Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 14, Number 2, pp. 189–199 (2019)
http://campus.mst.edu/adsa

Comparison of Smallest Eigenvalues for Fractional-Order Nonlocal Boundary Value Problems

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Abstract

For $1 < \alpha \leq 2$ a real number, we apply the theory of u_0-positive operators to establish the existence of smallest positive eigenvalues and their comparison for the αth-order Riemann–Liouville linear differential equations, $D_{0+}^{\alpha}y(t)+\lambda p(t)y(t) = 0$ and $D_{0+}^{\alpha}y(t) + \sigma q(t)y(t) = 0$, $0 < t < 1$, with each satisfying the nonlocal boundary conditions, $y(0) = \sum_{i=1}^{p} a_i y(\xi_i)$, $0 < \xi_1 < \cdots < \xi_p < 1$, and $y(1) = \sum_{j=1}^{r} b_j y(\eta_j)$, $0 < \eta_1 < \cdots < \eta_r < 1$.

AMS Subject Classifications: 26A33, 34A08, 34B05, 34B09, 34B10.
Keywords: Riemann–Liouville linear differential equation, fractional order, eigenvalue comparison, nonlocal boundary condition.

1 Introduction

For $1 < \alpha \leq 2$ a real number, we establish the existence of smallest positive eigenvalues and then their comparison for the αth-order Riemann–Liouville linear differential equations,

$$D_{0+}^{\alpha}y(t) + \lambda p(t)y(t) = 0, \quad 0 < t < 1, \tag{1.1}$$

Received January 23, 2019; Accepted February 12, 2019
Communicated by John Graef

$$D_{0+}^{\alpha} y(t) + \sigma q(t) y(t) = 0, \quad 0 < t < 1, \tag{1.2}$$

with each satisfying the nonlocal boundary conditions,

$$y(0) = \sum_{i=1}^{p} a_i y(\xi_i), \quad y(1) = \sum_{j=1}^{r} b_j y(\eta_j), \tag{1.3}$$

where $p, q \in C([0,1],[0,\infty))$ and neither p nor q vanishes identically on any nondegenerate compact subinterval of $[0,1]$, and where $0 < \xi_1 < \cdots < \xi_p < 1$, $0 < \eta_1 < \cdots < \eta_r < 1$, $a_i \geq 0$, $1 \leq i \leq p$, such that $0 < \sum_{i=1}^{p} a_i < 1$, and $b_j \geq 0$, $1 \leq j \leq r$, such that $0 < \sum_{j=1}^{r} b_j < 1$.

Application is made of the theory of u_0-positive operators with respect to a cone in a Banach space as developed in the book by Krasnosel'skii [35] and in the book by Krein and Rutman [36]. Applications of these methods have seen extensive use in papers devoted to eigenvalue problems for differential equations, finite difference equations and dynamic equations on time scales; to name a few references, we suggest some respective groups of papers such as [3–5, 8, 9, 17, 30, 31, 34, 38–40, 44], [2, 13, 14] and [1, 28, 29, 37].

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics. They are now commonly viewed as better tools for the description of hereditary properties of various materials and processes than the corresponding integer-order differential equations [10, 33, 41–43]. Much current research concerning boundary value problems for fractional differential equations has involved conditions for existence of positive solutions; see, for example [11, 12, 16, 18, 20–26, 47, 48] and references therein.

And while a few papers [6, 7, 15, 27, 45, 46] have been devoted to eigenvalue comparisons for fractional boundary value problems, this paper employs the Krein-Rutman theory for nonlocal (i.e., multi-point) fractional boundary value problems. The results of this paper should motivate some generalizations for nonlocal boundary value problems of higher fractional order.

Sign properties of a Green's function will be of fundamental importance in many of our arguments.

In Section 2, we state preliminary definitions and important results from the theory of u_0-positive operators with respect to a cone in a Banach space. Then, in Section 3, the results of Section 2 are applied in showing existence and comparison of smallest positive eigenvalues of (1.1)–(1.3) with smallest positive eigenvalues of (1.2)–(1.3).

2 Preliminaries

In this section, we state some definitions and theorems from cone theory on which the paper's main results depend.

Let $(B, ||\cdot||)$ be a real Banach space. A nonempty, closed subset P of B is a *cone* provided: (i) $\alpha u + \beta v \in P$, for all $\alpha, \beta \in [0, \infty)$ and for all $u, v \in P$, and (ii) $P \cap (-P) = \{0\}$. A cone P is *solid* if $P^\circ \neq \emptyset$, and P is *reproducing*, if $B = P - P$.

Remark 2.1. Krasnosel'skii [35] proved that every solid cone is reproducing.

A cone P induces a partial order, $\preceq$, on the Banach space B by: for $u, v \in B$, define $u \preceq v$ iff $v - u \in P$. Also, if $M, N : B \to B$ are bounded linear operators, define $M \preceq N$, if $Mu \preceq Nu$, for all $u \in P$.

An operator $L : B \to B$ is said to be u_0*-positive with respect to* P, if there exists $u_0 \in P \setminus \{0\}$ such that, for each $u \in P \setminus \{0\}$, there exist $k_1(u), k_2(u) \in (0, \infty)$ such that $k_1 u_0 \preceq Lu \preceq k_2 u_0$.

The comparison of eigenvalue results for our eigenvalue problems arise from applications of the following three theorems. The proof of the first theorem can be found in many of the papers listed in the bibliography of this paper. The proof of the second theorem can be found in Krasnosel'skii's book [35], and the proof of the third theorem can be found in the paper by Keener and Travis [32].

Theorem 2.2. *Let B be a real Banach space and let $P \subset B$ be a solid cone. If $M : B \to B$ is a linear operator such that $M : P \setminus \{0\} \to P^\circ$, then M is u_0-positive.*

Theorem 2.3. *Let B be a real Banach space and let $P \subset B$ be a reproducing cone. Let $M : B \to B$ be a compact, linear operator, which is u_0-positive. Then M has an essentially unique eigenvector in P, and the corresponding eigenvalue is simple, positive and larger than the absolute value of any other eigenvalue.*

Theorem 2.4. *Let B be a real Banach space and let $P \subset B$ be a cone. Let both $M, N : B \to B$ be bounded, linear operators, and assume that at least one of the operators is u_0-positive. If $M \preceq N$, $\lambda u_1 \preceq M u_1$ for some $u_1 \in P$ and some $\lambda > 0$, and $N u_2 \preceq \sigma u_2$ for some $u_2 \in P$ and some $\sigma > 0$, then $\lambda \leq \sigma$. Furthermore, $\lambda = \sigma$ implies u_1 is a scalar multiple of u_2.*

3 Comparison of Smallest Eigenvalues

We remark that, from the assumptions on a_i and b_i, we have $0 < \sum_{i=1}^{p} a_i \xi_i^{\alpha-1} < 1$ and $0 < \sum_{j=1}^{r} b_j \eta_j^{\alpha-1} < 1$. We also remark that 0 is not an eigenvalue of either (1.1)–(1.3) or (1.2)–(1.3), and so there is a Green's function, $G(t, s)$, for

$$-D_{0+}^{\alpha} y(t) = 0, \tag{3.1}$$

and satisfying the boundary conditions (1.3). We define

$$\Delta := \left(1 - \sum_{j=1}^{r} b_j\right)\left(\sum_{i=1}^{p} a_i \xi_i^{\alpha-1}\right) + \left(1 - \sum_{i=1}^{p} a_i\right)\left(1 - \sum_{j=1}^{r} b_j \eta_j^{\alpha-1}\right). \tag{3.2}$$

From the remark above, it follows that $\Delta > 0$.

Extending the arguments of Henderson and Luca [16, 19], we obtain by direct computation that the Green's function for (3.1)–(1.3) is given by,

$$\begin{aligned} G(t,s) := g(t,s) &+ \frac{1}{\Delta}\left[\left(1 - t^{\alpha-1}\right)\left(1 - \sum_{j=1}^{r} b_j\right) + \sum_{j=1}^{r} b_j\left(1 - \eta_j^{\alpha-1}\right)\right]\sum_{i=1}^{p} a_i g(\xi_i, s) \\ &+ \frac{1}{\Delta}\left[t^{\alpha-1}\left(1 - \sum_{i=1}^{p} a_i\right) + \sum_{i=1}^{p} a_i \xi_i^{\alpha-1}\right]\sum_{j=1}^{r} b_j g(\eta_j, s), \end{aligned}$$

for $(t,s) \in [0,1] \times [0,1]$, and where

$$g(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Properties of $G(t,s)$ that we will subsequently use include:

- For each fixed $0 < s < 1$, as a function of t, $G(t,s)$ satisfies the boundary conditions (1.3).
- $G(t,s) \ge g(t,s) \ge 0$ on $[0,1] \times [0,1]$.
- $G(t,s) > g(t,s) > 0$ on $(0,1) \times (0,1)$.

In applying Theorems 2.2 - 2.4, we will define the Banach space

$$B := C[0,1],$$

with norm

$$||u|| := \max_{0 \le t \le 1} |u(t)|,$$

and we define the cone

$$P := \{u \in B \mid u(t) \ge 0 \text{ on } [0,1]\}.$$

Lemma 3.1. *The cone P has nonempty interior and $Q := \{v \in B \mid v(t) > 0 \text{ on } [0,1]\} \subseteq P^\circ$.*

The following is immediate by Remark 2.1.

Corollary 3.2. *The cone P is reproducing.*

Now, we define linear operators $M, N : B \to B$ by

$$Mu(t) := \int_0^1 G(t,s)p(s)u(s)ds,\ t \in [0,1], \tag{3.3}$$

and

$$Nu(t) := \int_0^1 G(t,s)q(s)u(s)ds,\ t \in [0,1], \tag{3.4}$$

where $G(t,s)$ is the Green's function for (3.1)–(1.3).

Remark 3.3. By making the standard arguments, it is straightforward that both M and N are compact operators.

Remark 3.4. At this point we observe that

$$\Lambda u(t) = Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds,\ t \in [0,1],$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G(t,s)p(s)u(s)ds,\ t \in [0,1],$$

if and only if

$$-D_{0+}^{\alpha}u(t) = \frac{1}{\Lambda}p(t)u(t),\ 0 < t < 1, \text{ and } u(0) = \sum_{i=1}^{p} a_i u(\xi_i),\ u(1) = \sum_{j=1}^{r} b_j u(\eta_j).$$

Namely, the eigenvalues of (1.1)–(1.3) are multiplicative reciprocals of the eigenvalues of M, and conversely. Similarly, the eigenvalues of (1.2)–(1.3) are multiplicative reciprocals of the eigenvalues of N, and conversely.

Theorem 3.5. *The bounded linear operators M and N are u_0-positive.*

Proof. In proving the result for the operator M, it suffices, by Theorem 2.2, to show that $M : P \setminus \{0\} \to P^{\circ}$.

We first show that $M : P \to P$. Choose $u \in P$. Each of $u(t), G(t,s)$ and $p(t)$ is nonnegative valued, and so it follows that, for $0 \le t \le 1$,

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \ge 0.$$

Therefore, $Mu \in P$.

Now, choose $u \in P \setminus \{0\}$. Then by the assumptions on $p(t)$ stated in the Introduction, there exists $[c,d] \subseteq [0,1]$ such that $u(t)p(t) > 0$ on $[c,d]$. Moreover, from the properties above for the Green's function, $G(t,s) > 0$ on $(0,1) \times (0,1)$. Define

$$z(t) := \int_0^1 G(t,s)p(s)u(s)ds.$$

Then $z \in P$, and for $0 < t < 1$,

$$z(t) \geq \int_c^d G(t,s)p(s)u(s)ds > 0.$$

In addition, from properties of the Green's function, $z(t)$ satisfies the boundary conditions (3). Coupled with the stated properties in the Introduction on $a_i, i = 1, \ldots, p$, and on $b_j, j = 1, \ldots, r$,

$$z(0) = \sum_{i=1}^{p} a_i z(\xi_i) > 0 \text{ and } z(1) = \sum_{j=1}^{r} b_j z(\eta_j) > 0.$$

In particular, $z(t) > 0$ on $[0,1]$. That is, $z \in Q \subset P^\circ$, and so $M : P \setminus \{0\} \to P^\circ$. In a similar way, N is also u_0-positive. □

Theorem 3.6. . *The operator M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to belong to P°.*

Proof. M is a u_0-positive, compact linear operator, and so by Theorem 2.3, M has an essentially unique eigenvector, say $u \in P$, and an eigenvalue Λ having the properties in the statement of this theorem. Since $u \neq 0$, $Mu \in Q \subset P^\circ$, and so $u = M\left(\frac{1}{\Lambda}u\right) \in P^\circ$. □

Theorem 3.7. *Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ and Σ be the eigenvalues of Theorem 3.6 corresponding to M and N, respectively, with associated essentially unique eigenvectors, u_1 and u_2, that belong to P°. Then, $\Lambda \leq \Sigma$, and $\Lambda = \Sigma$ if and only if $p(t) = q(t)$, for all $t \in [0,1]$.*

Proof. With $p(t) \leq q(t)$ on $[0,1]$, then, for any $u \in P$ and for any $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G(t,s)[q(s) - p(s)]u(s)ds \geq 0,$$

and so $M \preceq N$. By Theorem 2.4, then $\Lambda \leq \Sigma$.

For the last part of the this theorem, if $p(t) \equiv q(t)$, then of course $\Lambda = \Sigma$. Next, for the purpose of contrapositive argument, suppose $p(t) \not\equiv q(t)$. Then there is some subinterval $[a,b] \subset [0,1]$ on which $p(t) < q(t)$. It follows that $(N - M)(u_1) \in Q \subset P^\circ$. So there exists an $\epsilon > 0$ such that $(N - M)(u_1) - \epsilon u_1 \in P^\circ$, from which

$$\Lambda u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \preceq Nu_1.$$

This implies $(\Lambda + \epsilon)u_1 \preceq Nu_1$, and since $N \preceq N$ and $Nu_2 = \Sigma u_2$, Theorem 2.4 implies that $\Lambda + \epsilon \leq \Sigma$, or $\Lambda < \Sigma$. □

By Remark 3.4 and Theorems 3.6 and 3.7, we can now state our main result concerning existence and comparison of smallest positive eigenvalues for (1.1)–(1.3) and (1.2)–(1.3).

Theorem 3.8. *Assume the hypotheses of Theorem 3.7. Then there exist smallest positive eigenvalues λ and σ of* (1.1)–(1.3) *and* (1.2)–(1.3)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problem, and the eigenvectors corresponding to λ and σ may be chosen to belong to P°. Finally, $\lambda \geq \sigma$, and $\lambda = \sigma$ if and only if $p(t) = q(t)$, for all $t \in [0, 1]$.*

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