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Boundedness, Periodicity and Stability in Nonlinear Delay Differential Equations

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Abstract

The objective of this paper is to show a new method for inverting first order ordinary differential equations with time-delay terms to obtain a new variation of parameters formula. Then, we will resort to the contraction mapping principle to obtain results concerning boundedness, stability and periodicity. We present several instances of these equations and examples.

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1 Introduction and Basic Example

It is customary in a nonlinear differential equation to add and subtract a convenient term that allows us to invert the equation in question and obtain a variation of parameters formula that can be used to obtain different results on the solutions. However, the added term will cause restrictions on the coefficients and, in turn, limit the class of equations that can be considered. To see this we begin by considering the following totally delayed nonlinear delay differential equation

$$\dot{x}(t) = a(t)l(x_r), \tag{1.1}$$

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where $\dot{x} = \frac{d}{dt}x$, $x_r(t) = x(t-r)$ for $r > 0$ constant and $l = l(t)$ is a continuous function that satisfies more conditions to be imposed later. Assume the existence of a continuous initial function $\psi : [-r, 0] \to \mathbb{R}$ and we rewrite (1.1) in the form

$$\begin{aligned} \dot{x}(t) &= a(t+r)l(x) - \frac{d}{dt}\int_{t-r}^{t} a(s+r)l(x(s))ds \\ &= a(t+r)x(t) - a(t+r)[x(t) - l(x(t))] \\ &- \frac{d}{dt}\int_{t-r}^{t} a(s+r)l(x(s))ds. \end{aligned} \tag{1.2}$$

Note that we added and subtracted $a(t+r)x$ so the inversion is possible. Thus, by the variation of parameters formula we obtain the integral

$$\begin{aligned} x(t) &= \psi(0)e^{\int_0^t a(s+r)ds} \\ &- \int_0^t e^{\int_s^t a(u+r)du} a(s+r)[x(s) - l(x(s))]ds \\ &- \int_0^t e^{\int_s^t a(u+r)du} \frac{d}{ds}\int_{s-r}^{s} a(u+r)l(x(u))du\, ds. \end{aligned} \tag{1.3}$$

The appearance of the term $x(s) - l(x(s))$ in (1.3) is a direct consequence of the borrowed term. In addition, to get any meaningful results one would have to assume that $l(x(t))$ is odd. Otherwise, $x(s) - l(x(s))$ will not define a contraction. For more on such inversion, we refer the reader to the book [1] and the article [2]. In [8], Raffoul considered the nonlinear functional delay differential equation that arises from population models

$$x'(t) = g(x(t)) - g(x(t-L)).$$

and used the same techniques to overcome some of the difficulties that arise from straight inversion as we saw above. Next, we invert our way by assuming $\nu : [0, \infty) \to \mathbb{R}$ to be a nonnegative function such that $0 < \int_0^\infty \nu(s)\, ds = m < \infty$. In order to solve (1.1), we apply an inversion technique that starts by multiplying both sides by $e^{\int_0^t \nu}$ and then integrates them. We simplify notation here and drop the ds at the end of the exponent.

$$\int_0^t e^{\int_0^s \nu} \dot{x}(s)ds = x(t)e^{\int_0^t \nu} - x(0) - \int_0^t x\nu e^{\int_0^s \nu}\, ds$$

Therefore (1.1) becomes

$$\begin{aligned} x(t)e^{\int_0^t \nu} &= x(0) + \int_0^t x\nu e^{\int_0^s \nu}\, ds + \int_0^t e^{\int_0^s \nu} a(s)l(x_r)\, ds, \\ x(t) &= x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu}\, ds + \int_0^t e^{-\int_s^t \nu} a(s)l(x_r)\, ds \end{aligned} \tag{1.4}$$

By imposing adequate conditions on $\nu(t), c(t)$ and $l(t)$, we will prove the existence of bounded solutions to (1.1).

1.1 The Fixed Point Theorem

In order to prove existence of bounded solutions to (1.4), we will use the right-hand side to define a contraction map on a complete metric space $\mathcal{S}$. The resulting unique fixed point will be the solution we are looking for. The subject of stability using fixed point arguments in delay differential equations is vast and we refer to [1–3, 6].

Let $K > 0$ be a constant and fix an initial continuous function $\Psi : [-r, 0] \to \mathbb{R}$ with $|\Psi(t)| < K$ for $t \in [-r, 0]$ and $|\Psi(0)| > 0$.

Define the space

$$\mathcal{S} := \{x : [-r, \infty) \to \mathbb{R} | x \in \mathcal{C}^1[-r, \infty), \|x\|_\infty \leq K, x \equiv \Psi \text{ on } [-r, 0]\}, \tag{1.5}$$

where $\mathcal{C}^1[-r, \infty)$ denotes the space of continuously differentiable functions and $\|\cdot\|_\infty$ is the supremum norm. By general principles it follows that $\mathcal{S}$ is a complete metric space.

Define a map $\mathfrak{P}$ on $\mathcal{S}$ by

$$\mathfrak{P}(x) := x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu}\, ds + \int_0^t e^{-\int_s^t \nu} a(s) l(x_r)\, ds. \tag{1.6}$$

The following lemma can be adapted to several different situations.

Lemma 1.1. *Let $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ be a triple of positive numbers such that $\epsilon_1 + \epsilon_2 + \epsilon_3 \leq 1 - \delta$ for a fixed $0 < \delta < 1$. Assume that $|\Psi(0)| \leq \epsilon_1 K$ and m is small enough that $|1 - e^{-m}| < \epsilon_2$. Assume that $l(y)$ is Lipschitz on $[-K, K]$ and satisfies $|l(y)| \leq C_l |y|$ for $y \in [-K, K]$ and for some positive constant C_l. Suppose that $a(s) \in L^1([0, \infty))$ and $\int_0^\infty |a(s)|\, ds \leq \frac{\epsilon_3}{C_l}$. Then the map $\mathfrak{P}$ has range in $\mathcal{S}$ and it is a contraction. This implies that* (1.1) *has a unique solution in $\mathcal{S}$.*

Proof. We use the hypotheses of the theorem to bound each of the summands on the right-hand side of (1.6). The first term satisfies the bound

$$\left| x(0)e^{-\int_0^t \nu} \right| \leq \epsilon_1 K,$$

since $|e^{-\int_s^t \nu}| \leq 1$. The second summand can be bounded by

$$\left| \int_0^t x\nu e^{-\int_s^t \nu}\, ds \right| \leq \|x\|_\infty \int_0^t \nu e^{-\int_s^t \nu}\, ds \leq (1 - e^{-m})\|x\|_\infty \leq \epsilon_2 K.$$

The conditions stated above allow us to bound the third summand by

$$\left| \int_0^t e^{-\int_s^t \nu} a(s) l(x_r)\, ds \right| \leq \epsilon_3 \|x\|_\infty \leq \epsilon_3 K.$$

Therefore,

$$|\mathfrak{P}(x)| \leq K\epsilon_1 + K\epsilon_2 + K\epsilon_3 \leq (1-\delta)K. \tag{1.7}$$

This implies $\|\mathfrak{P}(x)|\|_\infty \leq (1-\delta)K$ so $\mathfrak{P}$ has range in $\mathcal{S}$.

The contraction part follows similarly,

$$\begin{aligned} |\mathfrak{P}(x) - \mathfrak{P}(y)| &\leq |x-y| \left(\int_0^t \nu e^{-\int_s^t \nu} ds + C_l \int_0^t |a(s)|\, ds \right) \\ &\leq |x-y| \left(\epsilon_2 + C_l \frac{\epsilon_3}{C_l} \right) \\ &\leq (1-\delta)|x-y|. \end{aligned} \tag{1.8}$$

Therefore, $\|\mathfrak{P}(x) - \mathfrak{P}(y)|\|_\infty \leq (1-\delta)\|x-y\|_\infty$. and $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$ is a contraction. □

1.2 A General ODE with Constant Time-Delayed Terms.

Now we consider a slightly more general equation

$$\dot{x}(t) = a(t)g(x_r) + b(t)G(x_r), \tag{1.9}$$

where a, b, g, G are all continuous on $[-r, \infty)$. Let us denote the right-hand side of (1.9) by $L(x_r)$. On the same space $\mathcal{S}$ as before we define a new map

$$\hat{\mathfrak{P}}(x) := x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu}\, ds + \int_0^t e^{-\int_s^t \nu} L(x_r)\, ds. \tag{1.10}$$

Impose the same conditions on ν as in Lemma 1.1. The only difference now is the third term. We impose the following conditions:

C1- Assume g and G are Lipschitz on $[-K, K]$. Explicitly, there are positive constants C_g and C_G such that, for every $y \in [-K, K]$, $|g(y)| \leq C_g|y|$ and $|G(y)| \leq C_G|y|$.

C2- Assume $a(s), b(s) \in L^1([0,\infty))$, $\int_0^\infty |a(s)|\, ds \leq \frac{\epsilon_3}{2C_g}$ and $\int_0^\infty |b(s)|\, ds \leq \frac{\epsilon_3}{2C_G}$.

Lemma 1.2. *Using the notation and definitions from Lemma 1.1 and conditions C1–C2 above, we get that* $\hat{\mathfrak{P}} : \mathcal{S} \to \mathcal{S}$ *is a contraction.*

Proof. The only difference is the third summand in the definition of $\hat{\mathfrak{P}}$. We bound it as follows:

$$\left| \int_0^t e^{-\int_s^t \nu} L(x_r)\, ds \right| \leq K \left(C_g \frac{\epsilon_3}{2C_g} + C_G \frac{\epsilon_3}{2C_G} \right) = K\epsilon_3,$$

and the rest follows similarly to the proof of Lemma 1.1. □

As a consequence, we find that (1.9) has a unique bounded solution in $\mathcal{S}$.

2 The Main Inversion

Now we present a more general way of inverting (1.9) that starts by rewriting it as

$$\begin{aligned}\dot{x}(t) &= (a+b)g(x_r) + b(G(x_r) - g(x_r)) \\ &= -\frac{d}{dt}\int_{t-r}^{t} c(p+r)g(x(p))\,dp + c(t+r)g(x(t)) + bl(x_r),\end{aligned} \tag{2.1}$$

where $c = a+b$ and $l = G-g$. Now multiply both sides by $e^{\int_0^t \nu}$ and integrate as before to solve for $x(t)$. We get

$$\begin{aligned}x(t) =& x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu} ds \\ & - \int_0^t e^{-\int_s^t \nu}\left(\frac{d}{ds}\int_{s-r}^{s} c(p+r)g(x(p))\,dp\right) ds \\ & + \int_0^t c(s+r)g(x)e^{-\int_s^t \nu} ds + \int_0^t e^{-\int_s^t \nu} bl(x_r)ds.\end{aligned} \tag{2.2}$$

Apply integration by parts in the middle line to get

$$\begin{aligned}x(t) =& x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu} ds \\ & - F(t) + e^{-\int_0^t \nu}F(0) + \int_0^t \nu e^{-\int_s^t \nu} F(s)\,ds \\ & + \int_0^t c(s+r)g(x)e^{-\int_s^t \nu} ds + \int_0^t e^{-\int_s^t \nu} bl(x_r)ds,\end{aligned} \tag{2.3}$$

where $F(t) = \displaystyle\int_{t-r}^{t} c(p+r)g(x(p))dp.$

Again, we consider the same space $\mathcal{S}$ and define a new map

$$\begin{aligned}\mathfrak{P}(x) =& x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu} ds \\ & - F(t) + e^{-\int_0^t \nu}F(0) + \int_0^t \nu e^{-\int_s^t \nu} F(s)\,ds \\ & + \int_0^t c(s+r)g(x)e^{-\int_s^t \nu} ds + \int_0^t e^{-\int_s^t \nu} bl(x_r)ds,\end{aligned} \tag{2.4}$$

We need to impose a different set of conditions on the summands on the right-hand side of (2.4) in order to get a contraction mapping of $\mathcal{S}$. First label the terms on the right-hand side 1-7. The new bounds on the absolute values the terms are now:

D1- The first two summands are bounded exactly as in section 1.1. That is $x(0) = \Psi(0) \leq \epsilon_1 K$ and $\|\nu\|_{L^1} = m > 0$ such that $1 - e^{-m} \leq \epsilon_2$.

D2- To bound $F(t)$, assume $\int_0^\infty |c|\,dt \le \int_0^\infty |a|\,dt + \int_0^\infty |b|\,dt \le \dfrac{\epsilon_3}{C_g + C_G}$. Here C_g and C_G are the Lipschitz constants of the delayed functions. The third and fourth terms are each bounded above by $\epsilon_3 K$.

D3- The definitions above produce an upper bound for the fifth term equal to $m\epsilon_3 K$, where $m = \int_0^\infty |\nu| dt$.

D4- The sixth term is again bounded above by $\epsilon_3 K$.

D5- The upper bound on b above implies that the seventh term is also bounded above by $\epsilon_3 K$.

Therefore,

$$|\mathfrak{P}(x)| \le K(\epsilon_1 + \epsilon_2 + (m+4)\epsilon_3). \tag{2.5}$$

Theorem 2.1. *If the bounds D1–D5 above hold and there is $0 < \delta < 1$ such that $\epsilon_1 + \epsilon_2 + (m+4)\epsilon_3 \le 1 - \delta$ then the map $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$ is a contraction with a unique fixed point. This fixed point is a solution to* (1.9) *that belongs to $\mathcal{S}$.*

Example 2.2. Let $\alpha, \beta > 0$ such that

$$\dot{x} = e^{-\alpha t} g(x_r) + e^{-\beta t} G(x_r), \tag{2.6}$$

where α and β are big enough that $\alpha^{-1} + \beta^{-1} \le \dfrac{\epsilon_3}{C_g + C_G}$. This equation satisfies the previous conditions. The equation (2.6) is a first order differential equation with exponentially damped time-delayed terms. Theorem 2.1 implies the existence of a solution to (2.6) in $\mathcal{S}$.

3 Variable Time Delay

Here we consider $r = r(t)$ variable. We impose conditions on it later on. We want to find solutions to

$$\dot{x}(t) = a(t) g(x(t - r(t))) + b(t) G(x(t - r(t))). \tag{3.1}$$

We still denote $x(t - r(t))$ by x_r. The same strategy as before gives

$$\begin{aligned} \dot{x}(t) &= (a+b) g(x_r) + b(G(x_r) - g(x_r)) \\ &= c(t) g(x_r) + b l(x_r) \\ &= \frac{c g(x_r)}{1 - \dot{r}}(1 - \dot{r}) + b l(x_r). \end{aligned} \tag{3.2}$$

Denote $f = \dfrac{c}{1 - \dot{r}}$ and $f_r = f(t - r(t))$. Now we rewrite (3.2) as follows

$$
\begin{aligned}
\dot{x} &= f_r g(x_r)(1-\dot{r}) + (f(t) - f_r)g(x_r)(1-\dot{r}) + bl(x_r) \\
&= (f_r g(x_r)(1-\dot{r}) - f(t)g(x(t))) + f(t)g(x(t)) \\
&+ (f(t) - f_r)g(x_r)(1-\dot{r}) + bl(x_r) \\
&= -\left(\frac{d}{dt}\int_{t-r(t)}^{t} f(s)g(x(s))ds\right) + f(t)g(x(t)) \\
&+ (f(t) - f_r)g(x_r)(1-\dot{r}) + bl(x_r)
\end{aligned} \tag{3.3}
$$

To simplify, the upcoming expression assume $r(0) = 0$.

The inversion gives

$$
\begin{aligned}
x(t) &= x(0)e^{-\int_0^t \nu} + \int_0^t x\nu e^{-\int_s^t \nu} ds \\
&- \int_{t-r(t)}^{t} f(p)g(x(p))dp + \int_0^t \nu e^{-\int_s^t \nu} \int_{s-r(s)}^{s} f(p)g(x(p))dpds \\
&+ \int_0^t e^{-\int_s^t \nu} \left(f(s)g(x(s)) + (f(s) - f_r)g(x_r)(1-\dot{r}) + bl(x_r)\right) ds,
\end{aligned} \tag{3.4}
$$

The most important condition to be imposed on r is $|\dot{r}(t)| \leq \kappa < 1$ for some κ small and positive. This condition allows us to get uniform pointwise upper bounds on the function $f = \dfrac{c}{1-\dot{r}}$. From the definition of $\mathcal{S}$, we impose that $r(t) \leq r$ for all $t \geq 0$. Now we impose bounds on the different terms.

E1- The first two summands are bounded exactly as in section 1.1. That is $x(0) = \Psi(0) \leq \epsilon_1 K$ and $\|\nu\|_{L^1} = m > 0$ such that $1 - e^{-m} \leq \epsilon_2$.

E2- To bound $f(t)$, assume

$$
\begin{aligned}
\int_{-r}^{\infty} |f| \, dt &\leq (1-\kappa)^{-1} \int_{-r}^{\infty} |c| \, dt \\
&\leq (1-\kappa)^{-1} \left(\int_0^{\infty} |a| \, dt + \int_0^{\infty} |b| \, dt\right) \\
&\leq \frac{\epsilon_3}{(1-\kappa)(C_g + C_G)}.
\end{aligned}
$$

This is enough to bound all the terms in the last line as well.

Theorem 3.1. *Define a map $\mathfrak{P}$ on $\mathcal{S}$ by the right-hand side of* (3.4)*. Suppose there are constants $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ such that the bounds E1–E2 above are satisfied and such that $\epsilon_1 + \epsilon_2 + \dfrac{2\kappa + 4}{1-\kappa}\epsilon_3 \leq 1 - \delta$ for some $0 < \delta < 1$, then $\mathfrak{P} : \mathcal{S} \to \mathcal{S}$ is a contraction. This implies that* (3.1) *has a unique solution in $\mathcal{S}$.*

4 Periodicity

The objective of this section is to prove existence of periodic solutions to

$$\dot{x}(t) = a(t)g(x) + b(t)G(x), \tag{4.1}$$

where $x_L = x(t-L)$ and $a(t), b(t)$ are continuous periodic functions with period $L > 0$. We also assume $g = g(t,x) = g(t+L,x)$ and $G = G(t,x) = G(t+L,x)$. We assume g and G are Lipschitz on $[0,L]$ (and extended periodically) with constants C_g and C_G respectively. For more on periodicity we refer to [1, 4–7, 9]. Our aim is to prove that there is a solution to (4.1) contained in the space

$$\mathcal{T} := \{y \in \mathcal{C} | y(t+L) = y(t), \forall t \in \mathbb{R}\}. \tag{4.2}$$

It is well known that $\mathcal{T}$ is a Banach space with the maximum norm.

Choose a positive function $\nu(t) = \nu(t+L)$. Multiply both sides of (4.1) by $e^{\int_0^t \nu}$ and integrate by parts between $t-L$ and t. Manipulations similar to the previous cases give

$$\begin{aligned} x(t) = &\int_{t-L}^{t} x\nu e^{-\int_s^t \nu} ds - \left(1 - e^{-\int_{-L}^{0} \nu}\right) \eta(t) \\ &+ \int_{t-L}^{t} \eta(s)\nu e^{-\int_s^t \nu} ds + \int_{t-L}^{t} e^{-\int_s^t \nu} c(s)g(x(s))ds \\ &+ \int_{t-L}^{t} e^{-\int_s^t \nu} Q(s)ds, \end{aligned} \tag{4.3}$$

where $\eta(t) = \displaystyle\int_{t-L}^{t} c(s)g(x(s))\,ds$ and $Q(s) = b(s)l(x(s))$. Remember that we assume $x \in \mathcal{T}$. It follows that the right-hand side of (4.3) is periodic of period L. Now we impose conditions on the coefficients of (4.1).

F1- Assume $0 < m = \displaystyle\int_{t-L}^{t} \nu ds < \epsilon_1$.

F2- Assume $0 < \displaystyle\int_{t-L}^{t} |c(s)|\,ds \le \frac{\epsilon_2}{C_g + C_G}$.

In a similar fashion to the previous sections, we define a map $\mathfrak{P}$ using the right-hand side of (4.3) and prove the following theorem regarding the existence of a unique periodic solution.

Theorem 4.1. *Assume* $\{\epsilon_1, \epsilon_2\}$ *satisfy* $\epsilon_1 + \epsilon_1\epsilon_2 + 3\epsilon_2 \le 1 - \delta < 1$ *for some* $0 < \delta < 1$ *and the bounds F1–F2 above hold, then* $\mathfrak{P} : \mathcal{T} \to \mathcal{T}$ *is a contraction and* (4.1) *has a unique solution in* $\mathcal{T}$.

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