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On Oscillatory Second Order Nonlinear Neutral Impulsive Differential Systems with Variable Delay

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Abstract

In this work, sufficient conditions are obtained for oscillatory and asymptotic behavior of second-order neutral impulsive differential systems of the form

$$(E) \qquad \begin{cases} \big[r(t)\big(x(t)+p(t)x(\delta(t))\big)'\big]' + q(t)G\big(x(\sigma(t))\big) = 0, \ t \neq \tau_k, \ k \in \mathbb{N} \\ \Delta\big[r(\tau_k)\big(x(\tau_k)+p(\tau_k)x(\delta(\tau_k))\big)'\big] + h(\tau_k)G\big(x(\sigma(\tau_k))\big) = 0, \ k \in \mathbb{N} \end{cases}$$

for various ranges of the bounded neutral coefficient p. Here one can see, if the differential equations is oscillatory, then the discrete equation of similar type do not disturb the oscillatory behavior of the system (E), when impulse satisfies the discrete equation, and δ' is allowed to be oscillatory. Further, some illustrative examples showing applicability of the new results are included.

Keywords: Oscillation, nonoscillation, nonlinear, delay argument, impulse, second-order neutral impulsive differential systems.
AMS Subject Classifications: 34K.

1 Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in population dynamics, biotechnology,

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control theory, physics, chemistry, industrial robotic, economics, rhythmical beating and to mention a few. Recently, impulsive differential equations have become a very active area of research, since it is much richer than the corresponding theory of differential equations without impulse effect. The monographs by Bainov and Simeonov [1], Bainov and Covachev [2,4], Bainov and Simeonov [3], Dishliev et al. [7], Lakshmikantham et al. [10], and Samoilenko and Perestynk [12] are the excellent sources for applications and properties of various impulsive differential equations.

Tripathy and Santra [15] have established necessary and sufficient conditions for oscillatory and asymptotic behavior of first-order forced impulsive differential systems of the form

$$(E1) \qquad \begin{cases} \big(x(t)+p(t)x(t-\tau)\big)'+q(t)G\big(x(t-\sigma)\big)=f(t),\ t\neq\tau_k,\ k\in\mathbb{N} \\ \Delta\big(x(\tau_k)+p(\tau_k)x(\tau_k-\tau)\big)+h(\tau_k)G\big(x(\tau_k-\sigma)\big)=g(\tau_k),\ k\in\mathbb{N} \end{cases}$$

for various ranges of the neutral coefficient p. Tripathy, Santra and Pinelas [14, 16], have studied the homogeneous counter part of (E1) and established sufficient and necessary conditions for oscillation of all solutions for different ranges of $p(t)$. Tripathy and Santra [19] have studied oscillation and nonoscillation properties of second-order neutral impulsive differential systems with constant coefficients and constant delays of the form

$$\begin{cases} \big(x(t)+ry(t-\tau)\big)''+qx(t-\sigma)=0,\ t\neq\tau_k,\ k\in\mathbb{N} \\ \Delta\big(x(\tau_k)+rx(\tau_k-\tau)\big)'+px(\tau_k-\sigma)=0,\ k\in\mathbb{N} \end{cases}$$

by using pulsatile constant for $r\in\mathbb{R}-\{0\}$. Also, they have made an attempt to extend the constant coefficient results to variable coefficient result. For more information related to oscillation and asymptotic behavior of solutions to this type of systems, we refer the readers to [5, 6, 8, 11, 13, 14, 17, 18, 20]. Motivated by the above works, an attempt is made here to establish sufficient condition for oscillatory and asymptotic behavior of solutions of a class of nonlinear neutral second-order impulsive differential systems of the form

$$(E) \qquad \begin{cases} \big[r(t)\big(x(t)+p(t)x(\delta(t))\big)'\big]'+q(t)G\big(x(\sigma(t))\big)=0,\ t\neq\tau_k,\ k\in\mathbb{N} \\ \Delta\big[r(\tau_k)\big(x(\tau_k)+p(\tau_k)x(\delta(\tau_k))\big)'\big]+h(\tau_k)G\big(x(\sigma(\tau_k))\big)=0,\ k\in\mathbb{N}. \end{cases}$$

We suppose that the following assumptions holds:

(A1) Let τ_k, $k\in\mathbb{N}$ with $\tau_1<\tau_2<\ldots<\tau_k<\ldots$ and $\lim\limits_{k\to\infty}\tau_k=+\infty$ are fixed moments of impulsive effect;

(A2) Δ is the difference operator defined by $\Delta V(\tau_k)=V(\tau_k+0)-V(\tau_k-0)$;

(A3) $r\in \mathrm{C}\big([t_0,\infty),(0,\infty)\big)$ and $q,h\in \mathrm{C}\big([t_0,\infty),[0,\infty)\big)$, where q and h are not identically zero eventually;

(A4) $G \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ is non-decreasing, and satisfies $uG(u) > 0$ for $u \neq 0$;

(A5) $\delta, \sigma \in \mathrm{C}\big([t_0, \infty), \mathbb{R}\big)$ such that $\delta(t), \sigma(t) \leq t$ for $t \geq t_0$, $\delta(t), \sigma(t) \to \infty$ as $t \to \infty$ with differentiable and invertible δ when necessary.

The investigation on the asymptotic behavior of solutions depend on the following two possible conditions:

(C1) $\displaystyle\int^{\infty} \frac{1}{r(\eta)} d\eta = \infty$ if and only if $\displaystyle\sum_{k=1}^{\infty} \frac{1}{r(\tau_k)} = \infty$;

(C2) $\displaystyle\int^{\infty} \frac{1}{r(\eta)} d\eta < \infty$ if and only if $\displaystyle\sum_{k=1}^{\infty} \frac{1}{r(\tau_k)} < \infty$.

We denote by $C(\mathbb{R}, \mathbb{R})$ the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $PC(\mathbb{R}_+, \mathbb{R})$ the space of left continuous functions $U : \mathbb{R}_+ \to \mathbb{R}$ such that, for each $t \in \mathbb{R}_+$, $U(t-0) = U(t)$ and the right limit $U(t+0)$ exists, where $U(t-0) = \lim\limits_{h \to 0-} U(t+h)$ and $U(t+0) = \lim\limits_{h \to 0+} U(t+h)$.

Let $\rho(t) = \min\limits_{t \in \mathbb{R}_+}\{\delta(t), \sigma(t)\}$. We say that a real valued function $x(t)$ is a solution of the system (E), if there exists a number $T_0 \in \mathbb{R}$ such that $x \in PC\big([\rho(T_0), +\infty), \mathbb{R}\big)$, the function

$$z(t) = x(t) + p(t)x(\delta(t)) \tag{1.1}$$

and $r(t)z'(t)$ are continuously differentiable for $t \geq T_0$, $t \neq \tau_k$, $k \in \mathbb{N}$ and $x(t)$ satisfies (E) for all $t \geq T_0$.

Without further mentioning we will assume throughout this paper, that every solution $x(t)$ of the system (E) that is under consideration here, is continuable to the right and is nontrivial. That is, $x(t)$ is defined on some ray of the form $[T_x, +\infty)$ and for each $T \geq T_x$ it is fulfilled $\sup\{|x(t)| : t \geq T\} > 0$. Such a solution is called a regular solution of (E).

We say that a real valued function u defined on an interval $[a, +\infty)$ has some property eventually, if there is a number $b \geq a$ such that u has this property on the interval $[b, +\infty)$.

A regular solution $x(t)$ of equation (E) is said to be nonoscillatory, if there exists a number $t_0 \geq 0$ such that $x(t)$ is of constant sign for every $t \geq t_0$. Otherwise, it is called oscillatory. Also, note that a nonoscillatory solution is called eventually positive (eventually negative), if the constant sign that determines its nonoscillation is positive (negative). The system (E) is called oscillatory , if all its solutions are oscillatory.

Remark 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2 Sufficient Conditions for Oscillation

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of second order nonlinear neutral impulsive differential systems of the form (E).

2.1 Oscillation under Condition (C1).

Lemma 2.1. *Assume that (C1) and (A1)–(A5) hold. If x is an eventually positive solution of (E) such that the companion function z defined by* (1.1) *is also eventually positive, then z satisfies*

$$z'(t) > 0 \quad \textit{and} \quad (rz')'(t) < 0 \quad \textit{for all large } t. \tag{2.1}$$

Proof. Suppose that $x(t) > 0$ and $z(t) > 0$ for $t \geq t_1$, where $t \geq t_0$. By (A5), we may assume without loss of generality that $x(\sigma(t)) > 0$ for $t \geq t_1$. From (E) and (A4), it follows that

$$\begin{aligned} (rz')'(t) &= -q(t)G\left(x(\sigma(t))\right) < 0 \\ \Delta(rz')(\tau_k) &= -h(\tau_k)G\left(x(\sigma(\tau_k))\right) < 0 \end{aligned} \tag{2.2}$$

Consequently, rz' is non-increasing on $[t_1, \infty)$ and thus either $z'(t) < 0$ or $z'(t) > 0$ for $t \geq t_2$, where $t_2 \geq t_1$. If $z'(t) < 0$, then there exists $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_2$. A similar argument holds for the discrete equation, and we have $r(\tau_k)z'(\tau_k) \leq -\varepsilon$. Integrating the relation $z'(t) \leq -\dfrac{\varepsilon}{r(t)}$ over $[t_2, t) \subset [t_2, \infty)$, we obtain

$$z(t) - z(t_2) - \sum_{t_2 \leq \tau_k < t} z'(\tau_k) \leq -\varepsilon \int_{t_2}^{t} \frac{1}{r(\eta)} d\eta \quad \text{for } t \geq t_2,$$

that is,

$$z(t) \leq z(t_2) - \varepsilon \Big[\int_{t_2}^{t} \frac{1}{r(\eta)} d\eta + \sum_{t_2 \leq \tau_k < t} \frac{1}{r(\tau_k)} \Big]. \tag{2.3}$$

In view of (C1), letting $t \to \infty$ in (2.3) yields $z(t) \to -\infty$, which is a contradiction. Therefore, $z'(t) > 0$ for $t \geq t_2$. This completes the proof. □

Remark 2.2. It follows from Lemma 2.1 that $\lim\limits_{t\to\infty} z(t) > 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for all large t.

Lemma 2.3. *Assume that (C1) and (A1)–(A5) hold. If x is an eventually positive solution of (E) such that the companion function z defined by* (1.1) *is bounded, then z satisfies* (2.1) *for all large t.*

Proof. The proof can be obtained from the proof of Lemma 2.1. □

Theorem 2.4. *Let* $0 \leq p(t) \leq p < 1$ *for* $t \geq t_0$*, where* p *is a constant. Assume that (C1) and (A1)–(A5) hold. Furthermore, assume that*

$$\text{(A6)} \quad \int^{\infty} q(\eta)d\eta + \sum_{k=1}^{\infty} h(\tau_k) = \infty.$$

Then, every solution of (E) is oscillatory.

Proof. Suppose the contrary that x is a nonoscillatory solution of (E). Then, there exists $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \geq t_1$. Assume that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Clearly, z defined by (1.1) is positive on $[t_1, \infty)$. By Lemma 2.1 and Remark 2.2, there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for $t \geq t_2$, where $t_2 \geq t_1$. On the other hand, z being increasing implies that

$$\begin{aligned}(1-p)z(t) \leq &\big(1-p(t)\big)z(t) \leq z(t) - p(t)z(\delta(t)) \\ =&x(t) - p(t)p(\delta(t))x(\delta(\delta(t))) \leq x(t)\end{aligned}$$

for $t \geq t_3$, where $t_3 \geq t_2$. Consequently, $x(t) \geq (1-p)\varepsilon > 0$ for $t \geq t_3$. From (2.2), we have

$$\begin{aligned}(rz')'(t) + G\big((1-p)\varepsilon\big)q(t) \leq 0 \\ \Delta(rz')(\tau_k) + G\big((1-p)\varepsilon\big)h(\tau_k) \leq 0\end{aligned}$$

for $t \geq t_3$. Integrating the last system over the interval $[t_3, t) \subset [t_3, \infty)$, we get

$$G\big((1-p)\varepsilon\big) \int_{t_3}^{t} q(\eta)d\eta - \sum_{t_3 \leq \tau_k < t} \Delta(rz')(\tau_k) \leq - \big[(rz')(t)\big]_{t_3}^{t},$$

that is,

$$G\big((1-p)\varepsilon\big)\Big[\int_{t_3}^{t} q(\eta)d\eta + \sum_{t_3 \leq \tau_k < t} h(\tau_k)\Big] \leq r(t_3)z'(t_3)$$

for $t \geq t_3$. This contradicts (A6).

If $x(t) < 0$ for $t \geq t_1$, then we set $y(t) := -x(t)$ for $t \geq t_1$ in (E). Using (A4), we find

$$\begin{cases}\big[r(t)\big(y(t) + p(t)y(\delta(t))\big)'\big]' + q(t)H\big(y(\sigma(t))\big) = 0, \ t \neq \tau_k, \ k \in \mathbb{N} \\ \Delta\big[r(\tau_k)\big(y(\tau_k) + p(\tau_k)y(\delta(\tau_k))\big)'\big] + h(\tau_k)H\big(y(\sigma(\tau_k))\big) = 0, \ k \in \mathbb{N}\end{cases}$$

for $t \geq t_1$, where $H(u) := -G(-u)$ for $u \in \mathbb{R}$. Clearly, H also satisfies (A4). Then, proceeding as above, we find the same contradiction. This completes the proof. □

Theorem 2.5. *Let* $1 \leq p(t) \leq p$ *for* $t \geq t_0$*, where* p *is a constant. Assume that (C1) and (A1)–(A5) hold. Furthermore, assume that the following conditions hold:*

(A7) there exists $\lambda > 0$ *such that*

$$G(u) + G(v) \geq \lambda G(u+v) \quad \textit{for } u, v \geq 0$$

and

$$G(u) + G(v) \leq \lambda G(u+v) \quad \textit{for } u, v \leq 0;$$

(A8) there exists $\lambda > 0$ *such that*

$$G(uv) \leq G(u)G(v) \quad \textit{for } u, v \geq 0$$

and

$$G(uv) \geq G(u)G(v) \quad \textit{for } u, v \leq 0;$$

(A9) $\delta(\sigma(t)) = \sigma(\delta(t))$ *for* $t \geq t_0$*;*

(A10) $\displaystyle\int^{\infty} Q(\eta)d\eta + \sum_{k=1}^{\infty} H(\tau_k) = \infty$*, where*

$$Q(t) := \min\{q(t), q(\delta(t))\delta'(t)\} \textit{ and } H(\tau_k) := \min\{h(\tau_k), h(\delta(\tau_k))\delta(\tau_k)\}$$

for $t \geq t_0$*. Then, every solution of (E) is oscillatory.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive solution of (E). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Clearly, z defined by (1.1) is positive on $[t_1, \infty)$. By Lemma 2.1 and Remark 2.2, there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for $t \geq t_2$, where $t_2 \geq t_1$. Let us define

$$w(t) := r(t)z'(t) + G(p)r(\delta(t))z'(\delta(t)) \quad \text{for } t \geq t_3,$$

where $t_3 \geq t_2$. From (E), we compute that

$$\begin{aligned} 0 =& (rz')'(t) + q(t)G\big(x(\sigma(t))\big) + G(p)\delta'(t)\big[(rz')'(\delta(t)) + q(\delta(t))G\big(x(\sigma(\delta(t)))\big)\big] \\ =& w'(t) + q(t)G\big(x(\sigma(t))\big) + G(p)\delta'(t)q(\delta(t))G\big(x(\delta(\sigma(t)))\big) \end{aligned}$$

for $t \geq t_4$, where $t_4 \geq t_3$. Using (A7) and (A8), we obtain

$$\begin{aligned} 0 &\geq w'(t) + Q(t)\big[G\big(x(\sigma(t))\big) + G\big(px(\delta(\sigma(t)))\big)\big] \\ &\geq w'(t) + \lambda Q(t)G\big[x(\sigma(t)) + px(\delta(\sigma(t)))\big] \\ &\geq w'(t) + \lambda Q(t)G\big[x(\sigma(t)) + p(\sigma(t))x(\delta(\sigma(t)))\big] \\ &= w'(t) + \lambda Q(t)G\big(z(\sigma(t))\big) \end{aligned} \tag{2.4}$$

for $t \geq t_4$. Similarly, it is easy to find

$$\Delta w(\tau_k) + \lambda H(\tau_k)G\big(z(\sigma(\tau_k))\big) = 0 \tag{2.5}$$

for $t \geq t_4$. Consequently,

$$\begin{aligned} w'(t) + \lambda Q(t)G(\varepsilon) \leq 0 \\ \Delta w(\tau_k) + \lambda H(\tau_k)G(\varepsilon) \leq 0 \end{aligned}$$

for $t \geq t_4$, which upon integration over the interval $[t_4, t) \subset [t_4, \infty)$ yields that

$$\lambda G(\varepsilon)\Big[\int_{t_4}^{t} Q(\eta)d\eta + \sum_{t_4 \leq \tau_k < t} H(\tau_k)\Big] \leq w(t_4) \quad \text{for all } t \geq t_4.$$

This contradicts (A10). Thus, $x(t) > 0$ for $t \geq t_1$ cannot hold.

The case where x is an eventually negative solution is omitted since it can be dealt similarly. Thus, the theorem is proved. □

Let us give an important example for Theorem 2.5 where δ' is allowed to be oscillatory.

Example 2.6. Consider the impulsive system

$$\begin{cases} [x(t) + 2x(t - \sin(\frac{\pi}{2}t) - 1)]'' + x(t-4) = 0, \ t \neq \tau_k, \ k \in \mathbb{N} \\ \Delta[x(\tau_k) + 2x(\tau_k - \sin(\frac{\pi}{2}\tau_k) - 1)]' + x(\tau_k - 4) = 0, \ t \neq \tau_k, \ k \in \mathbb{N} \end{cases} \tag{2.6}$$

where $r(t) :\equiv 1$, $p(t) :\equiv 2$, $\delta(t) := t - \sin(\frac{\pi}{2}t) - 1$, $\tau_k = k$ for $k \in \mathbb{N}$, $q(t) :\equiv 1 :\equiv h(\tau_k)$, $\sigma(t) := t - 4$ and $G(u) := u$ for $t \geq 0$ and $u \in \mathbb{R}$. We have

$$\delta(\sigma(t)) = t - \sin(\tfrac{\pi}{2}t) - 5 = \sigma(\delta(t)) \quad \text{and} \quad \delta'(t) = 1 - \frac{\pi}{2}\cos(\frac{\pi}{2}t) \quad \text{for} \quad t \geq 0.$$

Note that δ' is oscillatory and $Q(t) = \min\{1, 1 - \frac{\pi}{2}\cos(\frac{\pi}{2}t)\}$ for $t \geq 0$. Obviously, Q is a periodic function with a period 4. Further, $\int_0^4 Q(\eta)d\eta = 2$, which shows that $\int^{\infty} Q(\eta)d\eta + \sum_{k=1}^{\infty} H(\tau_k) = \infty$. Then, all the assumptions of Theorem 2.5 holds. Hence, every solution of (2.6) oscillates.

Theorem 2.7. *Let $-1 \leq p(t) \leq 0$ for $t \geq t_0$. Assume that (C1) and (A1)–(A6) hold. Then, every unbounded solution of (E) oscillates.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive unbounded solution of (E). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is non-increasing and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2.1) holds for $t \geq t_3$. Clearly, $z(t) \leq x(t)$ for $t \geq t_3$ implies

$$\begin{aligned} (rz')'(t) + q(t)G\big(z(\sigma(t))\big) &\leq 0 \\ \Delta(rz')(\tau_k) + h(\tau_k)G\big(z(\sigma(t))\big) &\leq 0 \end{aligned} \tag{2.7}$$

for $t \geq t_3$, where $t_4 \geq t_3$. Further, by Lemma 2.1 and Remark 2.2, there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for $t \geq t_4$. Consequently, it follows from (2.7) that

$$\begin{aligned} (rz')'(t) + G(\varepsilon)q(t) &\leq 0 \\ \Delta(rz')(\tau_k) + G(\varepsilon)h(\tau_k) &\leq 0 \end{aligned}$$

for $t \geq t_4$. Integrating the last inequality over $[t_4, t) \subset [t_4, \infty)$, we have

$$G(\varepsilon)\Big[\int_{t_4}^{t} q(\eta) + \sum_{t_4 \leq \tau_k < t} h(\tau_k)\Big] d\eta \leq r(t_4)z'(t_4) \quad \text{for } t \geq t_4.$$

This contradicts (A6).

Case 2. Let $z(t) < 0$ for $t \geq t_2$. As x is unbounded, there exists $T \geq t_2$ such that $x(T) = \max\{x(\eta) : \ t_2 \leq \eta \leq T\}$. Then, from (1.1), we have $x(T) \leq z(T) + x(\delta(T)) < x(T)$, which is a contradiction.

The case where x is an eventually negative solution is very similar. Hence, the details are omitted. Thus, the proof is complete. □

Theorem 2.8. *Let $-1 < -p \leq p(t) \leq 0$ for $t \geq t_0$, where p is a constant. Assume that (C1) and (A1)–(A6) hold. Then, every bounded solution of (E) either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (E). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. By Lemma 2.3, there exists $t_2 \geq t_1$ such that $z'(t) > 0$ for $t \geq t_2$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. Proceeding as in Case 1 in the proof of Theorem 2.7, we get a contradiction.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. Then, $\lim\limits_{t\to\infty} z(t)$ exists. Thus, we have

$$\begin{aligned} 0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) = \limsup_{t\to\infty}[x(t) + p(t)x(\delta(t))] \geq \limsup_{t\to\infty}[x(t) - px(\delta(t))] \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}[-px(\delta(t))] = (1-p)\limsup_{t\to\infty} x(t), \end{aligned}$$

i.e., $\limsup\limits_{t\to\infty} x(t) = 0$ (since $0 < p < 1$) and hence $\lim\limits_{t\to\infty} x(t) = 0$ for $t \neq \tau_k$, $k \in \mathbb{N}$. We note that $\{x(\tau_k - 0)\}_{k=1}^{\infty}$ and $\{x(\tau_k + 0)\}_{k=1}^{\infty}$ are sequences of reals. Therefore,

$\liminf_{t\to\infty} x(t) = 0$ and $\limsup_{t\to\infty} x(t) = 0$ coincide with $\lim_{t\to\infty} x(\tau_k - 0) = 0$ and $\lim_{t\to\infty} x(\tau_k + 0) = 0$ respectively. As a result, $\lim_{t\to\infty} x(t) = 0$ for all t and τ_k, $k \in \mathbb{N}$.

The case where x is an eventually negative bounded solution is omitted since it can be dealt similarly. This completes the proof. □

Combining Theorem 2.5 and Theorem 2.7, we have the following corollary.

Corollary 2.9. *Let $-1 < -p \le p(t) \le 0$ for $t \ge t_0$, where p is a constant. Assume that (C1) and (A1)–(A6) hold. Then, every solution of (E) either oscillates or converges to zero asymptotically.*

Theorem 2.10. *Let $-p_1 \le p(t) \le -p_2 < -1$ for $t \ge t_0$, where p_1 and p_2 are constants. Assume that (C1) and (A1)–(A6) hold. Then, every bounded solution of (E) either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (E). Then, z defined by (1.1) is also bounded. By Lemma 2.3, z is increasing. Hence, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \ge t_2$. Proceeding as in Case 1 in the proof of Theorem 2.7, we get a contradiction.

Case 2. Let $z(t) < 0$ for $t \ge t_2$. In this case, $\lim_{t\to\infty} z(t)$ exists as a non-positive finite value. We claim that $\lim_{t\to\infty} z(t) = 0$. Otherwise, $\lim_{t\to\infty} z(t) < 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) < -\varepsilon$ for $t \ge t_1$. Then, we have $z(t) \ge p(t)x(\delta(t)) \ge -p_1 x(\delta(t))$ for $t \ge t_1$, which implies $x(t) \ge -\frac{1}{p_1} z(\delta^{-1}(t)) \ge \frac{\varepsilon}{p_1}$ for $t \ge t_1$. Consequently, (2.2) becomes

$$\begin{aligned} (rz')'(t) + q(t)G\Big(\frac{\varepsilon}{p_1}\Big) &\le 0 \\ \Delta(rz')(\tau_k) + h(\tau_k)G\Big(\frac{\varepsilon}{p_1}\Big) &\le 0 \end{aligned}$$

for $t \ge t_1$. Integrating the last inequality over the interval $[t_1, t) \subset [t_1, \infty)$, we get

$$G\Big(\frac{\varepsilon}{p_1}\Big)\Big[\int_{t_2}^{t} q(\eta)d\eta + \sum_{t_2 \le \tau_k < t} h(\tau_k)\Big] \le r(t_2)z'(t_2) \quad \text{for } t \ge t_1.$$

This contradicts (A6). Therefore, $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} z(t) \le \liminf_{t\to\infty} [x(t) - p_2 x(\delta(t))] \\ &\le \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} [-p_2 x(\delta(t))] \le (1 - p_2) \limsup_{t\to\infty} x(t), \end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$ (since $p_2 > 1$). Thus, $\liminf_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

Therefore, any nonoscillatory solution x of (E) converges to zero. This completes the proof of the theorem. $\square$

Example 2.11. Consider the impulsive system

$$\begin{cases} \left[t\big(x(t) - 3x(e^{-\pi}t)\big)'\right]' + \dfrac{4}{t}x(t) = 0,\ t \neq \tau_k,\ k \in \mathbb{N} \quad \text{for } t \geq 1, \\ \Delta\left[\tau_k\big(x(t) - 3x(e^{-\pi}\tau_k)\big)'\right] + \dfrac{4}{2k}x(\tau_k) = 0,\ t \neq \tau_k,\ k \in \mathbb{N} \end{cases} \tag{2.8}$$

where $r(t) := t$, $p(t) :\equiv -3$, $\delta(t) := e^{-\pi}t$, $q(t) := \dfrac{4}{t}$, $\tau_k = 2k$ for $k \in \mathbb{N}$, $h(\tau_k) = \dfrac{4}{2k}$, $\sigma(t) := t$ and $G(u) := u$ for $t \geq 1$ and $u \in \mathbb{R}$. It can be easily shown that Theorem 2.10 applies to (2.8). Thus, every bounded solution oscillates or converges to zero asymptotically. Obviously, $x(t) = \sin(\ln(t))$ for $t \geq 1$ is an oscillating solution.

2.2 Oscillation under Condition (C2).

Remark 2.12. If we set

$$R(t) := \int_t^\infty \frac{1}{r(\eta)} d\eta \quad \text{for } t \geq t_0, \tag{2.9}$$

then (C2) implies that $R(t) \to 0$ as $t \to \infty$.

Lemma 2.13. *Assume that (C2) and (A1)–(A5) hold. If x is an eventually positive solution of (E) such that the companion function z defined by* (1.1) *is eventually decreasing and positive, then there exists $\varepsilon > 0$ such that z satisfies*

$$\varepsilon R(t) \leq z(t) \quad \textit{for all large } t, \tag{2.10}$$

where R is defined in (2.9).

Proof. Suppose that $x(t), z(t) > 0$ and $z'(t) < 0$ for $t \geq t_1$, where $t \geq t_0$. By (A5), we may assume without loss of generality that $x(\sigma(t)) > 0$ for $t \geq t_1$. From (E) and (A4), we get (2.2). Consequently, rz' is non-increasing on $[t_1, \infty)$. Therefore, $r(s)z'(s) \leq r(t)z'(t)$ for $s \geq t \geq t_1$, which implies

$$z'(s) \leq \frac{r(t)z'(t)}{r(s)} \quad \text{for } s \geq t \geq t_1.$$

Consequently,

$$z(s) \leq z(t) + r(t)z'(t) \int_t^s \frac{1}{r(\eta)} d\eta \quad \text{for } s \geq t \geq t_1.$$

As rz' is non-increasing, we can find a constant $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_1$. As a result $z(s) \leq z(t) - \varepsilon \displaystyle\int_t^s \frac{1}{r(\eta)} d\eta$ for $s \geq t \geq t_1$. By letting $s \to \infty$, we get $0 \leq z(t) - \varepsilon R(t)$ for $t \geq t_1$, which proves (2.10). $\square$

Theorem 2.14. *Let $0 \le p(t) \le p$ for $t \ge t_0$, where p is a constant. Assume that (C2), (A1)–(A5) and (A7)–(A10) hold. Further, assume*

(A11)

$$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} Q(\zeta) G\big(\varepsilon R(\sigma(\zeta))\big) d\zeta d\eta + R(t_0) \sum_{k=1}^{\infty} H(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big) = \infty$$

for every $\varepsilon > 0$, and

$$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} Q(\zeta) G\big(\varepsilon R(\sigma(\zeta))\big) d\zeta d\eta + R(t_0) \sum_{k=1}^{\infty} H(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big) = -\infty$$

for every $\varepsilon < 0$, where Q and H are defined in (A10).

Then, every solution of (E) is oscillatory.

Proof. Without loss of generality, assume the contrary that x is an eventually positive solution of (E). Proceed as in the proof of Lemma 2.1 to obtain (2.2) for $t \ge t_1$, i.e., rz' is nonincreasing on $[t_2, \infty)$, where $t_2 \ge t_1$. Recall that z is positive on $[t_2, \infty)$. Thus, we have the following two cases.

Case 1. Let $z'(t) > 0$ for $t \ge t_2$. Then, we proceed as in Theorem 2.5 to get a contradiction.

Case 2. Let $z'(t) < 0$ for $t \ge t_2$. By Lemma 2.13, we have (2.10) for $t \ge t_3$, where $\varepsilon > 0$ and $t_3 \ge t_2$. Using (2.10) in (2.4) and (2.5), we have

$$\begin{aligned} w'(t) + \lambda Q(t) G\big(\varepsilon R(\sigma(t))\big) &\le 0 \\ \Delta w(\tau_k) + \lambda H(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big) &\le 0 \end{aligned}$$

for $t \ge t_3$, where $t_3 \ge t_2$. Integrating the last inequality over the interval $[t_3, t) \subset [t_3, \infty)$, we obtain

$$\begin{aligned} \lambda\Big[\int_{t_3}^{t} Q(\eta) G\big(\varepsilon R(\sigma(\eta))\big) d\eta + \sum_{t_3 \le \tau_k < t} H(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big)\Big] &\le -w(t) \qquad (2.11) \\ &\le -\big(1 + G(p)\big) r(t) z'(t), \end{aligned}$$

which implies

$$\frac{\lambda}{1+G(p)} \frac{1}{r(t)} \Big[\int_{t_3}^{t} Q(\eta) G\big(\varepsilon R(\sigma(\eta))\big) d\eta + \sum_{t_3 \le \tau_k < t} H(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big)\Big] \le -z'(t)$$

for $t \ge t_3$. Again integrating the last inequality over the interval $[t_3, t) \subset [t_3, \infty)$, we obtain

$$\frac{\lambda}{1+G(p)} \int_{t_3}^{t} \frac{1}{r(\eta)} \Big[\int_{t_3}^{\eta} Q(\zeta) G\big(\varepsilon R(\sigma(\zeta))\big) d\zeta + \sum_{t_3 \le \tau_k < t} H(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big)\Big] d\eta$$

$$\begin{aligned}&\leq \, -\, \big[z(\eta)\big]_{t_3}^{t} + \sum_{t_3 \leq \tau_k < t} \Delta z(\tau_k) \\ &= -\big[z(\eta)\big]_{t_3}^{t} + \sum_{t_3 \leq \tau_k < t} \big[z(\tau_k + 0) - z(\tau_k - 0)\big] \\ &\leq z(t_3) + \sum_{t_3 \leq \tau_k < t} z(\tau_k + 0)\end{aligned}$$

which contradicts (A11).

The case where x is eventually negative can be dealt similarly, and we omit the details here. This completes the proof. □

Example 2.15. Consider the impulsive systems

$$\begin{cases} \big[e^t(x(t) + 3e^{-t}x(t-3))'\big]' + e^{3t}\big(x(t-1)\big)^3 = 0, \ t \neq k, \ k \in \mathbb{N} \\ \Delta\big[e^k(x(\tau_k) + 3e^{-k}x(\tau_k - 3))'\big] + e^{3k}\big(x(\tau_k - 1)\big)^3 = 0, \ k \in \mathbb{N} \end{cases} \tag{2.12}$$

for $t \geq 3$, where $r(t) := e^t$, $R(t) := e^{-t}$, $p(t) := 3e^{-t}$, $\delta(t) := t-3$, $q(t) := e^{3t}$, $\tau_k = k$ for $k \in \mathbb{N}$, $h(\tau_k) = e^{3k}$, $\sigma(t) := t-1$ and $G(u) := u^3$ for $t \geq 3$ and $u \in \mathbb{R}$. Then, all the assumptions of Theorem 2.14 holds. Hence, every solution of (2.12) oscillates.

Theorem 2.16. *Let* $-1 \leq p(t) \leq 0$ *for* $t \geq t_0$. *Assume that (C2) and (A1)–(A6) hold. Furthermore, assume that*

(A12)

$$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} q(\zeta) G\big(\varepsilon R(\sigma(\zeta))\big) d\zeta d\eta + R(t_0) \sum_{k=1}^{\infty} h(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big) = \infty \tag{2.13}$$

for every $\varepsilon > 0$, *and*

$$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} q(\zeta) G\big(\varepsilon R(\sigma(\zeta))\big) d\zeta d\eta + R(t_0) \sum_{k=1}^{\infty} h(\tau_k) G\big(\varepsilon R(\sigma(\tau_k))\big) = -\infty$$

for every $\varepsilon < 0$. *Then, every unbounded solution of (E) oscillates.*

Proof. Without loss of generality, let x be an eventually positive unbounded solution of (E). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see that z and z' are of single sign on $[t_2, \infty)$, where $t_2 \geq t_1$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. Note that in this case, we have $z(t) \leq x(t)$ for $t \geq t_2$.

(a) Let $z'(t) > 0$ for $t \geq t_2$. We easily get (2.7). Then, proceeding as in Case 1 in the proof of Theorem 2.7, we get a contradiction.

(b) Let $z'(t) < 0$ for $t \geq t_2$. By Lemma 2.13, we have (2.10) for $t \geq t_3$, where $\varepsilon > 0$ and $t_3 \geq t_2$. Using $z(t) \leq x(t)$ for $t \geq t_2$ and (2.2), we get

$$(rz')'(t) + q(t)G\big(\varepsilon R(\sigma(t))\big) \leq 0$$
$$\Delta(rz')(\tau_k) + h(\tau_k)G\big(\varepsilon R(\sigma(\tau_k))\big) \leq 0$$

for $t \geq t_3$, where $t_3 \geq t_2$. The rest of the proof follows similar to Case 2 in the proof of Theorem 2.14.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. The proof is similar to Case 2 in the proof of Theorem 2.7.

The proof is therefore completed. □

Theorem 2.17. *Let $-1 < -p \leq p(t) \leq 0$ for $t \geq t_0$, where p is a constant. Assume that (C2), (A1)–(A6) and (A12) hold. Then, every bounded solution of (E) either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (E). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see that z and z' are of single sign on $[t_2, \infty)$, where $t_2 \geq t_1$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. In this case, we proceed as in Case 1 in the proof of Theorem 2.16 and get a contradiction.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. Recalling that z is monotonic, we follow the steps in Case 2 in the proof of Theorem 2.8 and see that $\lim_{t\to\infty} x(t) = 0$.

Hence, the proof of the theorem is complete. □

Combining Theorem 2.16 and Theorem 2.17, we have the following corollary.

Corollary 2.18. *Let $-1 < -p \leq p(t) \leq 0$ for $t \geq t_0$, where p is a constant. Assume that (C2), (A1)–(A6) and (A12) hold. Then, every solution of (E) either oscillates or converges to zero asymptotically.*

Theorem 2.19. *Let $-p_1 \leq p(t) \leq -p_2 < -1$ for $t \geq t_0$, where p_1 and p_2 are constants. Assume that (C2), (A1)–(A6) and (A12) hold. Further, assume that*

(A13) $$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} q(\zeta)d\zeta d\eta + R(t_0)\sum_{k=1}^{\infty} h(\tau_k) = \infty.$$

Then, every bounded solution of (E) either oscillates or converges to zero asymptotically.

Proof. Without loss of generality, let x be an eventually positive bounded solution of (E). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see that z and z' are of single sign on $[t_2, \infty)$, where $t_2 \geq t_1$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. In this case, we proceed as in Case 1 in the proof of Theorem 2.16 and get a contradiction.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. In this case, $\lim_{t\to\infty} z(t)$ exists as a non-positive finite value. We claim that $\lim_{t\to\infty} z(t) = 0$. Otherwise, $\lim_{t\to\infty} z(t) < 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) < -\varepsilon$ for $t \geq t_2$. Then, we have $z(t) \geq p(t)x(\delta(t)) \geq -p_1 x(\delta(t))$ for $t \geq t_2$, which implies $x(t) \geq -\dfrac{1}{p_1} z(\delta^{-1}(t)) \geq \dfrac{\varepsilon}{p_1}$ for $t \geq t_2$. Consequently, (2.2) becomes

$$\begin{aligned} (rz')'(t) + q(t)G\Big(\frac{\varepsilon}{p_1}\Big) &\leq 0 \\ \Delta(rz')(\tau_k) + h(\tau_k)G\Big(\frac{\varepsilon}{p_1}\Big) &\leq 0 \end{aligned}$$

for $t \geq t_2$. Integrating the last inequality over the interval $[t_2, t) \subset [t_2, \infty)$, we get

$$\big[(rz')(\eta)\big]_{t_2}^{t} - \sum_{t_2 \leq \tau_k < t} \Delta(rz')(\tau_k) + G\Big(\frac{\varepsilon}{p_1}\Big) \int_{t_2}^{t} q(\eta)d\eta \leq 0,$$

that is,

$$-r(t_2)z'(t_2) + G\Big(\frac{\varepsilon}{p_1}\Big)\Big[\int_{t_2}^{t} q(\eta)d\eta + \sum_{t_2 \leq \tau_k < t} h(\tau_k)\Big] \leq -r(t)z'(t) \quad \text{for } t \geq t_2.$$

Again, integrating the last inequality over the interval $[t_2, t) \subset [t_2, \infty)$ after dividing through by r, we get

$$\begin{aligned} -r(t_2)z'(t_2)\int_{t_2}^{t} \frac{1}{r(\eta)} d\eta + G\Big(\frac{\varepsilon}{p_1}\Big) \int_{t_2}^{t} \frac{1}{r(\eta)} \Big[\int_{t_2}^{\eta} q(\zeta)d\zeta + \sum_{t_2 \leq \tau_k < t} h(\tau_k)\Big] d\eta \\ \leq -\big[z(\eta)\big]_{t_2}^{t} + \sum_{t_2 \leq \tau_k < t} \Delta z(\tau_k) \\ \leq -z(t) - \sum_{t_2 \leq \tau_k < t} z(\tau_k - 0) \end{aligned}$$

for $t \geq t_2$, which contradicts $(A13)$ by (C2). Therefore, $\lim_{t\to\infty} z(t) = 0$. For the rest of the proof, we follow the steps in the last part of Case 2 of Theorem 2.10 to get $\lim_{t\to\infty} x(t) = 0$.

Hence, the proof of the theorem is complete. □

3 Final Comments

In this section, we will be giving two simple remarks to conclude the paper.

Remark 3.1. In Theorem 2.4–Theorem 2.19, G is allowed to be linear, sublinear or superlinear. A prototype of the function G satisfying (A4), (A7) and (A8) is

$$(1+\alpha|u|^{\beta})|u|^{\gamma}\operatorname{sgn}(u) \quad \text{for } u \in \mathbb{R},$$

where $\alpha \geq 1$ or $\alpha = 0$ and $\beta, \gamma > 0$ are reals. For verifying (A7), we may take help of the well-known inequality (see [9][p. 292])

$$u^p + v^p \geq h(p)(u+v)^p \quad \text{for } u, v > 0, \quad \text{where} \quad h(p) := \begin{cases} 1, & 0 \leq p \leq 1, \\ \dfrac{1}{2^{p-1}}, & p \geq 1. \end{cases}$$

Remark 3.2. If the nonlinear term G is an odd function (presented as in Remark 3.1), it suffices to verify only the first conditions in (A7), (A8), (A11) and (A12).

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