Initial Value Problems for Fractional Differential Equations of Riemann–Liouville Type

Muhammad N. Islam
University of Dayton
Department of Mathematics
Dayton, OH 45469 USA
mislam1@udayton.edu

Jeffrey T. Neugebauer
Eastern Kentucky University
Department of Mathematics and Statistics
jeffrey.neugebauer@eku.edu

Abstract
Consider fractional initial value problem
\[ D^q_{0+} x(t) = -f(t, x(t)), \quad \lim_{t \to 0^+} t^{1-q} x(t) = x_0 \neq 0, \]
where \( f : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) is continuous, and \( D^q_{0+} \) denotes the Riemann–Liouville differential operator of order \( q \in (0, 1) \). By studying an equivalent Volterra integral equation, we show the existence of a continuous solution on \((0, T]\) for some \( T > 0 \). We then show for a special case when \( f(t, x) = x + g(t, x) \) that if a continuous solution exists on \((0, \infty)\), then it is absolutely integrable on the same interval.

AMS Subject Classifications: 34A08, 34A12, 45D05, 45E10, 45G05.
Keywords: Fractional differential equation, Volterra integral equation, existence of solutions, uniqueness of solutions, absolute integrability.

1 Introduction

It is known that many real world problems can be modeled by the fractional initial value problem of Riemann–Liouville type:
\[ D^q_{0+} x(t) = -f(t, x(t)), \quad \lim_{t \to 0^+} t^{1-q} x(t) = x_0 \neq 0. \]
For various results on (1.1) in terms of theory and applications, we refer the interested reader to [4,6–8,10,11]. We also refer to [1–3] for more recent studies on the existence of solutions of the initial value problem (1.1).

In the present paper, we study (1.1) where \( f : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) is continuous, and \( D^{q}_{0+} \) denotes the Riemann–Liouville differential operator of order \( q \in (0, 1) \) defined by
\[
D^{q}_{0+}x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{0}^{t} (t-s)^{-q} x(s) ds.
\]
Here \( \Gamma : (0, \infty) \to \mathbb{R} \) is Euler’s Gamma function defined by
\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt.
\]

Under certain conditions, it is known that the initial value problem (1.1) is equivalent to the Volterra integral equation
\[
x(t) = x_0 t^{q-1} - \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, x(s)) ds.
\]
(1.2)

By equivalent, we mean that \( x \) is a solution of (1.1) if and only if \( x \) is a solution of (1.2). Some results of equivalence can be found in [1], where equivalence is shown on an interval \( (0, T] \). In Section 4, we will assume that (1.1) and (1.2) are equivalent on the interval \( (0, \infty) \). In Section 2, an example where (1.1) and (1.2) are equivalent on the interval \( (0, \infty) \) is given.

We start with some preliminaries and a motivating example in Section 2. In Section 3, we show the existence of a continuous solution of (1.2) on \( (0, T] \) for some \( T > 0 \). Then in Section 4, we obtain a result showing that if a continuous solution of (1.2) exists on \( (0, \infty) \), then it is absolutely integrable on the same interval, i.e., if \( x \) is a continuous solution of (1.2) on \( (0, \infty) \), then \( \int_{0}^{\infty} |x(t)| dt < \infty \). Absolute integrability of solutions when they exist is an important issue. We feel that our work of Section 4 in which we have obtained such a result is an important contribution to the research in Riemann–Liouville type equations. Although we consider a special case where \( f(t, x) = x + g(t, x) \), we hope that our work will motivate researchers to pursue more general cases.

2 A Motivating Example

Example 2.1. Consider the fractional differential equation
\[
D^{1/2}_{0+} x(t) = -\frac{\sqrt{\pi}}{2} \left( \sqrt{t} x(t) \right)^{3/2},
\]
(2.1)
satisfying the initial condition
\[
\lim_{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_{0}^{t} (t-s)^{-1/2} x(s) ds = \sqrt{\pi}.
\]
(2.2)
In [1], initial condition (2.2) is shown to be equivalent to an initial condition of the form
\[
\lim_{t \to 0^+} t^{-1/2} x(t) = 1.
\]
It is shown in [1] that the function
\[
x(t) = \frac{1}{\sqrt{t}(1 + t)}
\]
satisfies (2.1), (2.2) on the interval \((0, \infty)\) and also satisfies the integral equation
\[
x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2} \int_0^t (t - s)^{-1/2} (\sqrt{s} x(s))^{3/2} ds
\]
on \((0, \infty)\). Also, notice
\[
\int_0^\infty x(t) dt = \int_0^\infty \frac{1}{\sqrt{t}(1 + t)} dt
\]
\[
= 2 \int_0^\infty \frac{1}{1 + u^2} du
\]
\[
= 2 \arctan u|_0^\infty
\]
\[
= \pi.
\]
So \(x\) is absolutely integrable on \((0, \infty)\).

Motivated by this example, in Section 3, we give conditions when (1.2) has a unique solution on \((0, T^*]\) for some \(T^* > 0\), where Theorem 3.2 gives equivalence of (1.1) and (1.2). In Section 4, we assume (1.1) and (1.2) are equivalent on \((0, \infty)\). When (1.1) has a solution \(x\) on \((0, \infty)\), we give sufficient conditions that imply \(x\) is absolutely integrable on \((0, \infty)\).

3 Existence of Solutions

In this section, we present some results on the existence of a continuous solution of (1.1) without showing detailed proofs. These results can be derived from the results in [2].

**Definition 3.1.** For a given \(q \in (0, 1)\), a function \(\varphi : (0, T] \to \mathbb{R}\) is said to be a solution of (1.2) if \(\varphi\) is continuous, \(\varphi\) satisfies (1.2) on \((0, T]\), and \(t^{1-q}\varphi\) is continuous on \([0, T]\) with \(\lim_{t \to 0^+} t^{1-q}\varphi(t) = x_0\).

The following theorem given in [1] establishes some conditions under which (1.1) and (1.2) are equivalent.
Theorem 3.2. Let \( q \in (0, 1) \) and \( x_0 \neq 0 \). Let \( f(t, x) \) be a function that is continuous on the set \( B = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, \ x \in I\} \), where \( I \subseteq \mathbb{R} \) is an unbounded interval. Suppose \( x : (0, T] \to I \) is continuous and both \( x(t) \) and \( f(t, x(t)) \) are absolutely integrable on \((0, T]\). Then \( x(t) \) satisfies (1.1) on \((0, T]\) if and only if \( x(t) \) satisfies (1.2) on \((0, T]\).

We assume \( f(t, x) \) satisfies the following local Lipschitz condition.

(A1) For each \( T > 0 \), there exists a \( k = k(T) > 0 \) such that

\[
|f(t, x) - f(t, y)| \leq k|x - y|,
\]

for all \( x, y \in \mathbb{R}, 0 < t \leq T \).

Notice that (A1) implies

\[
|f(t, x)| \leq k|x| + |f(t,0)|.
\]

Define \( f_0(t) = f(t, 0) \). We also assume \( f_0 \in X \).

For a fixed \( T > 0 \) and for \( g(t) = t^{q-1} \), let \( X \) be the space of all continuous functions \( \varphi : (0, T] \to \mathbb{R} \) with

\[
|\varphi|_g = \sup_{0 < t \leq T} \frac{|\varphi(t)|}{g(t)} < \infty.
\]

It is shown in [2, Theorem 2.2] that \((X, |\cdot|_g)\) is a Banach space.

Lemma 3.3. If \( \varphi \in X \), then \( \varphi \) is absolutely integrable on \((0, T]\).

The proof of the following lemma can be found in [1, Lemma 4.6]

Lemma 3.4. Suppose \( \varphi : (0, T] \to \mathbb{R} \) is a continuous and absolutely integrable function on \((0, T]\). Then

\[
h(t) := \int_0^t (t - s)^{q-1} \varphi(s)ds
\]

is continuous and absolutely integrable on \((0, T]\).

Define a mapping \( P \) on \( X \) as follows. For \( \varphi \in X \),

\[
(P\varphi)(t) := x_0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, \varphi(s))ds.
\] (3.1)

Define

\[
b_\varphi(t) := \int_0^t (t - s)^{q-1} f(s, \varphi(s))ds.
\]

Since \( f_0 \in X \), (A1) and Lemma 3.4 imply \( b_\varphi \in X \). This implies \( P\varphi \in X \). So \( P : X \to X \).
Theorem 3.5. Suppose assumption (A1) holds and suppose $f_0 \in X$. Then there exists a $T^* > 0$ such that (1.1) has a unique continuous solution on $(0, T^*)$.

Proof. Let $\varphi, \psi \in X$. Then, by (A1),
\[
| (P\varphi)(t) - (P\psi)(t) | \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} | f(s, \varphi(s)) - f(s, \psi(s)) | ds \\
\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} k | \varphi(s) - \psi(s) | ds \\
\leq \frac{1}{\Gamma(q)} k | \varphi - \psi |_g \int_0^t (t-s)^{q-1} s^{q-1} ds \\
= \frac{1}{\Gamma(q)} k | \varphi - \psi |_g t^{q-1} \frac{\Gamma(q)}{\Gamma(2q)}.
\]
Therefore
\[
\frac{| (P\varphi)(t) - (P\psi)(t) |}{t^{q-1}} \leq k | \varphi - \psi |_g \frac{\Gamma(q)}{\Gamma(2q)} t^q.
\]
Let $T^* > 0$ be such that
\[
k \frac{\Gamma(q)}{\Gamma(2q)} t^q \leq k^* < 1
\]
for $0 < t \leq T^*$. Thus, we have
\[
| (P\varphi) - (P\psi) |_g \leq k^* | \varphi - \psi |_g.
\]
Since $k^* < 1$, the mapping $P : X \rightarrow X$ is a contraction for $0 < t \leq T^*$. Therefore there exists a unique continuous $\varphi \in X$ such that $P\varphi = \varphi$. Since $\varphi$ is continuous and both $\varphi(t)$ and $f(t, \varphi(t))$ are absolutely integrable on $(0, T^*)$, (1.1) and (1.2) are equivalent. Thus $\varphi(t)$ is a unique continuous solution of (1.1) on $(0, T^*)$.

4 Absolute Integrability of Solutions

In this section, we assume (1.1) and (1.2) are equivalent on $(0, \infty)$. Let
\[
C(t-s) = \frac{1}{\Gamma(q)} (t-s)^{q-1}.
\] (4.1)
Then, for $f(t, x) = x + g(t, x)$, the integral equation (1.2) becomes
\[
x(t) = x_0 t^{q-1} - \int_0^t C(t-s) [x(s) + g(s, x(s))] ds.
\] (4.2)
We assume $| g(t, x) | \leq h(t)$ for $t << 1$ and all $x \in \mathbb{R}$, with
\[
\lim_{t \to 0^+} t^{1-q} \int_0^t C(t-s) h(s) ds = 0.
\] (4.3)
First, we present some known results regarding (4.2) and the associated resolvent equation (see [9, pp. 189-193]). A function $x$ is a solution of (4.2) if and only if $x$ satisfies

$$x(t) = y(t) - \int_0^t R(t-s)g(s, x(s))ds, \quad (4.4)$$

where the function $y$ is given by

$$y(t) = x_0t^{q-1} - \int_0^t R(t-s)x_0s^{q-1}ds, \quad (4.5)$$

and the function $R$, known as the resolvent kernel of $C$, is the solution of the resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds. \quad (4.6)$$

The function $C$ defined in (4.1) is completely monotone on $(0, \infty)$. Thus by [9, Theorem 6.2], the associated resolvent kernel $R$ satisfies, for $t > 0$,

$$0 \leq R(t) \leq C(t), \quad R(t) \to 0 \text{ as } t \to \infty \quad (4.7)$$

and that

$$C \notin L^1(0, \infty) \implies \int_0^\infty R(t)dt = 1. \quad (4.8)$$

If $x$ satisfies (4.4), it must satisfy the condition $\lim_{t \to 0^+} t^{1-q}x(t) = x_0$. To see this, notice that (4.3) implies

$$\lim_{t \to 0^+} t^{1-q}\left| \int_0^t R(t-s)g(s, x(s))ds \right| \leq \lim_{t \to 0^+} t^{1-q}\int_0^t C(t-s)h(s)ds$$

$$= 0.$$

Also,

$$\lim_{t \to 0^+} t^{1-q}\int_0^t R(t-s)x_0s^{q-1}ds \leq \lim_{t \to 0^+} t^{1-q}\int_0^t \frac{1}{\Gamma(q)}(t-s)^{q-1}x_0s^{q-1}ds$$

$$= \lim_{t \to 0^+} \frac{\Gamma(q)}{\Gamma(2q)}x_0t^q$$

$$= 0.$$

So

$$\lim_{t \to 0^+} t^{1-q}x(t) = \lim_{t \to 0^+} t^{1-q}y(t) = x_0.$$

Thus if $x$ satisfies (4.4), then $x$ is a solution of (1.1).

Suppose there exists a continuous solution $x(t)$ of (4.2) on $(0, \infty)$. In this section, we show that $x(t)$ is absolutely integrable on $(0, \infty)$. 
Multiplying both sides of (4.6) by \( x_0 \Gamma(q) \) gives

\[ x_0 \Gamma(q) R(t) = x_0 \Gamma(q) C(t) - x_0 \Gamma(q) \int_0^t C(t-s) R(s) ds \]

\[ = x_0 \Gamma(q) \frac{1}{\Gamma(q)} t^{q-1} - x_0 \Gamma(q) \int_0^t \frac{1}{\Gamma(q)} (t-s)^{q-1} R(s) ds \]

\[ = x_0 t^{q-1} - x_0 \int_0^t (t-s)^{q-1} R(s) ds \]

\[ = x_0 t^{q-1} - \int_0^t R(t-s)x_0 s^{q-1} ds \]

\[ = y(t), \]

the last equality coming from (4.5). Therefore \( y(t) \) is a constant multiple of \( R(t) \). Since \( R(t) \) is continuous, so is \( y(t), 0 < t < \infty \). Also, by (4.7) and (4.8), it is clear that

\[ \int_0^\infty y(t) dt < \infty. \]

**Remark 4.1.** If \( g(t, x) \equiv 0 \), then (1.1) becomes the linear initial value problem

\[ D_0^q x(t) = -x(t), \quad \lim_{t \to 0^+} t^{1-q} x(t) = x_0 \neq 0. \]  

(4.9)

Here

\[ x(t) = y(t) = x_0 t^{q-1} - \int_0^t R(t-s)x_0 s^{q-1} ds. \]

Since \( y(t) \) is a constant multiple of \( R(t) \), this also implies \( x(t) \) is a constant multiple of \( R(t) \). Then (4.7) and (4.8) imply that \( x \) is absolutely integrable on \((0, \infty)\).

For a specific example, consider (4.9) with \( q = \frac{1}{2} \) and \( x_0 = \frac{1}{\sqrt{\pi}} \). In [11, p. 138], it is shown that the solution of the differential equation is given by

\[ x(t) = C \left( \frac{1}{\sqrt{\pi t}} - e^t \operatorname{erfc}(\sqrt{t}) \right), \]

for \( t > 0 \). The initial condition gives \( C = 1 \). Notice

\[ \int_0^\infty \left( \frac{1}{\sqrt{\pi t}} - e^t \operatorname{erfc}(\sqrt{t}) \right) dt = 1, \]

so \( x \) is absolutely integrable on \((0, \infty)\). In this case, \( x(t) = R(t) \), and so

\[ R(t) = \frac{1}{\sqrt{\pi t}} - e^t \operatorname{erfc}(\sqrt{t}), \]

which was shown in [5].
Assume \( g \) satisfies a global Lipschitz condition
\[ |g(t, x) - g(t, y)| \leq k|x - y| \text{ for all } t \in (0, \infty), \ x, y \in \mathbb{R}. \]
This condition implies that
\[ |g(t, x)| \leq k|x| + |g(t, 0)| \text{ for all } t \in (0, \infty), \ x \in \mathbb{R}. \]  
(4.10)

**Theorem 4.2.** Suppose for \( t << 1, \) \( |g(t, x)| \leq h(t) \) for all \( x \in \mathbb{R}, \) where \( h \) satisfies (4.3). Suppose \( g \) satisfies (4.10) with \( k < 1, \)
\[ \int_0^\infty |g(t, 0)| \, dt < \infty, \]
and
\[ \lim_{t \to 0^+} \int_0^t |g(s, 0)| \, ds = 0. \]
If there exists a solution \( x \) of equation (4.4) on \( (0, \infty), \) then \( x \) is absolutely integrable on \( (0, \infty). \)

**Proof.** Define, for the solution \( x \) and for \( t > 0, \)
\[ V(t) = \int_0^t \int_{t-s}^\infty R(u) \, du [k|x(s)| + |g(s, 0)|] \, ds. \]
(4.11)
Now
\[ \int_{t-s}^\infty R(u) \, du \leq \int_0^\infty R(u) \, du \leq 1. \]
So
\[ \int_0^t \int_{t-s}^\infty R(u) \, du [k|x(s)| + |g(s, 0)|] \, ds \leq \int_0^t [k|x(s)| + |g(s, 0)|] \, ds. \]
Since \( \lim_{t \to 0^+} t^{1-q} x(t) = x_0, \) there exists a \( T > 0 \) such that
\[ \frac{|x_0|}{2} t^{q-1} \leq |x(t)| \leq \frac{3|x_0|}{2} t^{q-1}, \quad t \in (0, 1). \]
So
\[
0 \leq V(t) \\
\leq \int_0^t [k|x(s)| + |g(s, 0)|] \, ds \\
\leq \int_0^t k \frac{3|x_0|}{2} s^{q-1} \, ds + \int_0^t |g(s, 0)| \, ds \\
\leq k \frac{3|x_0| t^q}{2q} + \int_0^t |g(s, 0)| \, ds.
\]
Since \( \lim_{t \to 0^+} \int_0^t |g(s, 0)| \, ds = 0 \), it follows that

\[
\lim_{t \to 0^+} V(t) = 0.
\]

So \( V \) can be defined on \([0, \infty)\) so that \( V(0) = 0 \).

Next,

\[
V'(t) = \int_0^\infty R(u) du [k|x(t)| + |g(t, 0)|] - \int_0^t R(t-s)[k|x(s)| + |g(s, 0)|] \, ds. \tag{4.12}
\]

Since \( \int_0^\infty R(u) du = 1 \), (4.12) implies that

\[
V'(t) = [k|x(t)| + |g(t, 0)|] - \int_0^t R(t-s)[k|x(s)| + |g(s, 0)|] \, ds. \tag{4.13}
\]

Now, from (4.4),

\[
|x(t)| \leq |y(t)| + \int_0^t R(t-s)|g(s, x(s))| \, ds \\
\leq |y(t)| + \int_0^t R(t-s)[k|x(s)| + |g(s, 0)|] \, ds.
\]

Therefore

\[
-\int_0^t R(t-s)[k|x(s)| + |g(s, 0)|] \, ds \leq |y(t)| - |x(t)|.
\]

So (4.13) gives

\[
V'(t) \leq k|x(t)| + |g(t, 0)| + |y(t)| - |x(t)| \\
= (k-1)|x(t)| + |y(t)| + |g(t, 0)|.
\]

Integrating from 0 to \( t \) yields

\[
V(t) - V(0) \leq (k-1) \int_0^t |x(s)| \, ds + \int_0^t |y(s)| \, ds + \int_0^t |g(s, 0)| \, ds.
\]

Since \( V'(t) \geq 0 \) and \( V(0) = 0 \), the previous inequality implies that

\[
(1-k) \int_0^t |x(s)| \, ds \leq \int_0^t |y(s)| \, ds + \int_0^t |g(s, 0)| \, ds. \tag{4.14}
\]

Since

\[
\int_0^\infty |y(t)| \, dt < \infty
\]
and
\[ \int_0^\infty |g(t,0)|dt < \infty, \]
it follows from (4.14) that
\[ \int_0^\infty |x(t)|dt < \infty, \]
proving \( x \) is absolutely integrable on \((0, \infty)\).

\[ \Box \]

**Example 4.3.** Consider the fractional differential equation
\[ D_0^q x(t) = \begin{cases} -x + t^{q-1} - \sin x, & 0 \leq t \leq 1, \\ -x + t^{q-2} - \sin x, & 1 \leq t, \end{cases} \quad \lim_{t \to 0^+} t^{1-q}x(t) = x_0 \neq 0. \quad (4.15) \]

Here
\[ f(t, x) = \begin{cases} x - t^{q-1} + \sin x, & 0 \leq t \leq 1, \\ x - t^{q-2} + \sin x, & 1 \leq t, \end{cases} \]
and
\[ g(t, x) = \begin{cases} -t^{q-1} + \sin x, & 0 \leq t \leq 1, \\ -t^{q-2} + \sin x, & 1 \leq t. \end{cases} \]

Now
\[ |f(t, x) - f(t, y)| \leq |x - y| + |\sin x - \sin y| \leq 2|x-y|, \]
so (A1) holds. Set \( f(t, 0) = f_0(t) \). Then
\[ \frac{|f_0(t)|}{t^{q-1}} = \begin{cases} 1, & 0 \leq t \leq 1, \\ t^{-1}, & 1 \leq t, \end{cases} \]

So \(|f_0|_q = 1 \) and \( f_0 \in X \). Therefore Theorem 3.5 gives the existence of a \( T^* > 0 \) such that (4.15) has a unique solution \( x \) on \((0, T^*)\).

Notice for small \( t \), \( |g(t, x)| \leq t^{q-1} + 1 \). So
\[ t^{1-q} \int_0^t \frac{1}{\Gamma(q)} (t-s)^{q-1}(s^{q-1} + 1)ds = \frac{\Gamma(q)}{\Gamma(2q)} t^q + \frac{1}{\Gamma(q+1)} t, \]
implying
\[ \lim_{t \to 0^+} t^{1-q} \int_0^t \frac{1}{\Gamma(q)} (t-s)^{q-1}(s^{q-1} + 1)ds = 0. \]
The Lipschitz condition holds since
\[ |g(t, x) - g(t, y)| = |\sin x - \sin y| \leq |x - y|. \]

Now,
\[ \int_0^\infty |g(t, 0)|dt = \int_0^1 t^{q-1}dt + \int_1^\infty t^{q-2}dt = \frac{1}{q} + \frac{1}{1-q} < \infty. \]
Finally, for small $t$,

$$
\int_0^t |g(s,0)|ds = \frac{t^q}{q}.
$$

Hence

$$
\lim_{t \to 0^+} \int_0^t |g(s,0)|ds = 0.
$$

Therefore, if there exists a solution $x^*$ of (4.15) that can be extended to $(0, \infty)$, Theorem 4.2 guarantees that $x^*$ is absolutely integrable on $(0, \infty)$.

References


