.3), where $P = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ $(d_i \neq 0,\ i = \overline{1,n})$, we can obtain that almost everywhere on $(t_0, +\infty)$

$$\begin{aligned}
\frac{d\|u(t)\|_\infty}{dt} &\leq \mu_\infty(F)\|u(t)\|_\infty + L_\infty(AG)\|Au_\tau(t)\|_\infty\left(\min_{i=\overline{1,n}} a_i\right)^{-1} \\
&\leq \left\{\mu_\infty(AF)\|u(t)\|_\infty + L_\infty(AG)\sup_{t-b\leq s\leq t}\|u(s)\|_\infty\right\}\left(\min_{i=\overline{1,n}} a_i\right)^{-1}.
\end{aligned}$$

Now the proof of the estimate (3.2) is completed as the proof of the estimate (3.1). ∎

Now, extending some results of [12], we present some sufficient conditions for existence and uniqueness of equilibrium of the network (1.2) and the impulsive network (1.3).

Theorem 3.2. Let $\Omega \subset \mathbb{R}^n$ and m_i be the MLC of f_i on Ω_i, the projection of Ω to the i-th axis. If r_i $(i = \overline{1,n})$ are positive real numbers such that either

$$m_j R_j \left(\omega_{jj} + \sum_{i \neq j}^{n} \frac{r_j}{r_i} |\omega_{ij}| \right) < 1, \quad j = \overline{1,n}, \tag{3.3}$$

or

$$R_j \sum_{i=1}^{n} \frac{r_i}{r_j} m_i |\omega_{ji}| < 1, \quad j = \overline{1,n}, \tag{3.4}$$

then, corresponding to each group of external input I_i, the equilibrium point of system (1.2) is unique in Ω. Furthermore, if $\Omega = \mathbb{R}^n$, then there exists an equilibrium point.

Proof. Assume that $P = \operatorname{diag}(r_1, r_2, \dots, r_n),\ r_j > 0$ $(j = \overline{1,n})$ and define the function $F:\ \Omega \to \mathbb{R}^n$ by

$$\big(F(u)\big)_i = -\frac{u_i}{R_i} + \sum_{j=1}^{n} \omega_{ij} f_j(u_j) + I_i, \quad i = \overline{1,n},$$

where $u = \big(u_1, u_2, \dots, u_n\big)^T, \quad F(u) = \big(F_1(u), F_2(u), \dots, F_n(u)\big)^T$. An equilibrium point u^* of system (1.2) corresponds to a solution of the equation

$$F(u) = 0. \tag{3.5}$$

Since the matrix P is nonsingular, the problem of finding a solution $u \in \Omega$ of equation (3.5) is equivalent to the problem of finding a solution $v \in P^{-1}(\Omega)$ of

$$P^{-1} F(Pv) = 0. \tag{3.6}$$

In fact, these solutions are related by $u = Pv$.

Let us suppose that (3.3) holds. In this case we follow closely [12].

We denote

$$\mu_j = \omega_{jj} + \sum_{i \neq j}^{n} \frac{r_j}{r_i} |\omega_{ij}| \quad \text{and } \mu_j^+ = \max\{0,\ \mu_j\}, \quad j = \overline{1,n}.$$

For each $i = \overline{1,n}$ the transfer function f_i is increasing, or equivalently

$$\big(f_i(t) - f_i(s)\big)\operatorname{sgn}(t - s) = |f_i(t) - f_i(s)| \quad \text{for all } t, s \in \mathbb{R}.$$

It follows that for all $x, y \in P^{-1}(\Omega)$ we have

$$
\begin{aligned}
&\langle P^{-1}F(Px) - P^{-1}F(Py), \operatorname{sgn}(x-y)\rangle \\
&= \sum_{i=1}^{n} \left\{ -\frac{x_i - y_i}{R_i} + \sum_{j=1}^{n} r_i^{-1}\omega_{ij}\left(f_j(r_jx_j) - f_j(r_jy_j)\right) \right\} \operatorname{sgn}(x_i - y_i) \\
&= -\sum_{i=1}^{n} \frac{|x_i - y_i|}{R_i} + \sum_{j=1}^{n}\sum_{i=1}^{n} r_i^{-1}\omega_{ij}\left(f_j(r_jx_j) - f_j(r_jy_j)\right)\operatorname{sgn}(x_i - y_i) \\
&= -\sum_{j=1}^{n} \frac{|x_j - y_j|}{R_j} + \sum_{j=1}^{n} \Bigg\{ r_j^{-1}\omega_{jj}\left[f_j(r_jx_j) - f_j(r_jy_j)\right]\operatorname{sgn}(x_j - y_j) \\
&\quad + \sum_{i \neq j}^{n} r_i^{-1}\omega_{ij}\left[f_j(r_jx_j) - f_j(r_jy_j)\right]\operatorname{sgn}(x_i - y_i) \Bigg\} \\
&\leq -\sum_{j=1}^{n} \frac{|x_j - y_j|}{R_j} + \sum_{j=1}^{n} \Bigg\{ r_j^{-1}\omega_{jj}|f_j(r_jx_j) - f_j(r_jy_j)| \\
&\quad + \sum_{i \neq j}^{n} r_i^{-1}|\omega_{ij}| \cdot |f_j(r_jx_j) - f_j(r_jy_j)| \Bigg\} \\
&\leq -\sum_{j=1}^{n} \frac{|x_j - y_j|}{R_j} + \sum_{j=1}^{n} |f_j(r_jx_j) - f_j(r_jy_j)| r_j^{-1}\mu_j \\
&\leq -\sum_{j=1}^{n} R_j^{-1}\big(1 - \mu_j^{+} m_j R_j\big)|x_j - y_j| \\
&\leq -\frac{\min\limits_{j=\overline{1,n}}\{1 - \mu_j^{+} m_j R_j\}}{\max\limits_{j=\overline{1,n}} R_j}\|x - y\|_1 .
\end{aligned}
$$

This result implies that $m(P^{-1}FP) < 0$. The operator $P^{-1}FP$ is one-to-one, therefore there is not more than one $v^* \in P^{-1}(\Omega)$ such that $P^{-1}FP(v^*) = 0$. The equilibrium point of (1.2) is unique because P is nonsingular. Furthermore, if $\Omega = \mathbb{R}^n$, then $P^{-1}FP$ is a homeomorphism of $\mathbb{R}^n$. Thus F is also a homeomorphism of $\mathbb{R}^n$ because P is nonsingular. Therefore there is a unique u^* in $\mathbb{R}^n$ such that $F(u^*) = 0$, that is, u^* is the unique equilibrium point of the system (1.2).

Now let us suppose that condition (3.4) holds. We can write the equation (3.6) in the form

$$v = \Phi(v),$$

where

$$\Phi_i(v) = R_i r_i^{-1} \sum_{j=1}^{n} \omega_{ij} f_j(r_j v_j) + R_i I_i,$$

$v = (v_1, \ldots, v_n)^T,\ \Phi(v) = (\Phi_1(v), \ldots, \Phi_n(v))^T$. For $x, y \in P^{-1}(\Omega) \subset \mathbb{R}^n$ we have

$$|\Phi_i(x) - \Phi_i(y)| \le R_i r_i^{-1} \sum_{j=1}^{n} |\omega_{ij}| |f_j(r_j x_j) - f_j(r_j y_j)|$$

$$\le R_i r_i^{-1} \sum_{j=1}^{n} |\omega_{ij}| m_j r_j |x_j - y_j|,$$

thus

$$\|\Phi(x) - \Phi(y)\|_\infty \le q \|x - y\|_\infty,$$

where

$$q = \max_{i=\overline{1,n}} R_i r_i^{-1} \sum_{j=1}^{n} |\omega_{ij}| m_j r_j = \max_{j=\overline{1,n}} R_j \sum_{i=1}^{n} \frac{r_i}{r_j} |\omega_{ji}| m_i < 1.$$

This shows that the operator Φ is a contraction, so it has not more than one fixed point in $P^{-1}(\Omega)$ and exactly one fixed point in $\mathbb{R}^n$. The proof of the theorem is complete. ■

It is easy to see that an equilibrium u^* of the impulsive system (1.3) must be an equilibrium of the system without impulses (1.2). So there is at most one equilibrium point $u^* = (u_1^*, \ldots, u_n^*)^T$ of (1.3) and it must satisfy

$$J_{ik}(u_i^*) = 0, \quad i = \overline{1,n}, \quad k \in \mathbb{Z}^+. \tag{3.7}$$

Theorem 3.3. Assume that u^* is an equilibrium point of system (1.3) and Ω is a neighbourhood of u^*. Suppose that there exists a set of positive numbers $r_i,\ i = \overline{1,n}$, satisfying either

$$m_j R_j \sum_{i=1}^{n} \frac{r_j}{r_i} |\omega_{ij}| < 1 \tag{3.8}$$

or (3.4), where m_i is the minimal Lipschitz constant (MLC) of f_i on Ω_i. Suppose further that the unique positive solution λ of the equation

$$\lambda \min_{i=\overline{1,n}} R_i - 1 + q e^{\lambda b} = 0$$

with

$$q = \max_{j=\overline{1,n}} \left(m_j R_j r_j \sum_{i=1}^{n} r_i^{-1} |\omega_{ij}| \right)$$

when (3.8) holds, or

$$q = \max_{j=\overline{1,n}} \left(R_j r_j^{-1} \sum_{i=1}^{n} m_i r_i |\omega_{ji}| \right)$$

when (3.4) holds, satisfies $\lambda > p\ln(1+c)$. If $u(t)$ is the trajectory of system (1.3) initiated from $u_0(s) \in \Omega$ with $s \in (t_0 - b, t_0]$, then

$$\|u(t) - u^*\| \le Me^{-\tilde{\lambda}(t-t_0)} \frac{\max\limits_{i=\overline{1,n}} r_i}{\min\limits_{i=\overline{1,n}} r_i} \sup_{t_0-b\le s\le t_0} \|u_0(s) - u^*\|, \tag{3.9}$$

where the vector norm $\|\cdot\|$ is respectively $\|\cdot\|_1$ and $\|\cdot\|_\infty$, and $\tilde{\lambda} \in (0, \lambda - p\ln(1+c))$.

Proof. We can first note that condition (3.8) implies (3.3), so the equilibrium of system (1.2) or (1.3) is unique, if any. Moreover, if $\Omega = \mathbb{R}^n$, then condition (3.8) or (3.4) implies the existence of a unique equilibrium of system (1.2). It is an equilibrium of the impulsive system (1.3) as well if and only if condition (3.7) holds.

We can write system (1.3) in the form (2.1), (2.2), where $F = -\mathrm{diag}(R_1^{-1}, \ldots, R_n^{-1})$ and $G:\ \Omega \to \mathbb{R}^n$ is defined by

$$\big(G(u)\big)_i = \sum_{j=1}^n \omega_{ij} f_j(u_j) + I_i, \quad i = \overline{1,n}.$$

By the change $u = Pv$, where $P = \mathrm{diag}(r_1, \ldots, r_n)$, we obtain another system of the form (2.1), (2.2):

$$\begin{cases} \dfrac{d}{dt} v(t) = Fv(t) + P^{-1}G(Pv_\tau(t)), \\ \Delta v_i(t_k) = r_i^{-1} J_{ik}(r_i v_i(t_k)), \quad i = \overline{1,n}, \quad k \in \mathbb{Z}^+. \end{cases} \tag{3.10}$$

It is easily seen that the new impulse operators $v_i \mapsto r_i^{-1} J_{ik}(r_i v_i)$ satisfy condition **H3** with the same constant c and $v^* = P^{-1}u^*$ is an equilibrium point of (3.10).

Let us denote $A = \mathrm{diag}(R_1, \ldots, R_n)$, thus $FA = AF = -I$. We shall apply to system (3.10) the two cases of Theorem 3.1 in dependence on which one of the conditions (3.8) or (3.4) holds.

First suppose that condition (3.8) holds. Then we have $m(FA) = -1$. Moreover, for all $x, y \in A^{-1}P^{-1}(\Omega)$ we have

$$\|P^{-1}G(PAx) - P^{-1}G(PAy)\|_1 = \sum_{i=1}^n r_i^{-1} \left| \sum_{j=1}^n \omega_{ij} \big[f_j(r_j R_j x_j) - f_j(r_j R_j y_j)\big] \right|$$

$$\le \sum_{i=1}^n r_i^{-1} \sum_{j=1}^n |\omega_{ij}| m_j r_j R_j |x_j - y_j| = \sum_{j=1}^n |x_j - y_j| m_j R_j \sum_{i=1}^n |\omega_{ij}| \frac{r_j}{r_i} \le q\|x - y\|_1,$$

and thus $L_1(P^{-1}GPA) \le q < 1$. Since

$$m(FA) + L_1(P^{-1}GPA) \le q - 1 < 0,$$

by Theorem 3.1 the trajectory $v(t)$ of (3.10) satisfies

$$\|v(t) - P^{-1}u^*\| \le Me^{-\tilde{\lambda}(t-t_0)} \sup_{t_0-b\le s\le t_0} \|v(s) - P^{-1}u^*\| \tag{3.11}$$

with 1-norm for all $t \ge t_0$. This estimate easily implies (3.9) with 1-norm.

Next suppose that condition (3.4) holds. We have $\mu_\infty(AF) = -1$ and for all $x, y \in P^{-1}(\Omega)$

$$\begin{aligned}&\left|\left(AP^{-1}G(Px)\right)_i - \left(AP^{-1}G(Py)\right)_i\right| \\ &\qquad = R_i r_i^{-1}\left|\sum_{j=1}^{n} \omega_{ij}\left[f_j(r_jx_j) - f_j(r_jy_j)\right]\right| \le R_i r_i^{-1}\sum_{j=1}^{n}|\omega_{ij}|m_jr_j|x_j - y_j|,\end{aligned}$$

thus

$$\|AP^{-1}G(Px) - AP^{-1}G(Py)\|_\infty \le q\|x - y\|_\infty \text{ and } L_\infty(AP^{-1}GP) = q < 1.$$

Since

$$\mu_\infty(AF) + L_\infty(AP^{-1}GP) = q - 1 < 0,$$

by Theorem 3.1 the solution $v(t)$ of (3.10) satisfies the estimate (3.11) with ∞-norm for all $t \ge t_0$. This estimate implies (3.9) with ∞-norm. ■

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