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On Two-Point Riemann–Liouville Type Nabla Fractional Boundary Value Problems

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Abstract

The present article deals with a particular class of standard two-point Riemann–Liouville type nabla fractional boundary value problems associated with Dirichlet boundary conditions. First, we construct the corresponding Green's function using the nabla Laplace transform and obtain its key properties. Next, by applying a suitable fixed point theorem, we establish sufficient conditions on the existence of solutions. Finally, we deduce the uniqueness of solutions by assuming the Lipschitz condition. We close with two examples to illustrate the applicability of established results.

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1 Introduction

In 2009, Atici & Eloe [7] initiated the study of two-point boundary value problems for delta fractional difference equations. They obtained sufficient conditions on the existence of positive solutions for the following two-point boundary value problem using Guo–Krasnoselskii fixed point theorem.

$$\begin{cases} -\big(\Delta^{\alpha}u\big)(t) = F(t+\nu-1, u(t+\nu-1)), \quad t \in \mathbb{N}_1^{b+1}, \\ u(\nu-2) = 0, \; u(\nu+b+1) = 0. \end{cases} \tag{1.1}$$

Here $1 < \nu \leq 2$ is a real number, $b \geq 2$ an integer and $F : [\nu, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

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Following their work, Goodrich [13] introduced several existence theorems for (1.1) using cone theoretic techniques. In 2014, Chen et al. [10] improved and generalized the results in [7, 13].

On the other hand, in 2014, Brackins [9] derived Green's function for a nonhomogeneous nabla fractional boundary value problem with homogeneous boundary conditions and obtained a few of its properties.

Theorem 1.1 (See [9])**.** *Let a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_1$, $1 < \alpha < 2$ and $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. The nabla fractional boundary value problem*

$$\begin{cases} -\left(\nabla_a^{\alpha} u\right)(t) = h(t), \quad t \in \mathbb{N}_{a+1}^{b}, \\ u(a) = u(b) = 0, \end{cases} \tag{1.2}$$

has the unique solution

$$u(t) = \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^{b}, \tag{1.3}$$

where

$$G(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \dfrac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}(t-a)^{\overline{\alpha-1}}, & t \in \mathbb{N}_a^{\rho(s)}, \\ \dfrac{1}{\Gamma(\alpha)} \left[\dfrac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}(t-a)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}} \right], & t \in \mathbb{N}_s^{b}. \end{cases} \tag{1.4}$$

Remark 1.2. (1.2) is the problem of solving a system of $(b-a+2)$ equations in $(b-a+1)$ unknowns. So, (1.2) is an over-determined boundary value problem. For example, take $b = a+3$ so that $b - a = 3 \in \mathbb{N}_1$. Then, (1.2) reduces to the problem of solving a system of 5 equations in 4 unknowns as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \alpha & -1 & 0 \\ 0 & \dfrac{\alpha(1-\alpha)}{2} & \alpha & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(a) \\ u(a+1) \\ u(a+2) \\ u(a+3) \end{pmatrix} = \begin{pmatrix} 0 \\ h(a+1) \\ h(a+2) \\ h(a+3) \\ 0 \end{pmatrix}. \tag{1.5}$$

Recently, Gholami et al. [11] have also derived Green's function for a nonhomogeneous nabla fractional boundary value problem with homogeneous boundary conditions and obtained a few of its properties.

Theorem 1.3 (See [11])**.** *Let $a \in \mathbb{N}_0$, $b \in \mathbb{N}_3$, $1 < \alpha < 2$ and $h : \mathbb{N}_{a+2}^{b+1} \to \mathbb{R}$. The nabla fractional boundary value problem*

$$\begin{cases} -\left(\nabla_a^{\alpha} u\right)(t) = h(t), \quad t \in \mathbb{N}_{a+2}^{b+1}, \\ u(a+1) = u(b+1) = 0, \end{cases} \tag{1.6}$$

has the unique solution

$$u(t) = \sum_{s=a+2}^{b+1} G(t,s)h(s), \quad t \in \mathbb{N}_{a+1}^{b+1}, \tag{1.7}$$

where Green's function $G(t,s)$ is obtained by replacing a by $a+1$ and b by $b+1$ in (1.4).

Remark 1.4. (1.6) is also the problem of solving a system of $(b-a+2)$ equations in $(b-a+1)$ unknowns. So, (1.6) is also an over-determined boundary value problem. For example, take $a=0$ and $b=3$ so that (1.6) reduces to the problem of solving a system of 5 equations in 4 unknowns as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & -1 & 0 & 0 \\ \dfrac{\alpha(1-\alpha)}{2} & \alpha & -1 & 0 \\ \dfrac{\alpha(1-\alpha)(2-\alpha)}{6} & \dfrac{\alpha(1-\alpha)}{2} & \alpha & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(1) \\ u(2) \\ u(3) \\ u(4) \end{pmatrix} = \begin{pmatrix} 0 \\ h(2) \\ h(3) \\ h(4) \\ 0 \end{pmatrix}. \tag{1.8}$$

Remark 1.5. Brackins [9] and Gholami et al. [11] chose two different approaches to solve the respective boundary value problems. From (1.5) and (1.8), we observe that both the boundary value problems (1.2) and (1.6) are same, and of course, the corresponding Green's functions as well as the solutions are same.

Theorem 1.6 (See [9]). *The Green's function $G(t,s)$ defined in* (1.4) *satisfies the following properties:*

1. $G(a,s) = G(b,s) = 0$ *for all* $s \in \mathbb{N}_{a+1}^{b}$.
2. $G(t,a+1) = 0$ *for all* $t \in \mathbb{N}_{a}^{b}$.
3. $G(t,s) > 0$ *for all* $(t,s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^{b}$.
4. $\max\limits_{t \in \mathbb{N}_{a+1}^{b-1}} G(t,s) = G(s-1,s)$ *for all* $s \in \mathbb{N}_{a+2}^{b}$.
5. $\sum\limits_{s=a+1}^{b} G(t,s) \le \lambda$, $(t,s) \in \mathbb{N}_{a}^{b} \times \mathbb{N}_{a+1}^{b}$, *where*

$$\lambda = \Big(\frac{b-a-1}{\alpha\Gamma(\alpha+1)}\Big)\Big(\frac{(\alpha-1)(b-a)+1}{\alpha}\Big)^{\overline{\alpha-1}}.$$

Based on the Remarks 1.2 and 1.4, we introduce a standard nonhomogeneous nabla fractional boundary value problem with homogeneous boundary conditions as follows:

$$\begin{cases} -\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{1.9}$$

where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, $1 < \alpha < 2$ and $h : \mathbb{N}^{b}_{a+2} \to \mathbb{R}$.

Remark 1.7. (1.9) is the problem of solving a system of $(b-a+1)$ equations in $(b-a+1)$ unknowns. So, (1.9) is a standard boundary value problem. For example, take $b = a+3$ so that $b - a = 3 \in \mathbb{N}_2$. Then, (1.9) reduces to the problem of solving a system of 4 equations in 4 unknowns as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \dfrac{\alpha(1-\alpha)}{2} & \alpha & -1 & 0 \\ \dfrac{\alpha(1-\alpha)(2-\alpha)}{6} & \dfrac{\alpha(1-\alpha)}{2} & \alpha & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(a) \\ u(a+1) \\ u(a+2) \\ u(a+3) \end{pmatrix} = \begin{pmatrix} 0 \\ h(a+2) \\ h(a+3) \\ 0 \end{pmatrix}. \tag{1.10}$$

Since the determinant of the coefficient matrix is nonzero, the system (1.10) has a unique solution and so is true for (1.9).

Motivated by the Remark 1.7, in this article, we establish sufficient conditions on existence and uniqueness of solutions of the following standard two-point boundary value problem for a nonlinear nabla fractional difference equation:

$$\begin{cases} -\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) = f(t, u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = A, \ u(b) = B, \end{cases} \tag{1.11}$$

where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$; A, $B \in \mathbb{R}$; $1 < \alpha < 2$ and $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}$.

2 Preliminaries

Throughout, we shall use the following notations, definitions and known results of nabla fractional calculus [12]. Denote the set of all real numbers by $\mathbb{R}$. Define

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\} \text{ and } \mathbb{N}^{b}_{a} := \{a, a+1, a+2, \dots, b\}$$

for any $a, b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1 (See [8]). The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by

$$\rho(t) = \max\{a, (t-1)\}, \quad t \in \mathbb{N}_a.$$

Definition 2.2 (See [15, 17])**.** The Euler gamma function is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can be extended to the half-plane $\Re(z) \le 0$ except for $z \neq 0, -1, -2, \ldots$

Definition 2.3 (See [12])**.** For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)}, \quad 0^{\overline{r}} := 0.$$

Definition 2.4 (See [8])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) := u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) := \Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.5 (See [12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-N} u\big)(t) := \frac{1}{(N-1)!} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{N-1}} u(s), \quad t \in \mathbb{N}_{a+1}.$$

We define $\big(\nabla_a^{-0} u\big)(t) = u(t)$ for all $t \in \mathbb{N}_{a+1}$.

Definition 2.6 (See [12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\nu-1}} u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.7 (See [12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \le N$. The Riemann–Liouville type ν^{th}-order nabla difference of u is given by

$$\big(\nabla_a^{\nu} u\big)(t) := \Big(\nabla^N\big(\nabla_a^{-(N-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Theorem 2.8 (See [4]). *Assume $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$, $\nu \notin \mathbb{N}_1$, and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu < N$. Then,*

$$\big(\nabla_a^{\nu} u\big)(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{-\nu-1}} u(s), \quad t \in \mathbb{N}_{a+1}.$$

Theorem 2.9 (See [14]). *We observe the following properties of gamma and generalized rising functions.*

1. $\Gamma(t) > 0$ *for all* $t > 0$.
2. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
3. *If* $t \leq r$, *then* $t^{\overline{\alpha}} \leq r^{\overline{\alpha}}$.
4. *If* $\alpha < t \leq r$, *then* $r^{\overline{-\alpha}} \leq t^{\overline{-\alpha}}$.

Theorem 2.10 (See [2]). *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ, $\mu + \nu$ and $\mu - \nu$ are nonnegative integers. Then,*

$$\nabla_a^{-\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\overline{\mu+\nu}}, \quad t \in \mathbb{N}_a,$$
$$\nabla_a^{\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\overline{\mu-\nu}}, \quad t \in \mathbb{N}_a.$$

Definition 2.11 (See [12]). Assume $u : \mathbb{N}_{a+1} \to \mathbb{R}$. Then, the nabla Laplace transform of u is defined by

$$\mathcal{L}_a\big[u(t)\big] := \sum_{k=1}^{\infty}(1-s)^{k-1}u(a+k),$$

for those values of s such that this infinite series converges.

Definition 2.12 (See [12]). For u, $v : \mathbb{N}_{a+1} \to \mathbb{R}$, we define the nabla convolution product of u and v by

$$(u * v)(t) := \sum_{s=a+1}^{t} u(t - \rho(s) + a)v(s), \quad t \in \mathbb{N}_{a+1}.$$

Theorem 2.13 (See [12]). *We observe the following properties of the nabla Laplace transform.*

1. *For ν not an integer, we have that*

$$\mathcal{L}_a\big[(t-a)^{\overline{\nu}}\big] = \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \quad |s-1| < 1.$$

2. *Assume u, $v : \mathbb{N}_{a+1} \to \mathbb{R}$ and their nabla Laplace transforms converge for $|s-1| < r$ for some $r > 0$. Then,*

$$\mathcal{L}_a\big[(u * v)(t)\big] = \mathcal{L}_a\big[u(t)\big]\mathcal{L}_a\big[v(t)\big], \quad |s-1| < r.$$

3. *Assume $\nu > 0$ and the nabla Laplace transform of $u : \mathbb{N}_{a+1} \to \mathbb{R}$ converges for $|s-1| < r$ for some $r > 0$. Then,*

$$\mathcal{L}_a\big[\big(\nabla_a^{-\nu} u\big)(t)\big] = s^{-\nu}\mathcal{L}_a\big[u(t)\big], \quad |s-1| < \min\{1, r\}.$$

4. *Given $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $n \in \mathbb{N}_1$, we have that*

$$\mathcal{L}_{a+n}\big[u(t)\big] = \Big(\frac{1}{1-s}\Big)^n \mathcal{L}_a\big[u(t)\big] - \sum_{k=1}^{n} \frac{u(a+k)}{(1-s)^{n-k+1}}.$$

5. *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $1 < \nu < 2$ be given. Then, we have*

$$\mathcal{L}_a\big[\big(\nabla_a^{\nu} u\big)(t)\big] = s^{\nu}\mathcal{L}_a\big[u(t)\big].$$

Theorem 2.14 (See [12]). *Assume $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \leq N$. Then, a general solution of*

$$\big(\nabla_{\rho(a)}^{\nu} u\big)(t) = 0, \quad t \in \mathbb{N}_{a+N},$$

is given by

$$u(t) = C_1(t-a+1)^{\overline{\nu-1}} + C_2(t-a+1)^{\overline{\nu-2}} + \ldots + C_N(t-a+1)^{\overline{\nu-N}}, \quad t \in \mathbb{N}_a,$$

where $C_1, C_2, \ldots, C_N \in \mathbb{R}$.

3 Construction of Green's Function

Let $1 < \alpha < 2$. First, we consider an α^{th}-order nabla fractional initial value problem and give a formula for its solution using Theorem 2.13.

Lemma 3.1. *Assume A_0, $A_1 \in \mathbb{R}$ and $g : \mathbb{N}_{a+2} \to \mathbb{R}$. Then, the unique solution of the fractional initial value problem*

$$\begin{cases} \big(\nabla_{\rho(a)}^{\alpha} u\big)(t) = g(t), \quad t \in \mathbb{N}_{a+2}, \\ u(a) = A_0, \ u(a+1) = A_1, \end{cases} \tag{3.1}$$

is given by

$$u(t) = \Big[\frac{A_1 - (\alpha-1)A_0}{\Gamma(\alpha)}\Big](t-a+1)^{\overline{\alpha-1}} + \Big[\frac{\alpha A_0 - A_1}{\Gamma(\alpha-1)}\Big](t-a+1)^{\overline{\alpha-2}} + \big(\nabla_{a+1}^{-\alpha} g\big)(t), \tag{3.2}$$

for $t \in \mathbb{N}_a$.

Proof. Applying the nabla Laplace transform $\mathcal{L}_{a+1}$ on both sides of (3.1), we have that

$$\mathcal{L}_{a+1}\big[\big(\nabla^{\alpha}_{\rho(a)}u\big)(t)\big] = \mathcal{L}_{a+1}\big[g(t)\big]. \tag{3.3}$$

Now, we use Theorem 2.13 to rewrite the left and right hand parts of (3.3) in terms of $\mathcal{L}_{\rho(a)}\big[u(t)\big]$ and $\mathcal{L}_{\rho(a)}\big[g(t)\big]$, respectively. Consider

$$\begin{aligned}\mathcal{L}_{a+1}\big[\big(\nabla^{\alpha}_{\rho(a)}u\big)(t)\big] &= \frac{1}{(1-s)^2}\mathcal{L}_{\rho(a)}\big[\big(\nabla^{\alpha}_{\rho(a)}u\big)(t)\big] - \frac{1}{(1-s)^2}\big[\big(\nabla^{\alpha}_{\rho(a)}u\big)(t)\big]_{t=a} \\ &\quad - \frac{1}{(1-s)}\big[\big(\nabla^{\alpha}_{\rho(a)}u\big)(t)\big]_{t=a+1} \\ &= \frac{s^{\alpha}}{(1-s)^2}\mathcal{L}_{\rho(a)}\big[u(t)\big] + \Big[\frac{\alpha}{(1-s)} - \frac{1}{(1-s)^2}\Big]A_0 \\ &\quad - \frac{A_1}{(1-s)}. \end{aligned} \tag{3.4}$$

Also, consider

$$\mathcal{L}_{a+1}\big[g(t)\big] = \frac{1}{(1-s)^2}\mathcal{L}_{\rho(a)}\big[g(t)\big] - \frac{g(a)}{(1-s)^2} - \frac{g(a+1)}{(1-s)}. \tag{3.5}$$

Using (3.4) and (3.5) in (3.3) and then multiplying by $(1-s)^2$ on both sides, we get that

$$\begin{aligned} s^{\alpha}\mathcal{L}_{\rho(a)}\big[u(t)\big] &= \big[1-\alpha(1-s)\big]A_0 + (1-s)A_1 + \mathcal{L}_{\rho(a)}\big[g(t)\big] \\ &\quad - g(a) - (1-s)g(a+1). \end{aligned} \tag{3.6}$$

Multiplying by $s^{-\alpha}$ on both sides of (3.6), it follows that

$$\begin{aligned} \mathcal{L}_{\rho(a)}\big[u(t)\big] &= \Big[\frac{(1-\alpha)}{s^{\alpha}} + \frac{\alpha}{s^{\alpha-1}}\Big]A_0 + \Big[\frac{1}{s^{\alpha}} - \frac{1}{s^{\alpha-1}}\Big]A_1 + s^{-\alpha}\mathcal{L}_{\rho(a)}\big[g(t)\big] \\ &\quad - \frac{g(a)}{s^{\alpha}} - \Big[\frac{1}{s^{\alpha}} - \frac{1}{s^{\alpha-1}}\Big]g(a+1). \end{aligned} \tag{3.7}$$

Applying the inverse nabla Laplace transform $\mathcal{L}^{-1}_{\rho(a)}$ on both sides of (3.7) and then using Theorem 2.13, we have that

$$\begin{aligned} u(t) &= \Big[\frac{(1-\alpha)(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \frac{\alpha(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\Big]A_0 \\ &\quad + \Big[\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\Big]A_1 + \big(\nabla^{-\alpha}_{\rho(a)}g\big)(t) \\ &\quad - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}g(a) - \Big[\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\Big]g(a+1) \\ &= \Big[\frac{A_1 - (\alpha-1)A_0}{\Gamma(\alpha)}\Big](t-a+1)^{\overline{\alpha-1}} + \Big[\frac{\alpha A_0 - A_1}{\Gamma(\alpha-1)}\Big](t-a+1)^{\overline{\alpha-2}} \end{aligned}$$

$$
\begin{aligned}
&+ \big(\nabla_{\rho(a)}^{-\alpha} g\big)(t) \\
&- \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} g(a) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} g(a+1) \\
&= \Big[\frac{A_1 - (\alpha-1)A_0}{\Gamma(\alpha)}\Big](t-a+1)^{\overline{\alpha-1}} + \Big[\frac{\alpha A_0 - A_1}{\Gamma(\alpha-1)}\Big](t-a+1)^{\overline{\alpha-2}} \\
&+ \big(\nabla_{a+1}^{-\alpha} g\big)(t).
\end{aligned}
$$

Thus, we have (3.2). □

Next, we use Lemma 3.1 to deduce the unique solution of the nabla fractional boundary value problem (1.9).

Theorem 3.2. *The nabla fractional boundary value problem* (1.9) *has the unique solution*

$$
u(t) = \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3.8}
$$

where Green's function $G(t,s)$ *is as in* (1.4).

Proof. Comparing (1.9) and (3.1), we have that $A_0 = 0$ and $g = -h$. Then, from (3.2), it follows that

$$
\begin{aligned}
u(t) &= A_1\Big[\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\Big] - \big(\nabla_{a+1}^{-\alpha} h\big)(t) \\
&= A_1 \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} (t-s+1)^{\overline{\alpha-1}} h(s).
\end{aligned} \tag{3.9}
$$

Using $u(b) = 0$ in (3.9), we obtain the value of A_1 as

$$
A_1 = \frac{1}{(b-a)^{\overline{\alpha-1}}} \sum_{s=a+2}^{b} (b-s+1)^{\overline{\alpha-1}} h(s). \tag{3.10}
$$

Substituting the value of A_1 from (3.10) in (3.9), we get that

$$
\begin{aligned}
u(t) &= \frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)} \sum_{s=a+2}^{b} (b-s+1)^{\overline{\alpha-1}} h(s) - \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} (t-s+1)^{\overline{\alpha-1}} h(s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Big[\frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}(t-a)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big] h(s) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b} \Big[\frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}(t-a)^{\overline{\alpha-1}}\Big] h(s)
\end{aligned}
$$

$$= \sum_{s=a+2}^{b} G(t,s)h(s).$$

Thus, we have (3.8). □

Now, we wish to obtain an expression for the unique solution of the following two-point nabla fractional boundary value problem associated with nonhomogeneous boundary conditions:

$$\begin{cases} -\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = A, \; u(b) = B. \end{cases} \tag{3.11}$$

For this purpose, we need the following Lemma.

Lemma 3.3. *The solution of the nabla fractional boundary value problem*

$$\begin{cases} -\big(\nabla^{\alpha}_{\rho(a)}w\big)(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ w(a) = A, \; w(b) = B, \end{cases} \tag{3.12}$$

is

$$w(t) = A\Big(\frac{b-t}{b-a}\Big)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + B\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}, \quad t \in \mathbb{N}^{b}_{a}. \tag{3.13}$$

Proof. Using Theorem 2.14, the general solution of the equation $-\big(\nabla^{\alpha}_{\rho(a)}w\big)(t) = 0$ is given by

$$w(t) = C_1(t-a+1)^{\overline{\alpha-1}} + C_2(t-a+1)^{\overline{\alpha-2}}, \quad t \in \mathbb{N}^{b}_{a}, \tag{3.14}$$

where C_1 and C_2 are arbitrary constants. Using $w(a) = A$ and $w(b) = B$ in (3.14), we have that

$$(\alpha-1)C_1 + C_2 = \frac{A}{\Gamma(\alpha-1)},$$
$$C_1(b-a+1)^{\overline{\alpha-1}} + C_2(b-a+1)^{\overline{\alpha-2}} = B.$$

Solving the above system of equations for C_1 and C_2, we get that

$$C_1 = \frac{1}{(b-a)^{\overline{\alpha-1}}}\Big[B - \frac{A}{\Gamma(\alpha-1)}(b-a+1)^{\overline{\alpha-2}}\Big],$$

and

$$C_2 = \frac{A}{\Gamma(\alpha-1)} - \frac{(\alpha-1)}{(b-a)^{\overline{\alpha-1}}}\Big[B - \frac{A}{\Gamma(\alpha-1)}(b-a+1)^{\overline{\alpha-2}}\Big].$$

Substituting the values of C_1 and C_2 in (3.14), it follows that

$$w(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}\Big[B - \frac{A}{\Gamma(\alpha-1)}(b-a+1)^{\overline{\alpha-2}}\Big]$$

$$
\begin{aligned}
&+ \frac{A(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} - \frac{(\alpha-1)(t-a+1)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-1}}}\Big[B - \frac{A}{\Gamma(\alpha-1)}(b-a+1)^{\overline{\alpha-2}}\Big] \\
&= \frac{A}{\Gamma(\alpha-1)}\Big[(t-a+1)^{\overline{\alpha-2}} - \frac{(t-a)^{\overline{\alpha-1}}}{(b-a)}\Big] + B\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}} \\
&= A\Big(\frac{b-t}{b-a}\Big)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + B\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}}.
\end{aligned}
$$

Thus, we have (3.13). □

Theorem 3.4. *Let $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$. The nabla fractional boundary value problem* (3.11) *has the unique solution*

$$
u(t) = w(t) + \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3.15}
$$

where Green's function $G(t,s)$ is as in (1.4) *and w is given by* (3.13)*.*

In order to derive sufficient conditions on the existence of solutions for (1.11), we use the following Lemma.

Lemma 3.5. $|w(t)| \le 2\max\{|A|,|B|\}$ *for each $t \in \mathbb{N}_a^b$.*

Proof. From Theorem 2.9, it follows that

$$
(t-a+1)^{\overline{\alpha-2}} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(t-a+1)} > 0,
$$

$$
(t-a)^{\overline{\alpha-1}} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(t-a)} \ge 0,
$$

and

$$
(b-a)^{\overline{\alpha-1}} = \frac{\Gamma(b-a+\alpha-1)}{\Gamma(b-a)} > 0,
$$

for $t \in \mathbb{N}_a^b$. Since $(\alpha-2) < 1 \le (t-a+1)$ and $(t-a) \le (b-a)$, using Theorem 2.9, we have

$$
(t-a+1)^{\overline{\alpha-2}} \le 1^{\overline{\alpha-2}} \text{ and } (t-a)^{\overline{\alpha-1}} \le (b-a)^{\overline{\alpha-1}},
$$

implying that

$$
\begin{aligned}
|w(t)| &\le \frac{|A|}{\Gamma(\alpha-1)}\Big(\frac{b-t}{b-a}\Big)(t-a+1)^{\overline{\alpha-2}} + |B|\frac{(t-a)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}} \\
&\le \frac{|A|}{\Gamma(\alpha-1)}1^{\overline{\alpha-2}} + |B| \\
&= |A| + |B| \\
&\le 2\max\{|A|,|B|\}.
\end{aligned}
$$

Hence the proof is complete. □

4 Existence of Positive Solutions

In this section, we establish sufficient conditions on the existence of at least one and two positive solutions for the following standard nonlinear nabla fractional boundary value problem using Guo–Krasnoselskii fixed point theorem for cones:

$$\begin{cases} -\big(\nabla^{\alpha}_{\rho(a)}u\big)(t) = f(t,u(t)), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0,\ u(b) = 0, \end{cases} \tag{4.1}$$

where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, $1 < \alpha < 2$ and $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Theorem 4.1 (See [3])**.** *[Guo–Krasnoselskii fixed point theorem] Let $\mathcal{B}$ be a Banach space and $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Further, assume that $A : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ is a completely continuous operator. If, either*

1. $\|Au\| \leq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_2$*; or*
2. $\|Au\| \geq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_2$*;*

holds, then A has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In addition to Theorem 1.6, we shall also need the following property of Green's function.

Theorem 4.2. *There exists a number $\gamma \in (0, 1)$ such that*

$$\min_{t \in \mathbb{N}^{b-1}_{a+1}} G(t,s) \geq \gamma \max_{t \in \mathbb{N}^{b-1}_{a+1}} G(t,s) = \gamma G(s-1,s), \quad s \in \mathbb{N}^{b}_{a+2}.$$

Proof. Brackins [9] showed that for fixed $s \in \mathbb{N}^{b}_{a+1}$, $G(t,s)$ increases from $G(a,s) = 0$ to a positive value at $t = s - 1$ and then decreases to $G(b,s) = 0$. Now, for $s \in \mathbb{N}^{b}_{a+2}$, consider

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & t \in \mathbb{N}^{\rho(s)}_{a+1}, \\ \dfrac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \dfrac{(t-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & t \in \mathbb{N}^{b-1}_{s}. \end{cases}$$

For $t \in \mathbb{N}^{\rho(s)}_{a+1}$ and $s \in \mathbb{N}^{b}_{a+2}$, we have

$$\frac{G(t,s)}{G(s-1,s)} = \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} \geq \frac{1^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} = \frac{\Gamma(\alpha)}{(b-a-1)^{\overline{\alpha-1}}}. \tag{4.2}$$

For $t \in \mathbb{N}_s^{b-1}$ and $s \in \mathbb{N}_{a+2}^{b-1}$, we know that $G(t,s)$ is a decreasing function of t. Then, we have

$$\begin{aligned}
\frac{G(t,s)}{G(s-1,s)} &\geq \frac{G(b-1,s)}{G(s-1,s)} \\
&= \frac{(b-a-1)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(b-s)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}} \\
&= \frac{(b-a-1)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}\Big[1 - \frac{(b-s)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}\frac{(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\Big] \\
&= \frac{(b-a-1)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}\Big[1 - \Big(\frac{b-s}{b-s+\alpha-1}\Big)\Big(\frac{b-a+\alpha-2}{b-a-1}\Big)\Big] \\
&= \frac{(b-a)^{\overline{\alpha-2}}}{(s-a)^{\overline{\alpha-2}}}\Big(\frac{\alpha-1}{b-s+\alpha-1}\Big) \\
&\geq \frac{(b-a)^{\overline{\alpha-2}}}{2^{\overline{\alpha-2}}}\Big(\frac{\alpha-1}{b-a+\alpha-3}\Big) \\
&= \frac{(b-a)^{\overline{\alpha-2}}}{(b-a+\alpha-3)\Gamma(\alpha-1)}.
\end{aligned} \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\min_{t \in \mathbb{N}_{a+1}^{b-1}} G(t,s) \geq \gamma G(s-1,s), \quad s \in \mathbb{N}_{a+2}^{b},$$

where

$$\gamma = \min\Big\{\frac{\Gamma(\alpha)}{(b-a-1)^{\overline{\alpha-1}}}, \frac{(b-a)^{\overline{\alpha-2}}}{(b-a+\alpha-3)\Gamma(\alpha-1)}\Big\}.$$

Clearly, $0 < \gamma < 1$. □

We observe that, by Theorem 3.2, u is a solution of (4.1) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+2}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{4.4}$$

Define the operator

$$\big(Au\big)(t) := \sum_{s=a+2}^{b} G(t,s)f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{4.5}$$

It is evident from (4.4) and (4.5) that u is a fixed point of A if and only if u is a solution of (4.1). Denote by

$$\mathcal{B} = \{u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = u(b) = 0\} \subseteq \mathbb{R}^{b-a+1}.$$

Clearly, $\mathcal{B}$ is a Banach space equipped with the maximum norm

$$\|u\| := \max_{t \in \mathbb{N}_a^b} |u(t)|,$$

and also, $A : \mathcal{B} \to \mathcal{B}$. Define the cone

$$\mathcal{K} := \Big\{ u \in \mathcal{B} : u(t) \geq 0 \text{ for all } t \in \mathbb{N}_a^b \text{ and } \min_{t \in \mathbb{N}_{a+1}^{b-1}} u(t) \geq \gamma \|u\| \Big\}.$$

We shall obtain sufficient conditions for the existence of a fixed point of A. First, note that A is a summation operator on a discrete finite set. Hence, A is trivially completely continuous. We state the following hypotheses which will be used later. Take

$$\eta := \frac{1}{\sum_{s=a+2}^{b} G(s-1, s)} = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \frac{(b-a)^{\overline{2\alpha - 1}}}{(b-a)^{\overline{\alpha - 1}}}.$$

Let $t_0 \in \mathbb{N}_{a+1}^{b-1}$ such that

$$\min_{t \in \mathbb{N}_{a+1}^{b-1}} G(t, s) = G(t_0, s), \text{ for all } s \in \mathbb{N}_{a+2}^{b}.$$

Then, it follows from Theorem 4.2 that

$$G(t_0, s) \geq \gamma G(s-1, s), \quad s \in \mathbb{N}_{a+2}^{b}. \tag{4.6}$$

(H1) $f(t, \xi) \geq 0$, $(t, \xi) \in \mathbb{N}_a^b \times [0, \infty)$;

(H2) There exists a number $r_1 > 0$ such that $f(t, u) \leq \eta r_1$, whenever $0 \leq u \leq r_1$;

(H3) There exists a number $r_2 > 0$ such that $f(t, u) \geq \frac{\eta r_2}{\gamma}$, whenever $\gamma r_2 \leq u \leq r_2$;

(H4) Assume that

$$\lim_{u \to 0^+} \min_{t \in \mathbb{N}_a^b} \frac{f(t, u)}{u} = \infty, \quad \lim_{u \to \infty} \min_{t \in \mathbb{N}_a^b} \frac{f(t, u)}{u} = \infty.$$

(H5) Assume that

$$\lim_{u \to 0^+} \min_{t \in \mathbb{N}_a^b} \frac{f(t, u)}{u} = 0, \quad \lim_{u \to \infty} \min_{t \in \mathbb{N}_a^b} \frac{f(t, u)}{u} = 0.$$

Lemma 4.3. *Assume (H1) holds. Then, $A : \mathcal{K} \to \mathcal{K}$.*

Proof. Let $u \in \mathcal{K}$. Clearly, $\big(Au\big)(t) \geq 0$ for $t \in \mathbb{N}_a^b$. Now, consider

$$\begin{aligned}
\min_{t \in \mathbb{N}_{a+1}^{b-1}} \big(Au\big)(t) &= \min_{t \in \mathbb{N}_{a+1}^{b-1}} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \\
&\geq \gamma \sum_{s=a+2}^{b} G(s-1,s) f(s,u(s)) \\
&= \gamma \max_{t \in \mathbb{N}_{a+1}^{b-1}} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \\
&= \gamma \max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \\
&= \gamma \|Au\|,
\end{aligned}$$

implying that $Au \in \mathcal{K}$. $\square$

Theorem 4.4. *Assume f satisfies (H1), (H2) and (H3). Then,* (4.1) *has at least one positive solution.*

Proof. We know that $A : \mathcal{K} \to \mathcal{K}$ is completely continuous. Define the set

$$\Omega_1 := \{u \in \mathcal{K} : \|u\| < r_1\}.$$

Clearly, $\Omega_1 \subseteq \mathcal{B}$ is an open set with $0 \in \Omega_1$. Since $\|u\| = r_1$ for $u \in \partial\Omega_1$, (H2) holds for all $u \in \partial\Omega_1$. So, it follows that

$$\begin{aligned}
\|Au\| = \max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) &\leq \sum_{s=a+2}^{b} G(s-1,s) f(s,u(s)) \\
&\leq \eta r_1 \sum_{s=a+2}^{b} G(s-1,s) \\
&= r_1 = \|u\|,
\end{aligned}$$

implying that $\|Au\| \leq \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_1$. On the other hand, define the set

$$\Omega_2 := \{u \in \mathcal{K} : \|u\| < r_2\}.$$

Clearly, $\Omega_2 \subseteq \mathcal{B}$ is an open set and $\overline{\Omega}_1 \subseteq \Omega_2$. Since $\|u\| = r_2$ for $u \in \partial\Omega_2$, (H3) holds for all $u \in \partial\Omega_2$. Using (4.6), we have

$$\|Au\| \geq |Au(t_0)| = \sum_{s=a+2}^{b} G(t_0,s) f(s,u(s)) \geq \gamma \sum_{s=a+2}^{b} G(s-1,s) f(s,u(s))$$

$$\begin{aligned} &\geq \eta r_2 \sum_{s=a+2}^{b} G(s-1,s) \\ &= r_2 = \|u\|, \end{aligned}$$

implying that $\|Au\| \geq \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_2$. Hence, by Theorem 4.1, A has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, say u_0, satisfying $r_1 < \|u_0\| < r_2$. □

Theorem 4.5. *Assume f satisfies (H1), (H2) and (H4). Then,* (4.1) *has at least two positive solutions.*

Proof. Fix $t_1 \in \mathbb{N}_{a+1}^{b-1}$ and choose $M > 0$ such that

$$M\gamma \sum_{s=a+2}^{b} G(t_1,s) > 1. \tag{4.7}$$

By (H4), there exists an $r > 0$ such that $r < p$ and $f(t,u) \geq Mu$ for all $0 \leq u \leq r$ and $t \in \mathbb{N}_a^b$. Define the set

$$\Omega_r := \{u \in \mathcal{K} : \|u\| < r\}.$$

Using (4.7), we have

$$\begin{aligned} \|Au\| \geq |Au(t_1)| = \sum_{s=a+2}^{b} G(t_1,s)f(s,u(s)) &\geq M \sum_{s=a+2}^{b} G(t_1,s)|u(s)| \\ &\geq M\gamma\|u\| \sum_{s=a+2}^{b} G(t_1,s) > \|u\|, \end{aligned}$$

implying that $\|Au\| > \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_r$. Next, for the same $M > 0$, we can find a number $R_1 > 0$ such that $f(t,u) \geq Mu$ for all $u \geq R_1$ and $t \in \mathbb{N}_a^b$. Choose R so that

$$R > \max\Big\{p, \frac{R_1}{\gamma}\Big\}.$$

Define the set

$$\Omega_R := \{u \in \mathcal{K} : \|u\| < R\}.$$

Clearly, $\|Au\| > \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_R$. Finally, define the set

$$\Omega_p := \{u \in \mathcal{K} : \|u\| < p\}.$$

It follows that $\|Au\| \leq \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_p$.

Hence, we conclude that A has at least two fixed points say $u_1 \in \Omega_p \setminus \mathring{\Omega}_r$ and $u_2 \in \Omega_R \setminus \mathring{\Omega}_p$, where $\mathring{\Omega}$ denotes the interior of the set Ω. In particular, (4.1) has at least two positive solution, say u_1 and u_2, satisfying $0 < \|u_1\| < p < \|u_2\|$. □

Theorem 4.6. *Assume f satisfies (H1), (H3) and (H5). Then,* (4.1) *has at least two positive solutions.*

Proof. By (H5), for any $\varepsilon > 0$, there is an $M > 0$ such that $f(t,u) \le M + \varepsilon u$ for $u \in \mathcal{K}$ and $t \in \mathbb{N}_a^b$. Then,

$$\big|\big(Au\big)(t)\big| = \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) \le \sum_{s=a+2}^{b} G(s-1,s)\big[M + \varepsilon u(s)\big].$$

Since $\varepsilon > 0$ is arbitrary,

$$\big|\big(Au\big)(t)\big| \le M \sum_{s=a+2}^{b} G(s-1,s) = \frac{M}{\eta},$$

implying that

$$\|Au\| \le \frac{M}{\eta}.$$

Pick $R > p$ sufficiently large so that

$$\frac{M}{\eta} < R.$$

Define the set

$$\Omega_R := \{u \in \mathcal{K} : \|u\| < R\}.$$

Then, $\|Au\| < R = \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_R$. Again, by (H5), there exists an $r > 0$ such that $r < p$ and $f(t,u) < \eta u$ for $0 \le u \le r$, $u \in \mathcal{K}$ and $t \in \mathbb{N}_a^b$. Define the set

$$\Omega_r := \{u \in \mathcal{K} : \|u\| < r\}.$$

Then, it follows that

$$\begin{aligned}\|Au\| = \max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) &\le \sum_{s=a+2}^{b} G(s-1,s)f(s,u(s)) \\ &< \eta \sum_{s=a+2}^{b} G(s-1,s)|u(s)| \\ &\le \|u\|,\end{aligned}$$

implying that $\|Au\| < \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_r$. Define the set

$$\Omega_p := \{u \in \mathcal{K} : \|u\| < p\}.$$

Clearly, $\|Au\| > \|u\|$ whenever $u \in \mathcal{K} \cap \partial\Omega_p$.

Hence, we conclude that A has at least two fixed points say $u_1 \in \Omega_p \setminus \mathring{\Omega}_r$ and $u_2 \in \Omega_R \setminus \mathring{\Omega}_p$, where $\mathring{\Omega}$ denotes the interior of the set Ω. In particular, (4.1) has at least two positive solution, say u_1 and u_2, satisfying $0 < \|u_1\| < p < \|u_2\|$. □

5 Existence of Solutions

In this section, we present several existence results using various fixed point theorems. By Theorem 3.4, we observe that u is a solution of (1.11) if and only if u is a solution of the summation equation

$$u(t) = w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{5.1}$$

Define the operator

$$\big(Tu\big)(t) := w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{5.2}$$

It is evident from (5.1) and (5.2) that u is a fixed point of T if and only if u is a solution of (1.11).

Theorem 5.1 (See [3])**.** *[Brouwer fixed point theorem] Let $\mathcal{K}$ be a nonempty compact convex subset of $\mathbb{R}^n$ and T be a continuous mapping of $\mathcal{K}$ into itself. Then, T has a fixed point in $\mathcal{K}$.*

Theorem 5.2 (See [3])**.** *[Leray–Schauder fixed point theorem] Let Ω be an open subset of $\mathbb{R}^n$ with $0 \in \Omega$. Then, every completely continuous mapping $T : \overline{\Omega} \to \mathbb{R}^n$ has at least one of the two following properties:*

1. *There exists an $u \in \overline{\Omega}$ such that $Tu = u$.*

2. *There exist a $v \in \partial\Omega$ and $\mu \in (0,1)$ such that $v = \mu T v$.*

Then, T has a fixed point u in Ω.

Theorem 5.3 (See [3])**.** *[Krasnoselskii–Zabreiko fixed point theorem] Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a completely continuous mapping. If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded linear mapping such that 1 is not an eigenvalue of L and*

$$\lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0,$$

then T has a fixed point in $\mathbb{R}^n$.

We shall use the fact that $\mathbb{R}^{b-a+1}$ is a Banach space equipped with the maximum norm defined by

$$\|u\| := \max_{t \in \mathbb{N}_a^b} |u(t)|.$$

Theorem 5.4. *Assume $f(t,u)$ is continuous with respect to u for each $t \in \mathbb{N}_a^b$. Assume there exist positive constants L and M such that*

$$\max\{|A|, |B|\} \leq L, \tag{5.3}$$

$$\max_{(t,u)\in\mathbb{N}_a^b\times[-3L,3L]} |f(t,u)| = M, \tag{5.4}$$

and

$$\lambda \leq \frac{L}{M}. \tag{5.5}$$

Then, (1.11) *has a solution.*

Proof. Define the set

$$\mathcal{K} := \Big\{ u : \mathbb{N}_a^b \to \mathbb{R} \text{ and } \|u\| \leq 3L \Big\}.$$

Clearly, $\mathcal{K}$ is a nonempty compact convex subset of $\mathbb{R}^{b-a+1}$. Now, we claim that $T : \mathcal{K} \to \mathcal{K}$. To see this, let $u \in \mathcal{K}$ and $t \in \mathbb{N}_a^b$. Consider

$$\begin{aligned}
\big|(Tu)(t)\big| &= \Big| w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \Big| \\
&\leq |w(t)| + \sum_{s=a+2}^{b} G(t,s) |f(s,u(s))| \\
&\leq 2\max\{|A|,|B|\} + M \sum_{s=a+2}^{b} G(t,s) \\
&\leq 2L + M\lambda \leq 3L,
\end{aligned}$$

implying that $Tu \in \mathcal{K}$ and hence $T : \mathcal{K} \to \mathcal{K}$. Since T is a summation operator on a discrete finite set, T is trivially continuous on $\mathcal{K}$. So, it follows by Theorem 5.1 that the operator T has a fixed point. This means that (1.11) has a solution, say u_0, with $\|u_0\| \leq 3L$. □

Theorem 5.5. *Assume $f(t,u)$ is continuous with respect to u for each $t \in \mathbb{N}_a^b$ and is bounded on $\mathbb{N}_a^b \times \mathbb{R}$. Then,* (1.11) *has a solution.*

Proof. Choose

$$P > \sup_{(t,u)\in\mathbb{N}_a^b\times\mathbb{R}} |f(t,u)|.$$

Pick L large enough so that

$$\max\{|A|,|B|\} \leq L \text{ and } \lambda \leq \frac{L}{M}.$$

For the M defined in Theorem 5.4, $M \leq P$, so that

$$\lambda \leq \frac{L}{M},$$

and, by Theorem 5.4, (1.11) has a solution. $\square$

Theorem 5.6. *Assume $f(t,u)$ is continuous with respect to u for each $t \in \mathbb{N}_a^b$. Assume there exists a continuous function $\sigma : \mathbb{N}_a^b \to \mathbb{R}^+$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f(t,u)| \leq \sigma(t)\psi(|u|), \quad (t,u) \in \mathbb{N}_a^b \times \mathbb{R}. \tag{5.6}$$

Moreover, assume there exists a positive constant γ such that

$$\frac{\gamma}{2\max\{|A|,|B|\} + \lambda\|\sigma\|\psi(\gamma)} > 1. \tag{5.7}$$

Then, (1.11) *has a solution.*

Proof. Define the set

$$\Omega := \{u : \mathbb{N}_a \to \mathbb{R} \text{ and } \|u\| < \gamma\}.$$

Clearly, Ω is an open subset of $\mathbb{R}^{b-a+1}$ with $0 \in \Omega$ and $T : \overline{\Omega} \to \mathbb{R}^{b-a+1}$. Since T is a summation operator on a discrete finite set, T is trivially completely continuous on $\overline{\Omega}$. Now, suppose there exist a $v \in \Omega$ and $\mu \in (0,1)$ such that

$$v = \mu T v. \tag{5.8}$$

Using the definition of T and Lemma 3.5 in (5.8), we obtain

$$\begin{aligned}
|v(t)| &\leq |w(t)| + \sum_{s=a+2}^{b} G(t,s)|f(s,v(s))| \\
&\leq 2\max\{|A|,|B|\} + \sum_{s=a+2}^{b} G(t,s)\sigma(s)\psi(|v(s)|) \\
&\leq 2\max\{|A|,|B|\} + \|\sigma\|\psi(\|v\|)\sum_{s=a+2}^{b} G(t,s) \\
&\leq 2\max\{|A|,|B|\} + \lambda\|\sigma\|\psi(\gamma),
\end{aligned}$$

implying that

$$\|v\| \leq 2\max\{|A|,|B|\} + \lambda\|\sigma\|\psi(\gamma).$$

Hence,

$$\frac{\gamma}{2\max\{|A|,|B|\} + \lambda\|\sigma\|\psi(\gamma)} \leq 1.$$

This is a contradiction to (5.7). So, it follows by Theorem 5.2 that the operator T has a fixed point in Ω. This means that (1.11) has a solution, say u_1, with $\|u_1\| < \gamma$. $\square$

Theorem 5.7. *Assume $f(t,u)$ is continuous with respect to u for each $t \in \mathbb{N}_a^b$. Assume there exists a continuous function $\phi : \mathbb{N}_a^b \to \mathbb{R}$ such that*

$$\lim_{\|u\|\to\infty} \frac{f(t,u)}{u} = \phi(t), \quad t \in \mathbb{N}_a^b, \tag{5.9}$$

and

$$\|\phi\| < \frac{1}{\lambda}. \tag{5.10}$$

Then, (1.11) *has a solution.*

Proof. Clearly, $T : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$. Since T is a summation operator on a discrete finite set, we have T is trivially completely continuous on $\mathbb{R}^{b-a+1}$. Consider a linear bounded mapping $L : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$ defined by

$$\big(Lu\big)(t) := \sum_{s=a+2}^{b} G(t,s)\phi(s)u(s), \quad t \in \mathbb{N}_a^b. \tag{5.11}$$

Obviously, $\|Lu\| < \|u\|$. To see this, let $u \in \mathbb{R}^n$ and $t \in \mathbb{N}_a^b$. Consider

$$\begin{aligned} \big|\big(Lu\big)(t)\big| &\le \sum_{s=a+2}^{b} G(t,s)|\phi(s)||u(s)| \\ &\le \|\phi\|\|u\| \sum_{s=a+2}^{b} G(t,s) \le \lambda\|\phi\|\|u\| < \|u\|, \end{aligned}$$

implying that $\|Lu\| < \|u\|$ and hence 1 is not an eigenvalue of L. From (5.9), we know that for every $\varepsilon > 0$ there exists a number N such that, for every $t \in \mathbb{N}_a^b$,

$$|f(t,u(t)) - \phi(t)u(t)| < \varepsilon \text{ whenever } \|u\| > N. \tag{5.12}$$

For every $t \in \mathbb{N}_a^b$, we have

$$\begin{aligned} \big|\big(Tu\big)(t) - \big(Lu\big)(t)\big| &\le |w(t)| + \sum_{s=a+2}^{b} G(t,s)\big|f(s,u(s)) - \phi(s)u(s)\big| \\ &< 2\max\{|A|,|B|\} + \varepsilon \sum_{s=a+2}^{b} G(t,s) \\ &\le 2\max\{|A|,|B|\} + \lambda\varepsilon, \end{aligned}$$

implying that

$$\frac{\|Tu - Lu\|}{\|u\|} < \frac{2\max\{|A|,|B|\} + \lambda\varepsilon}{N}.$$

Consequently, we obtain

$$\lim_{\|u\|\to\infty} \frac{\|Tu - Lu\|}{\|u\|} = 0.$$

So, it follows by Theorem 5.3 that the operator T has a fixed point in $\mathbb{R}^{b-a+1}$. This means that (1.11) has a solution. □

6 Uniqueness of Solutions

In this section, we deduce the existence of a unique solution to (1.11) by assuming the Lipschitz condition on f.

Theorem 6.1 (See [3])**.** *[Contraction Mapping Theorem] Let S be a closed subset of $\mathbb{R}^n$. Assume $T : S \to S$ is a contraction mapping, that is, there exists a μ, $0 \leq \mu < 1$, such that*

$$\|Tu - Tv\| \leq \mu\|u - v\|,$$

for all u, $v \in S$. Then, T has a unique fixed point w in S.

Theorem 6.2. *Assume $f(t, u)$ satisfies the Lipschitz condition with respect to u with Lipschitz constant K. That is,*

$$\|f(t, u) - f(t, v)\| \leq K\|u - v\|, \tag{6.1}$$

for all (t, u), $(t, v) \in \mathbb{N}_a^b \times \mathbb{R}$. If

$$0 < K\lambda < 1, \tag{6.2}$$

then (1.11) *has a unique solution.*

Proof. Let u, $v \in \mathbb{R}^{b-a+1}$ and $t \in \mathbb{N}_a^b$. Consider

$$\begin{aligned}
\big|\big(Tu\big)(t) - \big(Tv\big)(t)\big| &\leq \sum_{s=a+2}^{b} G(t, s)|f(s, u(s)) - f(s, v(s))| \\
&\leq K \sum_{s=a+1}^{b} G(t, s)|u(s) - v(s)| \\
&\leq K\|u - v\| \sum_{s=a+2}^{b} G(t, s) \\
&\leq K\lambda\|u - v\|,
\end{aligned}$$

implying that

$$\big\|Tu - Tv\big\| \leq K\lambda\|u - v\|.$$

Thus, T is a contraction on $\mathbb{R}^{b-a+1}$ due to (6.2). So, it follows by Theorem 6.1 that the operator T has a unique fixed point in $\mathbb{R}^{b-a+1}$. This means that (1.11) has a unique solution. □

Theorem 6.3. *Assume there exist positive constants η and β such that*

$$\|f(t, u) - f(t, v)\| \leq \beta\|u - v\|, \tag{6.3}$$

for all (t,u), $(t,v) \in \mathbb{N}_a^b \times [-\eta, \eta]$. *Take*

$$m_1 = \max_{t \in \mathbb{N}_a^b} |f(t,0)| \tag{6.4}$$

and

$$m_2 = \max_{(t,u) \in \mathbb{N}_a^b \times [-\eta,\eta]} |f(t,u)|. \tag{6.5}$$

Assume

$$0 < \beta\lambda < 1. \tag{6.6}$$

If

$$\eta \geq \frac{m_1\lambda + 2\max\{|A|,|B|\}}{1-\beta\lambda} \tag{6.7}$$

or

$$2\max\{|A|,|B|\} + m_2\lambda \leq \eta, \tag{6.8}$$

then (1.11) *has a unique solution.*

Proof. Clearly, T is a contraction on $\mathbb{R}^{b-a+1}$ due to (6.6). Define the set

$$\mathcal{A} = \Big\{u : \mathbb{N}_a^b \to \mathbb{R} \text{ and } \|u\| \leq \eta\Big\}.$$

Clearly, $\mathcal{A} \subseteq \mathbb{R}^{b-a+1}$. Now, we claim that $T : \mathcal{A} \to \mathcal{A}$. To see this, let $u \in \mathcal{A}$ and $t \in \mathbb{N}_a^b$. Suppose that (6.7) holds. Consider

$$\begin{aligned}
\big|T(0)(t)\big| &= \Big|w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,0)\Big| \\
&\leq |w(t)| + \sum_{s=a+2}^{b} G(t,s)|f(s,0)| \\
&\leq 2\max\{|A|,|B|\} + m_1 \sum_{s=a+2}^{b} G(t,s) \\
&\leq 2\max\{|A|,|B|\} + m_1\lambda,
\end{aligned}$$

implying that

$$\big\|T(0)\big\| \leq 2\max\{|A|,|B|\} + m_1\lambda.$$

Consequently, we have

$$\begin{aligned}
\|Tu\| &= \|Tu - T(0) + T(0)\| \\
&\leq \|Tu - T(0)\| + \|T(0)\| \\
&\leq \beta\lambda\|u - 0\| + 2\max\{|A|,|B|\} + m_1\lambda \\
&\leq \beta\lambda\eta + 2\max\{|A|,|B|\} + m_1\lambda \leq \eta,
\end{aligned}$$

implying that $T : \mathcal{A} \to \mathcal{A}$. On the other hand, suppose that (6.8) holds. Let $u \in \mathcal{A}$ and $t \in \mathbb{N}_a^b$. Consider

$$\begin{aligned}\big|\big(Tu\big)(t)\big| &= \Big|w(t) + \sum_{s=a+2}^{b} G(t,s) f(s,u(s))\Big| \\ &\leq |w(t)| + \sum_{s=a+2}^{b} G(t,s)|f(s,u(s))| \\ &\leq 2\max\{|A|,|B|\} + m_2 \sum_{s=a+2}^{b} G(t,s) \\ &\leq 2\max\{|A|,|B|\} + m_2\lambda \leq \eta,\end{aligned}$$

implying that

$$\big\|Tu\big\| \leq \eta,$$

and hence $T : \mathcal{A} \to \mathcal{A}$. So, it follows by Theorem 6.1 that the operator T has a unique fixed point in $\mathbb{R}^{b-a+1}$. This means that (1.11) has a unique solution, say u_2, with $\|u_2\| \leq \eta$. □

7 Examples

Example 7.1. Consider the boundary value problem

$$\begin{cases} -\big(\nabla^{1.5}_{\rho(0)} u\big)(t) = \dfrac{1}{u^2 + t^2 + 9}, \quad t \in \mathbb{N}_2^{10}, \\ u(0) = 1, \ u(10) = 2. \end{cases} \tag{7.1}$$

Here $\alpha = 1.5$, $f(t,u) = \dfrac{1}{u^2 + t^2 + 9}$, $a = 0$, $b = 10$, $A = 1$, $B = 2$ and $L \geq 2$. Also,

$$\lambda = \frac{9}{(1.5)\Gamma(2.5)} (12)^{\overline{0.5}} = 15.4738,$$

and

$$M = \max_{(t,u)\in\mathbb{N}_0^{10}\times[-3L,3L]} |f(t,u)| = \frac{1}{9}.$$

Since $\lambda \leq \dfrac{L}{M}$, by Theorem 5.4, the boundary value problem (7.1) has at least one solution, u_0, satisfying $|u_0(t)| \leq 3L$ for every $t \in \mathbb{N}_0^{10}$.

Example 7.2. Consider the boundary value problem

$$\begin{cases} -\big(\nabla^{1.5}_{\rho(0)} u\big)(t) = t + (0.05)\sin u, \quad t \in \mathbb{N}_2^{10}, \\ u(0) = 1, \ u(10) = 2. \end{cases} \tag{7.2}$$

Here $\alpha = 1.5$, $f(t,u) = t + (0.05)\sin u$, $a = 0$, $b = 10$, $A = 1$ and $B = 2$. Clearly, f satisfies Lipschitz condition with respect to the second argument on $\mathbb{N}_0^{10} \times \mathbb{R}$ with Lipschitz constant $K = 0.05$. Also,

$$\lambda = \frac{9}{(1.5)\Gamma(2.5)}(12)^{\overline{0.5}} = 15.4738.$$

Since $0 < K\lambda < 1$, by Theorem 6.2, the boundary value problem (7.1) has a unique solution on $\mathbb{N}_0^{10} \times \mathbb{R}$.

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