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Periodic Oscillations in a Network Model with Discrete and Distributed Delays

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Abstract

This paper investigates a network model incorporating discrete and distributed signal transmission delays. The existence of periodic oscillations for such a n-node neural network has been derived. Some criteria are provided to guarantee the existence of periodic oscillations. Our simulation examples are presented to demonstrate the correctness of the results.

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1 Introduction

Recently, the network models with discrete and distributed delays have been considered by many researchers [1, 2, 4–7, 12–17, 19–32, 34–36]. For example, Hajihosseini et al. [9] have discussed the following three-node network model with distributed delay:

$$\begin{cases} y_1'(t) = -y_1(t) + \displaystyle\int_0^{+\infty} F(r)\tanh(y_2(t-r))dr, \\ y_2'(t) = -y_2(t) + \displaystyle\int_0^{+\infty} F(r)\tanh(y_3(t-r))dr, \\ y_3'(t) = -y_3(t) + w_1 \displaystyle\int_0^{+\infty} F(r)\tanh(y_1(t-r))dr \\ +w_2 \displaystyle\int_0^{+\infty} F(r)\tanh(y_2(t-r))dr. \end{cases} \tag{1.1}$$

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where w_1 and w_2 are parameters, $F(r) = \mu^2 re^{-\mu r}(\mu > 0)$ is a strong kernel. The existence of Hopf bifurcation and stability of the bifurcating periodic solutions by taking μ as a bifurcating parameter have been discussed. Zhao has considered the existence and stability of periodic solutions for a delay population model as follows:

$$\begin{cases} N_i' = N_i[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)lnN_j(t) - \sum_{j=1}^{n} b_{ij}(t)lnN_j(t-\tau_{ij}(t)) \\ -\sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)lnN_j(s)ds] \end{cases} \tag{1.2}$$

where $\int_0^\infty K_{ij}(s)ds = 1$, and $\int_0^\infty sK_{ij}(s)ds < +\infty$. By contraction principle, some results on the existence of positive periodic solutions have been obtained [33]. I. Ncube has investigated the Hopf bifurcation in a multiple-delayed neural network with discrete and distributed delays as the following $(i = 0, 1, 2)$ [18]:

$$x_i' = -x_i + a\int_{-\infty}^{t} k(t-s)f(x_i(s))ds + b[f(x_{i-1}(t-\tau_n) + f(x_{i+1}(t-\tau_n))], \tag{1.3}$$

where $f(x) = \tanh(x)$, and $k(s) = re^{-rs}$ is a weak kernel. In this model, the distributed signal transmission delay has introduced in the self-connection of the network and maintained the usual discrete signal transmission delay in the nearest-neighbor connection. Noting that $k(s)$ is a weak kernel. The author set $w_i(t) = \int_{-\infty}^{t} k(t-s)f(x_i(s))ds$, and the model (1.3) becomes

$$\begin{cases} x_0'(t) = -x_0(t) + ax_3(t) + b[f(x_2(t-\tau_n)) + f(x_1(t-\tau_n))], \\ x_1'(t) = -x_1(t) + ax_4(t) + b[f(x_0(t-\tau_n)) + f(x_2(t-\tau_n))], \\ x_2'(t) = -x_2(t) + ax_5(t) + b[f(x_1(t-\tau_n)) + f(x_0(t-\tau_n))], \\ x_3'(t) = r[f(x_0(t)) - x_3(t)], \\ x_4'(t) = r[f(x_1(t)) - x_4(t)], \\ x_5'(t) = r[f(x_2(t)) - x_5(t)]. \end{cases} \tag{1.4}$$

It has been established that two cases of a single Hopf bifurcation may occur at the trivial equilibrium. Liu *et al.* have investigated a network model with discrete and distributed delays of the form

$$x_i' = -d_ix_i + \sum_{j=1}^{n} a_{ij}g_j(x_j) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau_{ij})) + \sum_{j=1}^{n} c_{ij}\int_0^\infty g_j(x_j(t-s))p_{ij}(s)ds. \tag{1.5}$$

By using the Mawhin-like coincidence theorem, M-matrix theory, differential inequality techniques and constructing of a Lyapunov functional, the authors established some new criteria for the existence and exponential stability of the periodic solutions [17]. This paper extends model (1.3) to a more general case for $i = 1, 2, \cdots, n$:

$$x_i' = -c_i x_i + a_i \int_{-\infty}^{t} k(t-s) f(x_i(s)) ds + \sum_{j=1, j\neq i}^{n} b_{ij} f[x_j(t-\tau_j)], \tag{1.6}$$

where the passive decay rates c_i are positive constants, $k(s) = re^{-rs}$ $(r > 0)$ is a weak kernel, a_i, b_{ij} $(i, j = 1, 2, \cdots, n)$ represent the synaptic strengths of the connection. $f(x_i)$ $(i = 1, 2, \cdots, n)$ are activation functions. Model (1.6) indicated that not only the distributed signal transmission delay has been introduced to the self-connection of the network, but also the discrete signal transmission delays have been introduced to all connections. Our goal is to inspect the existence of periodic oscillation of the model (1.6). Let $w_i(t) = \displaystyle\int_{-\infty}^{t} k(t-s) f(x_i(s)) ds (i = 1, 2, \cdots, n)$. Then

$$w_i'(t) = r\left[f(x_i(t)) - w_i(t)\right],\ i = 1, 2, \cdots, n.$$

Setting $w_1(t) = x_{n+1}(t), w_2(t) = x_{n+2}(t), \cdots, w_n(t) = x_{2n}(t)$, we obtain the equivalent system of (1.6) as follows:

$$\begin{cases} x_1'(t) = -c_1 x_1(t) + a_{n+1} x_{n+1}(t) + \displaystyle\sum_{j=2}^{n} b_{1j} f(x_j(t-\tau_j)), \\ x_2'(t) = -c_2 x_2(t) + a_{n+2} x_{n+2}(t) + \displaystyle\sum_{j=1, j\neq 2}^{n} b_{2j} f(x_j(t-\tau_j)), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1}'(t) = -c_{n-1} x_{n-1}(t) + a_{2n-1} x_{2n-1}(t) + \displaystyle\sum_{j=1, j\neq_{n-1}}^{n} b_{n-1,j} f(x_j(t-\tau_j)), \\ x_n'(t) = -c_n x_n(t) + a_{2n} x_{2n}(t) + \displaystyle\sum_{j=1}^{n-1} b_{n,j} f(x_j(t-\tau_j)), \\ x_{n+1}'t) = r[f(x_1(t)) - x_{n+1}(t)], \\ x_{n+2}'(t) = r[f(x_2(t)) - x_{n+2}(t)], \\ x_{n+3}'(t) = r[f(x_3(t)) - x_{n+3}(t)], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{2n}'(t) = r[f(x_n(t)) - x_{2n}(t)]. \end{cases} \tag{1.7}$$

The linearized system of (1.7) about zero point is the following:

$$\begin{cases} x_1'(t) = -c_1x_1(t) + a_{n+1}x_{n+1}(t) + \sum_{j=2}^{n} b_{1j}\gamma_j x_j(t-\tau_j), \\ x_2'(t) = -c_2x_2(t) + a_{n+2}x_{n+2}(t) + \sum_{j=1,j\neq 2}^{n} b_{2j}\gamma_j x_j(t-\tau_j), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1}'(t) = -c_{n-1}x_{n-1}(t) + a_{2n-1}x_{2n-1}(t) + \sum_{j=1,j\neq n-1}^{n} b_{n-1,j}\gamma_j x_j(t-\tau_j), \\ x_n'(t) = -c_nx_n(t) + a_{2n}x_{2n}(t) + \sum_{j=1}^{n-1} b_{n,j}\gamma_j x_j(t-\tau_j), \\ x_{n+1}'t) = -rx_{n+1}(t) + r\gamma_1 x_1(t), \\ x_{n+2}'(t) = -rx_{n+2}(t) + r\gamma_2 x_2(t), \\ x_{n+3}'(t) = -rx_{n+3}(t) + r\gamma_3 x_3(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{2n}'(t) = -rx_{2n}(t) + r\gamma_n x_n(t), \end{cases} \tag{1.8}$$

where $\gamma_j = \dfrac{df(x_j)}{dx_j}|x_j = 0$ $(j = 1, 2, \cdots, n)$. System (1.8) also can be written in matrix form

$$X'(t) = AX(t) + BX(t-\tau), \tag{1.9}$$

where

$$X(t) = [x_1(t), x_2(t), \cdots, x_{2n}(t)]^T,$$

and

$$X(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_n(t-\tau_n), 0, 0, \cdots, 0]^T.$$

Both A and B are $2n \times 2n$ matrices. It is known that bifurcation can induce periodic solutions. However, one must deal with a higher order algebraic equation in using the bifurcation method. For example, Hajihosseini $et\ al.$ [9] have investigated double eight degree algebraic equations (see [9, equations (34), (35), page 996]). Liao $et\ al.$ discussed a eight degree algebraic equation to consider a two-node network model with distributed delay (see [15, equation (42), page 550]). Ncube dealt with a six degree equation for system (1.4) (see [18, equation (15), page 144]). Similarly, we must discuss a $2n$ degree algebraic equation if one still follows the bifurcating method for system (1.7). It is extremely hard to deal with a $2n$ degree algebraic equation if c_i, b_{ij}, r_i $(i, j = 1, 2, \cdots, n)$ are different real numbers. In order to discuss the existence of periodic oscillation for system (1.7) we will adopt Chafee's criterion [3]: For a class time delay system which has a unique unstable equilibrium point, and all solutions of the system are bounded, this particular instability of the unique equilibrium point and the boundedness of the solutions will lead the system to generate a limit cycle, namely, a nonconstant periodic solution. System (1.7) is in keeping with the requirements of Chafee's criterion we refer the reader to the appendix of [8]. In this paper we adopt the following norms of vectors and matrices [10]: $\|x(t)\| = \sum_{i=1}^{2n} |x_i(t)|$, $\|A\| = \max_{1\le j\le 2n} \sum_{i=1}^{2n} |a_{ij}|$, the measure $\mu(A)$ of a matrix A is defined by $\mu(A) = \lim_{\theta\to 0^+} \frac{\|I+\theta A\|-1}{\theta}$, which for the chosen norms reduces to $\mu(A) = \max_{1\le j\le 2n} [a_{jj} + \sum_{i=1, i\ne j}^{2n} |a_{ij}|]$.

Definition 1.1. The trivial solution of system (1.6) is unstable if there exists at least one component of the trivial solution which is unstable.

2 Preliminaries

For activation functions $f(x_i)$, we assume that $f(x_i)$ $(i = 1, 2, \cdots, n)$ are continuous bounded functions and satisfy the following condition:

$$f(0) = 0, uf(u) > 0 (u \ne 0). \tag{2.1}$$

Also we assume that there exists the derivative of $f(x_i)$ around the zero point, and

$$\gamma_j = \frac{df(x_j)}{dx_j}|_{x_j=0} (j = 1, 2, \cdots, n).$$

The general activation functions such as $\tanh(x)$, $\arctan(x)$, and $\frac{1}{2}(|x+1| - |x-1|)$ satisfy condition (2.1).

Lemma 2.1. *All solutions of system* (1.6) *are bounded.*

Proof. From condition (2.1), the activation functions are bounded, assume that

$$|f(x_i)| \le N_i,$$

then we have

$$\int_{-\infty}^{t} k(t-s) f(x_i(s)) ds \le \int_{-\infty}^{t} k(t-s) N_i ds = N_i \int_{-\infty}^{t} r e^{-r(t-s)} ds = N_i. \tag{2.2}$$

From (1.6), noting that c_i $(i = 1, 2, \cdots, n)$ are positive real numbers, we obtain

$$\frac{d|x_i(t)|}{dt} \le -c_i |x_i(t)| + |a_i| N_i + \sum_{j=1, j\ne i}^{n} |b_{ij}| N_j. \tag{2.3}$$

Thus, for $i = 1, 2, \cdots, n$, we have

$$|x_i(t)| \leq \frac{|a_i|N_i + \sum_{j=1, j\neq i}^{n} |b_{ij}|N_j}{c_i}. \tag{2.4}$$

This means that the solutions of system (1.6) are uniformly bounded. This concludes the proof. □

Lemma 2.2. *If matrix C is not a positive definite matrix, then system* (1.6) *or equivalent system* (1.7) *has a unique equilibrium point, where*

$$C = \begin{pmatrix} a_{n+1} & b_{12} & b_{13} & \cdots & b_{1,n-1} & b_{1n} \\ b_{21} & a_{n+2} & b_{23} & \cdots & b_{2,n-1} & b_{2n} \\ b_{31} & b_{32} & a_{n+3} & \cdots & b_{3,n-1} & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & a_{2n-1} & b_{n-1,n} \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{n,n-1} & a_{2n} \end{pmatrix}.$$

Proof. Note that system (1.7) is an equivalent system of (1.6). An equilibrium point $x^* = (x_1^*, x_2^*, x_3^*, \cdots, x_{2n}^*)^T$ of system (1.7) is a solution of the following algebraic equation:

$$\begin{cases} -c_1 x_1^* + a_{n+1} x_{n+1}^* + \sum\limits_{j=2}^{n} b_{1j} f(x_j^*) = 0, \\ -c_2 x_2^* + a_{n+2} x_{n+2}^* + \sum\limits_{j=1, j\neq 2}^{n} b_{2j} f(x_j^*) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -c_{n-1} x_{n-1}^* + a_{2n-1} x_{2n-1}^* + \sum\limits_{j=1, j\neq n-1}^{n} b_{n-1,j} f(x_j^*) = 0, \\ -c_n x_n^* + a_{2n} x_{2n}^* + \sum\limits_{j=1}^{n-1} b_{n,j} f(x_j^*) = 0, \\ r[f(x_1^*) - x_{n+1}^*] = 0, \\ r[f(x_2^*) - x_{n+2}^*] = 0, \\ r[f(x_3^*) - x_{n+3}^*] = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r[f(x_n^*) - x_{2n}^*] = 0. \end{cases} \tag{2.5}$$

From $r[f(x_i^*) - x_{n+i}^*] = 0$ $(i = 1, 2, \cdots, n)$, we have $f(x_i^*) = x_{n+i}^*$ $(i = 1, 2, \cdots, n)$.

Hence system (2.5) changes to the following:

$$\begin{cases} -c_1x_1^* + a_{n+1}f(x_1^*) + \sum_{j=2}^{n} b_{1j}f(x_j^*) = 0, \\ -c_2x_2^* + a_{n+2}f(x_2^*) + \sum_{j=1,j\neq 2}^{n} b_{2j}f(x_j^*) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -c_{n-1}x_{n-1}^* + a_{2n-1}f(x_{n-1}^*) + \sum_{j=1,j\neq n-1}^{n} b_{n-1,j}f(x_j^*) = 0, \\ -c_nx_n^* + a_{2n}f(x_n^*) + \sum_{j=1}^{n-1} b_{nj}f(x_j^*) = 0. \end{cases} \tag{2.6}$$

system (2.6) can be written in matrix form

$$CF(Z^*) = DZ^*, \tag{2.7}$$

where $F(Z^*) = (f(x_1^*), f(x_2^*), \cdots, f(x_n^*))^T$, $Z^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$, and matrix D is a diagonal matrix, $D = \mathrm{diag}(c_1, c_2, \cdots, c_n)$. From condition (2.1), when $x_i^* > 0$ then $f(x_i^*) > 0$, while $x_i^* < 0$ then $f(x_i^*) < 0$ $(i = 1, 2, \cdots, n)$. Noting that $c_i > 0$ $(i = 1, 2, \cdots, n)$, and C is not a positive definite matrix. Therefore, positivity of the right-hand side of (2.7) cannot guarantee positivity of $CF(Z^*)$. On the other hand, negativity of the right-hand side for some $x_i^* < 0, i \in (1, 2, \cdots, n)$, cannot guarantee negativity of $CF(Z^*)$. Condition $f(0) = 0$ implies that system (2.7), which is equivalent to (2.5) has a unique trivial solution. Thus, system (1.7) has a unique equilibrium point. It is exactly the zero point. Obviously, the zero point is not only the equilibrium point of system (2.5) but also the equilibrium point of system (1.9). This concludes the proof. □

3 Existence of Limit Cycles

Since system (1.7) is an equivalent version of system (1.6), hence there exists a periodic oscillatory solution of system (1.6) if and only if system (1.7) has a periodic oscillatory solution. Therefore, in the following we only deal with system (1.7) in which the results hold for system (1.6).

Theorem 3.1. *Assume that system* (1.7) *has a unique equilibrium point for selecting parameters. If the following conditions hold*

$$(\| B \|)e\tau_* \exp(-\tau_*|\mu(A)|) > 1, \tag{3.1}$$

$$(\| B \|)e\tau^* \exp(-\tau^*|\mu(A)|) > 1, \tag{3.2}$$

where $\tau_* = \min\{\tau_i\}$ *and* $\tau^* = \max\{\tau_i\}$, $i = 1, 2, \cdots, n$. $\| B \| = \max_{1\le j\le n} \sum_{i=1}^{n} |b_{ij}\gamma_j|$, $\mu(A) = \max_{1\le i\le n}\{-c_i + |r\gamma_i|, -r + |a_{n+i}|\}$. *Then the unique equilibrium point of system* (1.7) *is unstable, implying that system* (1.7) *generates a limit cycle, namely, a periodic solution.*

Proof. We shall prove that the unique equilibrium point of system (1.7), which is exactly the zero point, is unstable. It is known that the equilibrium points of system (1.7) and system (1.9), which is a linearized system of (1.7), have the same instability. Thus, in order to prove the instability of the equilibrium point of system (1.7), we prove that the equilibrium point is unstable in system (1.9). Consider two special cases of system (1.9) as follows:

$$X'(t) = AX(t) + BX(t - \tau_*), \tag{3.3}$$

$$X'(t) = AX(t) + BX(t - \tau^*), \tag{3.4}$$

where $\tau_* = \min\{\tau_1, \tau_2, \cdots, \tau_n\}$ and $\tau^* = \max\{\tau_1, \tau_2, \cdots, \tau_n\}$. Noting that $|x_i(t)| = x_i(t)$ as $x_i(t) > 0$ and $|x_i(t)| = -x_i(t)$ as $x_i(t) < 0$ $(i = 1, 2, \cdots, 2n)$. From (3.3), when each $x_i(t) > 0$ we have

$$\frac{d|X(t)|}{dt} = AX(t) + BX(t - \tau_*), \tag{3.5}$$

when each $x_i(t) < 0$ one can obtain

$$\frac{d|X(t)|}{dt} = A(-X(t)) + B(-X(t - \tau_*)). \tag{3.6}$$

Therefore, we have

$$\frac{d(\sum_{i=1}^{2n}|x_i(t)|)}{dt} \le \mu(A)\sum_{i=1}^{2n}|x_i(t)| + \| B \| \sum_{i=1}^{2n}|x_i(t - \tau_*)|. \tag{3.7}$$

Specifically, for the scalar time delay differential equation $(y(t) = \sum_{i=1}^{2n}|x_i(t)|)$

$$\frac{dy(t)}{dt} = \mu(A)y(t) + \| B \| y(t - \tau_*), \tag{3.8}$$

if the unique equilibrium point of system (3.8) is stable, then the characteristic equation associated with (3.8) given by

$$\lambda = \mu(A) + \| B \| e^{-\lambda\tau_*}, \tag{3.9}$$

will have a real negative root say λ_0, and we have from (3.9)

$$|\lambda_0| \ge \| B \| e^{|\lambda_0|\tau_*} - |\mu(A)|. \tag{3.10}$$

Using the formula $e^x \geq ex$ for $x \geq 0$ one can get

$$1 \geq \frac{\| B \| e^{|\lambda_0|\tau_*}}{|\mu(A)| + |\lambda_0|} = \frac{\| B \| \tau_* e^{-|\mu(A)|\tau_*} e^{(|\mu(A)|+|\lambda_0|)\tau_*}}{(|\mu(A)| + |\lambda_0|)\tau_*} \geq (\| B \| e\tau_*)e^{-\tau_*|\mu(A)|}. \quad (3.11)$$

The last inequality contradicts the equation (3.1). Hence, our claim regarding the instability of the equilibrium point of system (3.8) is valid. Based on the comparison theorem of differential equation [11] we have $\sum_{i=1}^{2n} |x_i(t)| \leq y(t)$. According to the definition of instability of the trivial solution, for arbitrary $\varepsilon > 0$, there exists a sequence $\{t_k\}_{k=1}^{+\infty}$ such that $|y(t_k)| > \varepsilon$, where $y(t)$ represents the trivial solution of system (3.8). Since $\sum_{i=1}^{2n} |x_i(t)| \leq y(t)$, this means that there exists a subsequence $\{t_{k_j}\}$ of sequence $\{t_k\}$ such that $\sum_{i=1}^{2n} |x_i(t_{k_j})| = y(t_{k_j})$. Therefore, there exists at least one $x_i(t)$, without loss of generality, we assume that $|x_1(t_{k_j})| > \frac{\varepsilon}{2n}$. Since ε is an arbitrary sufficiently small positive number, $\frac{\varepsilon}{2n}$ is also an arbitrary sufficiently small positive number. Thus, $x_1(t)$ is unstable in system (3.3). The instability of the component $x_1(t)$ implies that the trivial solution of (3.3) is unstable. Similarly, we know that the trivial solution of (3.4) is unstable in which all time delays equal to τ^*. Noting that for any time delay τ_i we have $\tau_* \leq \tau_i \leq \tau^*$ $(i = 1, 2, \cdots, n)$. The instability of the trivial solution of systems (3.3) and (3.4) implies that the trivial solution of system (1.9) for any time delays is unstable. The instability of the trivial solution of linearized system (1.9) again implies the instability of the trivial solution of the original system (1.7). Since all solutions of system (1.7) are bounded, the instability of the unique equilibrium point together with the boundedness of the solutions lead system (1.7) to generate a limit cycle based on [3], namely, a periodic solution. This concludes the proof. □

Theorem 3.2. *Assume that system* (1.7) *has a unique equilibrium point for selecting parameters. If the following condition holds*

$$\| B \| + \mu(A) > 0, \quad (3.12)$$

where $\| B \| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |b_{ij}\gamma_j|, \mu(A) = \max_{1 \leq i \leq n} \{-c_i + |r\gamma_i|, -r + |a_{n+i}|\}$. *Then the unique equilibrium point of system* (1.7) *is unstable, which implies that system* (1.7) *generates a limit cycle, namely, a periodic solution.*

Proof. We still prove that the trivial solutions both of systems (3.3) and (3.4) are unstable. The characteristic equation of system (3.8) is the following

$$\lambda = \mu(A) + \| B \| e^{-\lambda\tau_*}, \quad (3.13)$$

namely

$$\lambda - \mu(A) - \| B \| e^{-\lambda\tau_*} = 0. \tag{3.14}$$

Equation (3.14) is a transcendental equation. Generally speaking we can not find all solutions of this equation. However, we prove that there exists a positive characteristic root of equation (3.14) under the restrictive condition (3.12). Indeed, let $g(\lambda) = \lambda - \mu(A) - \| B \| e^{-\lambda\tau_*}$, then $g(\lambda)$ is a continuous function of λ. Obviously, $g(0) = -\mu(A) - \| B \| = -(\mu(A) + \| B \|) < 0$, and $\lim_{\lambda\to+\infty} e^{-\lambda\tau_*} = 0$. Therefore, there exists $\tilde{\lambda}(> 0)$ such that $g(\tilde{\lambda}) = \tilde{\lambda} - \mu(A) - \| B \| e^{-\tilde{\lambda}\tau_*} > 0$. According to the well known Intermediate Value Theorem, there exists a positive value of λ say $\lambda_1, \lambda_1 \in (0, \tilde{\lambda})$ such that $\lambda_1 - \mu(A) - \| B \| e^{-\lambda_1\tau_*} = 0$. In other words, equation (3.14) has a positive characteristic root. Therefore, the trivial solution of system (3.8) is unstable, implying that the trivial solution of system (3.3) is unstable. Similarly, we change τ_* to τ^* in system (3.8), then one can obtain that the trivial solution of system (3.4) is also unstable. Since for all τ_i we have $\tau_* \le \tau_i \le \tau^*$ $(i = 1, 2, \cdots, n)$, the instability of trivial solutions of system (3.3) and (3.4) implies that the trivial solution of system (1.9) is unstable. This suggests that the unique equilibrium point of system (1.7) is unstable. According to the Chafee's criterion, system (1.7) has a limit circle, namely, a periodic solution. This concludes the proof. □

Theorem 3.3. *Assume that system* (1.7) *has a unique equilibrium point for selecting parameters. Let $\theta_1, \theta_2, \cdots, \theta_{2n}$ represent the eigenvalues of matrix A, and $\rho_1, \rho_2, \cdots, \rho_{2n}$ the eigenvalues of matrix B. Assume that matrix A has a positive real eigenvalue, or there is at least one eigenvalue, say θ_j which has a positive real part* $\mathrm{Re}(\theta_j) > 0$, *satisfying that* $\max_{1\le i\le 2n}(|\mathrm{Re}(\rho_i)| + |\mathrm{Im}(\rho_i)|) < \mathrm{Re}(\theta_j)$. *Then the unique equilibrium point of system* (1.7) *is unstable, which implies that system* (1.7) *generates a limit cycle, namely, a periodic solution.*

Proof. Consider system (3.3), the characteristic equation of system (3.3) is the follows:

$$\det(\lambda I_{ij} - a_{ij} - b_{ij}e^{-\lambda\tau_*}) = 0, \tag{3.15}$$

where I_{ij} is an identity matrix. Since the eigenvalues of matrix A are $\theta_1, \theta_2, \cdots, \theta_{2n}$, and the eigenvalues of matrix B are $\rho_1, \rho_2, \cdots, \rho_{2n}$, so equation (3.15) changes to the following

$$\prod_{i=1}^{2n}(\lambda - \theta_i - \rho_i e^{-\lambda\tau_*}) = 0. \tag{3.16}$$

We are led to an investigation of the nature of the roots of the equation:

$$\lambda = \theta_i + \rho_i e^{-\lambda\tau_*}, i = 1, 2, \cdots, 2n. \tag{3.17}$$

Without loss of generality, assume that θ_1 is a complex number which has a positive real part $\mathrm{Re}(\theta_1)$, then we have

$$\lambda = \theta_1 + \rho_1 e^{-\lambda\tau_*}. \tag{3.18}$$

Assume that $\lambda = \sigma + i\omega$. $\theta_1 = \theta_{11} + i\theta_{12}$, $\rho_1 = \rho_{11} + i\rho_{12}$, where $\sigma = \mathrm{Re}(\lambda), \omega = \mathrm{Im}(\lambda), \theta_{11} = \mathrm{Re}(\theta_1), \theta_{12} = \mathrm{Im}(\theta_1)$, $\rho_{11} = \mathrm{Re}(\rho_1)$, $\rho_{12} = \mathrm{Im}(\rho_1)$. From (3.18) we get

$$\sigma + i\omega = \theta_{11} + i\theta_{12} + (\rho_{11} + i\rho_{12})e^{-(\sigma+i\omega)\tau_*}. \tag{3.19}$$

Separating the real and imaginary parts, we have

$$\sigma = \theta_{11} + \rho_{11}e^{-\sigma\tau_*}\cos(\omega\tau_*) + \rho_{12}e^{-\sigma\tau_*}\sin(\omega\tau_*). \tag{3.20}$$

$$\omega = \theta_{12} - \rho_{11}e^{-\sigma\tau_*}\sin(\omega\tau_*) + \rho_{12}e^{-\sigma\tau_*}\cos(\omega\tau_*). \tag{3.21}$$

We show that equation (3.20) has a positive real root. Let

$$f(\sigma) = \sigma - \theta_{11} - \rho_{11}e^{-\sigma\tau_*}\cos(\omega\tau_*) - \rho_{12}e^{-\sigma\tau_*}\sin(\omega\tau_*). \tag{3.22}$$

Obviously, $f(\sigma)$ is a continuous function of σ, and $f(\sigma) \le \sigma - \theta_{11} + (|\rho_{11}| + |\rho_{12}|)e^{-\sigma\tau_*}$. So we have $f(0) = -\theta_{11} + (|\rho_{11}| + |\rho_{12}|) < 0$. Obviously, there exists a suitably large $\tilde{\sigma}(> 0)$ such that $f(\tilde{\sigma}) = \tilde{\sigma} - \mathrm{Re}(\theta_1) - \mathrm{Re}(\rho_1)e^{-\tilde{\sigma}\tau_*}\cos(\omega\tau_*) - \mathrm{Im}(\rho_1)e^{-\tilde{\sigma}\tau_*}\sin(\omega\tau_*) > 0$ since $\lim\limits_{\sigma\to+\infty} e^{-\sigma\tau_*} = 0$. Again by means of the Intermediate Value Theorem, there exists a $\bar{\sigma} \in (0, \tilde{\sigma})$ such that $f(\bar{\sigma}) = \bar{\sigma} - \theta_{11} - \rho_{11}e^{-\bar{\sigma}\tau_*}\cos(\omega\tau_*) - \rho_{12}e^{-\bar{\sigma}\tau_*}\sin(\omega\tau_*) = 0$. This means that the characteristic value λ has a positive real part. The same as we can prove that the characteristic value λ also has a positive real part if θ_1 is a positive real number. Therefore, the trivial solution of system (3.3) is unstable. Similarly, we can prove that the trivial solution of system (3.4) is also unstable. This means that the trivial solution of system (1.9) for any time delays $\tau_i(\tau_* \le \tau_i \le \tau^*)$ $(i = 1, 2, \cdots, n)$ is unstable, implying that the trivial solution of system (1.7) is unstable. Based on Chafee's criterion, there exists a limit cycle of system (1.7), namely, a periodic solution. The proof is completed. □

Theorem 3.4. *Assume that system* (1.7) *has a unique equilibrium point for selecting parameters. Let matrix $M = A + B$. Suppose that matrix M has a positive real eigenvalue, or there is at least one eigenvalue which has a positive real part. Then the unique equilibrium point of system* (1.7) *is unstable, implying that system* (1.7) *generates a limit cycle.*

Proof. Consider a special case in system (1.9): $\tau = 0$. Then we have

$$X'(t) = AX(t) + BX(t) = (A + B)X(t) = MX(t). \tag{3.23}$$

The characteristic equation corresponding to system (3.23) is the following

$$\det(\lambda I_{ij} - M) = 0. \tag{3.24}$$

Since matrix M has a positive real eigenvalue or has a eigenvalue which has a positive real part. This means that the characteristic equation (3.24) has a positive real eigenvalue or has an eigenvalue which has a positive real part. Therefore, the trivial solution of system (3.23) is unstable. Noting that not only system (3.23) has the trivial solution, but the system (1.9) also has the trivial solution. Since the trivial solution $X(t)$ of system (3.23) is unstable, we show that the trivial solution of system (1.9) is also unstable. It is known that $X(t)$ is equal to $X(t-\tau^*)$ when t is sufficiently large. Similar to Theorem 3.1, we can find a sequence $\{t_1, t_2, \cdots, t_k, \cdots\}$ ($t_1 > \tau^*$ is sufficiently large) such that $|X(t_k)| > \varepsilon$, where ε is an arbitrary small positive real number. When t_1 is sufficiently large we have $X(t_k) \sim X(t_k - \tau^*)$. Therefore $|X(t_k)|$ also is the solution of system (1.9). Hence the trivial solution of system (1.9) is unstable. This implies that the unique equilibrium point of system (1.7) is unstable, and hence system (1.7) has a limit cycle, namely, a periodic solution. This concludes the proof. □

4 Simulation Result

This simulation was performed by means of the equivalent system (1.7) of (1.6). Firstly consider $n = 3$ for the following system.

Example 4.1.

$$\begin{cases} x_1^{'}(t) = -c_1x_1(t) + a_4x_4(t) + b_{12}f(x_2(t-\tau_2)) + b_{13}f(x_3(t-\tau_3)), \\ x_2^{'}(t) = -c_2x_2(t) + a_5x_5(t) + b_{21}f(x_1(t-\tau_1)) + b_{23}f(x_3(t-\tau_3)), \\ x_3^{'}(t) = -c_3x_3(t) + a_6x_6(t) + b_{31}f(x_1(t-\tau_1)) + b_{32}f(x_2(t-\tau_2)), \\ x_4^{'}t) = r[f(x_1(t)) - x_4(t)], \\ x_5^{'}(t) = r[f(x_2(t)) - x_5(t)], \\ x_6^{'}(t) = r[f(x_3(t)) - x_6(t)]. \end{cases} \tag{4.1}$$

We select the activation function as $f(x) = \tanh(x)$. Then $f'(x) = 1 - \tanh^2(x)$, and $f'(0) = 1$. The parameters $r = 0.5$, $c_1 = 0.75$, $c_2 = 0.85$, $c_3 = 0.65$; $b_{12} = -1.65$, $b_{13} = 2.8$, $b_{21} = 3.5$, $b_{23} = -1.85$, $b_{31} = 1.65$, $b_{32} = -2.8$; $a_4 = -1.2$, $a_5 = 1.5$, $a_6 = -1.6$. The characteristic values of matrix C are -4.3292, $1.5146 \pm 2.1248i$. Therefore, C is not a positive definite matrix. Thus, system (4.1) has a unique equilibrium point, namely, the zero point. $\mu(A) = 1.1$, $\| B \|= 5.15$, when we select time delay as $\tau_1 = 1.5$, $\tau_2 = 1.8$, $\tau_3 = 1.6$, $\| B \| e\tau_* \exp(-\tau_*|\mu(A)|) = 5.15e * 1.5 * \exp(-1.5 * 1.1) = 4.0328 > 1$ and $\| B \| e\tau^* \exp(-\tau^*|\mu(A)|) = 5.15e{*}1.8{*}\exp(-1.8{*}1.1) = 3.4791 > 1$ hold, respectively. Based on Theorem 3.1, system (4.1) generates a periodic oscillation (see Figure 4.1). When we change the value of r from 0.5 to 1.5, the oscillatory behavior still maintained (Figure 4.2). Then we change the activation function from $\tanh(x)$ to $\arctan(x)$, we see that the oscillatory amplitude and frequency only slightly change (see Figures 4.3 and 4.4).

Then we consider $n = 5$ for the following system.

Example 4.2.

$$
\begin{cases}
x_1'(t) = -c_1x_1(t) + a_6x_6(t) + b_{12}f(x_2) + b_{13}f(x_3) + b_{14}f(x_4) + b_{15}f(x_5),\\
x_2'(t) = -c_2x_2(t) + a_7x_7(t) + b_{21}f(x_1) + b_{23}f(x_3) + b_{24}f(x_4) + b_{25}f(x_5)\\
x_3'(t) = -c_3x_3(t) + a_8x_8(t) + b_{31}f(x_1) + b_{32}f(x_2) + b_{34}f(x_4) + b_{35}f(x_5),\\
x_4'(t) = -c_4x_4(t) + a_9x_9(t) + b_{41}f(x_1) + b_{42}f(x_2) + b_{43}f(x_3) + b_{45}f(x_5),\\
x_5'(t) = -c_5x_5(t) + a_{10}x_{10}(t) + b_{51}f(x_1) + b_{52}f(x_2) + b_{53}f(x_3) + b_{54}f(x_4),\\
x_6'(t) = r[f(x_1(t)) - x_6(t)],\\
x_7'(t) = r[f(x_2(t)) - x_7(t)],\\
x_8'(t) = r[f(x_3(t)) - x_8(t)],\\
x_9'(t) = r[f(x_1(t)) - x_9(t)],\\
x_{10}'(t) = r[f(x_4(t)) - x_{10}(t)].
\end{cases}
\tag{4.2}
$$

The activation functions $f(x_i) = f(x_i(t-\tau_i))$, $i = 1, 2, \cdots, 5$. We select the activation function as $f(x) = \arctan(x)$. Then $f'(x) = 1/(1+x^2)$, and $f'(0) = 1$. The parameters $c_1 = 0.65$, $c_2 = 0.75$, $c_3 = 0.85$, $c_4 = 0.55$, $c_5 = 0.45$; $b_{12} = -0.25$, $b_{13} = 2.85$, $b_{14} = -1.5$, $b_{15} = 2.4$, $b_{21} = 2.5$, $b_{23} = 1.15$, $b_{24} = 5.45$, $b_{25} = -0.8$, $b_{31} = 2.6$, $b_{32} = 2.8$, $b_{34} = -9.25$, $b_{35} = -4.25$, $b_{41} = 1.35$, $b_{42} = -0.2$, $b_{43} = 8.85$, $b_{45} = -0.45$, $b_{51} = 0.45$, $b_{52} = 0.28$, $b_{53} = 0.85$, $b_{54} = 0.45$; $a_6 = 2$, $a_7 = 5.5$, $a_8 = -1.6$, $a_9 = -1.5$, $a_{10} = -1.6$. $r = 0.5$ and 1.8, respectively. The characteristic values of matrix C are 6.4941, 1.7895, -1.8353, $-1.8242 \pm 9.2098i$. Therefore, C is not a positive definite matrix. System (4.2) has a unique equilibrium point, namely, the zero point. The characteristic values of matrix $A+B = M$ are 1.7284, 0.3292, $-1.0910 \pm 8.8924i$, $-1.7050 \pm 1.2386i$, $-0.6060 \pm 0.6698i$, $-0.5019 \pm 0.0809i$. There exists a positive characteristic value (1.7284) of matrix M. Based on Theorem 3.4, the trivial solution of system (4.2) is unstable. System (1.7) has a limit cycle, namely, a periodic solution (see Figures 4.5 and 4.6).

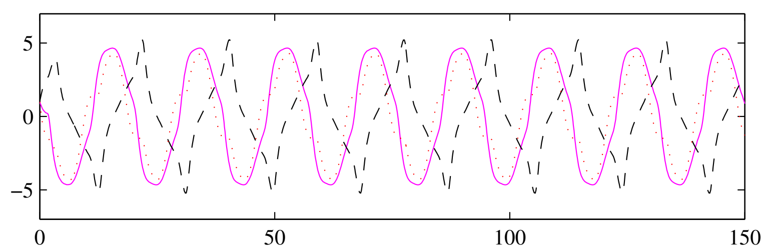
Finally we consider $n = 8$ for the following system.

Example 4.3.

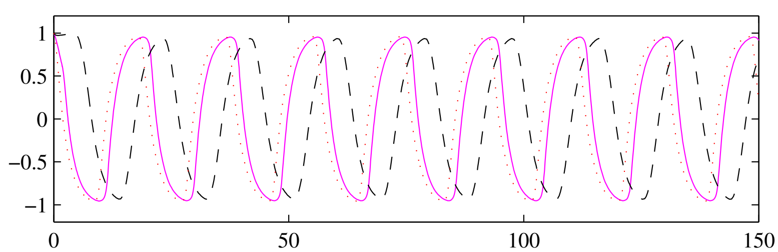
$$\begin{cases} x_1'(t) = -c_1x_1(t) + a_9x_9(t) + b_{12}f(x_2) + b_{13}f(x_3) + b_{14}f(x_4) \\ \qquad\quad +b_{15}f(x_5) + b_{16}f(x_6) + b_{17}f(x_7) + b_{18}f(x_8), \\ x_2'(t) = -c_2x_2(t) + a_{10}x_{10}(t) + b_{21}f(x_1) + b_{23}f(x_3) + b_{24}f(x_4) \\ \qquad\quad +b_{25}f(x_5) + b_{26}f(x_6) + b_{27}f(x_7) + b_{28}f(x_8), \\ x_3'(t) = -c_3x_3(t) + a_{11}x_{11}(t) + b_{31}f(x_1) + b_{32}f(x_2) + b_{34}f(x_4) \\ \qquad\quad +b_{35}f(x_5) + b_{36}f(x_6) + b_{37}f(x_7) + b_{38}f(x_8), \\ x_4'(t) = -c_4x_4(t) + a_{12}x_{12}(t) + b_{41}f(x_1) + b_{42}f(x_2) + b_{43}f(x_3) \\ \qquad\quad +b_{45}f(x_5) + b_{46}f(x_6) + b_{47}f(x_7) + b_{48}f(x_8), \\ x_5'(t) = -c_5x_5(t) + a_{13}x_{13}(t) + b_{51}f(x_1) + b_{52}f(x_2) + b_{53}f(x_3) \\ \qquad\quad +b_{54}f(x_4) + b_{56}f(x_6) + b_{57}f(x_7) + b_{58}f(x_8), \\ x_6'(t) = -c_6x_6(t) + a_{14}x_{14}(t) + b_{61}f(x_1) + b_{62}f(x_2) + b_{63}f(x_3) \\ \qquad\quad +b_{64}f(x_4) + b_{65}f(x_5) + b_{67}f(x_7) + b_{68}f(x_8), \\ x_7'(t) = -c_7x_7(t) + a_{15}x_{15}(t) + b_{71}f(x_1) + b_{72}f(x_2) + b_{73}f(x_3) \\ \qquad\quad +b_{74}f(x_4) + b_{75}f(x_5) + b_{76}f(x_6) + b_{78}f(x_8), \\ x_8'(t) = -c_8x_8(t) + a_{16}x_{16}(t) + b_{81}f(x_1) + b_{82}f(x_2) + b_{83}f(x_3) \\ \qquad\quad +b_{84}f(x_4) + b_{85}f(x_5() + b_{86}f(x_6) + b_{87}f(x_7), \\ x_9'(t) = r[f(x_1(t)) - x_9(t)], \\ x_{10}'(t) = r[f(x_2(t)) - x_{10}(t)], \\ x_{11}'(t) = r[f(x_3(t)) - x_{11}(t)], \\ x_{12}'(t) = r[f(x_4(t)) - x_{12}(t)], \\ x_{13}'(t) = r[f(x_5(t)) - x_{13}(t)], \\ x_{14}'(t) = r[f(x_6(t)) - x_{14}(t)], \\ x_{15}'(t) = r[f(x_7(t)) - x_{15}(t)], \\ x_{16}'(t) = r[f(x_8(t)) - x_{16}(t)]. \end{cases} \tag{4.3}$$

The activation functions $f(x_i) = f(x_i(t-\tau_i))$, $i = 1, 2, \cdots, 8$. We select the activation functions as $f(x_i) = 0.5\times(|x_i+1|-|x_i-1|)(i = 1, 2)$, $f(x_j) = \arctan(x_j)(j = 3, 4, 5)$, $f(x_k) = \tanh(x_k)(k = 6, 7, 8)$. Then we always have $f'(x_i) = 1(i = 1, 2, \cdots, 8)$. The parameters $r = 0.2$, $c_1 = 0.65$, $c_2 = 0.75$, $c_3 = 0.85$, $c_4 = 0.55$, $c_5 = 0.45$, $c_6 = 0.75$, $c_7 = 0.72$, $c_8 = 0.95$; $b_{12} = -0.25$, $b_{13} = 2.85$, $b_{14} = -1.5$, $b_{15} = 2.4$, $b_{16} = -4.75$, $b_{17} = 1.45$, $b_{18} = -1.25$; $b_{21} = 2.5$, $b_{23} = -7.15$, $b_{24} = -2.45$, $b_{25} = 4.85$, $b_{26} = -1.75$, $b_{27} = 1.65$, $b_{28} = -1.5$; $b_{31} = -2.6$, $b_{32} = 2.8$, $b_{34} = -3.45$, $b_{35} = -1.25$, $b_{36} = -0.85$, $b_{37} = 2.75$, $b_{38} = -4.25$, $b_{41} = 1.35$, $b_{42} = -0.2$, $b_{43} = 4.85$, $b_{45} = -0.45$, $b_{46} = -0.55$, $b_{47} = 0.45$, $b_{48} = -2.25$; $b_{51} = 0.45$, $b_{52} = 3.28$, $b_{53} = 0.85$, $b_{54} = 4.45$, $b_{56} = -0.75$, $b_{57} = -1.45$, $b_{58} = 1.25$; $b_{61} = 0.45$, $b_{62} = 0.28$, $b_{63} = 0.85$, $b_{64} = 4.45$, $b_{65} = -0.75$, $b_{67} = -1.45$, $b_{68} = 1.25$; $b_{71} = 0.45$, $b_{72} = 0.28$, $b_{73} = 0.85$, $b_{74} = -3.45$, $b_{75} = -0.75$, $b_{76} = -1.45$, $b_{78} = 1.25$; $b_{81} = 0.45$, $b_{82} = 0.28$, $b_{83} = 0.85$, $b_{84} = 6.95$, $b_{85} = -0.75$, $b_{86} = -1.45$, $b_{87} = 1.25$, $a_9 = 2$, $a_{10} = 5.5$, $a_{11} = -1.6$, $a_{12} = 2.5$, $a_{13} = -2.6$, $a_{14} = 2.4$, $a_{15} = -1.8$, $a_{16} = 5.9$. The characteristic values of matrix C are 6.2772, 4.2884, 2.0249, -4.4726, $3.4370 \pm 7.8512i$, $-1.3459 \pm 0.5648i$.

Therefore, C is not a positive definite matrix. System (4.3) thus has a unique equilibrium point which is exactly the zero point. Obviously, $\| B \| + \mu(A) > 0$. When we select time delay as $\tau_1 = 2.5$, $\tau_2 = 2.6$, $\tau_3 = 2.7$, $\tau_4 = 2.4$, $\tau_5 = 2.5$, $\tau_6 = 2.8$, $\tau_7 = 2.3$, $\tau_8 = 2.4$, and $r = 0.2$. The conditions of Theorem 3.2 are satisfied, system (4.3) generates a periodic oscillation (see Figure 4.7). When we change the value of r from 0.2 to 0.8, the oscillatory behavior still maintained (Figure 4.8). In order to inspect the effects of the time delays for the system, we increase the time delays to $\tau_1 = 3.5$, $\tau_2 = 3.6$, $\tau_3 = 3.7$, $\tau_4 = 3.4$, $\tau_5 = 3.5$, $\tau_6 = 3.8$, $\tau_7 = 3.3$, $\tau_8 = 3.4$, and decreased the time delays as $\tau_1 = 1.5$, $\tau_2 = 1.6$, $\tau_3 = 1.7$, $\tau_4 = 1.4$, $\tau_5 = 1.5$, $\tau_6 = 1.8$, $\tau_7 = 1.3$, $\tau_8 = 1.4$. We see that the oscillatory amplitude is almost the same as before. However, the oscillatory frequency is changed (see Figures 4.9 and 4.10).



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.

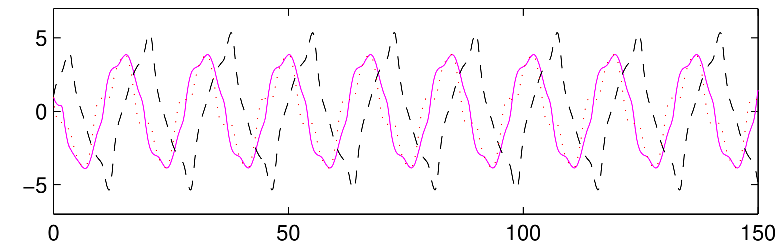


(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.

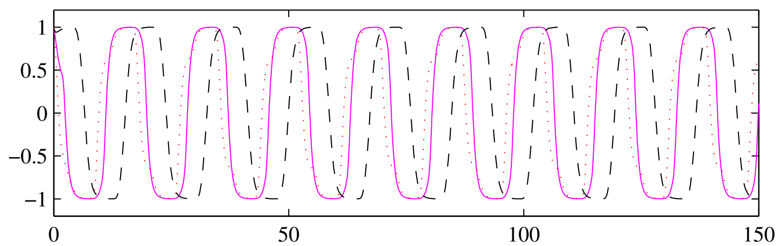
Figure 4.1: Periodic oscillation of the solutions, delays:

$$[1.5, 1.8, 1.6].$$

$r = 0.5$, activation function: $\tanh(x)$.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.

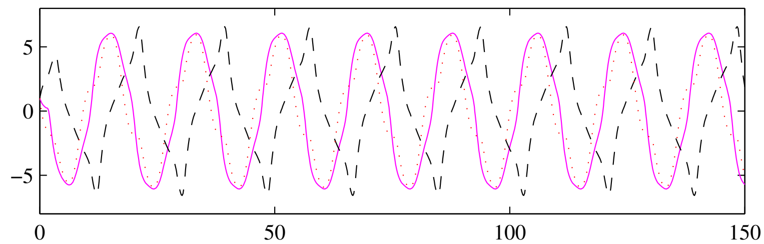


(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.

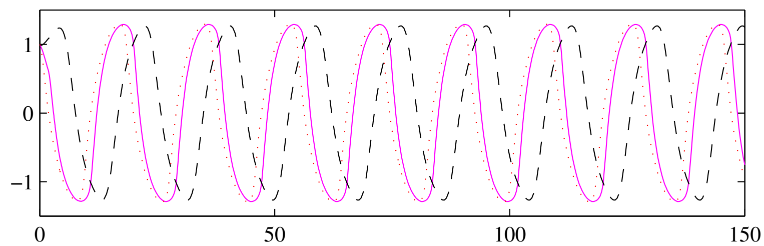
Figure 4.2: Periodic oscillation of the solutions, delays:

$$[1.5, 1.8, 1.6].$$

$r = 1.5$, activation function: $\tanh(x)$.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.

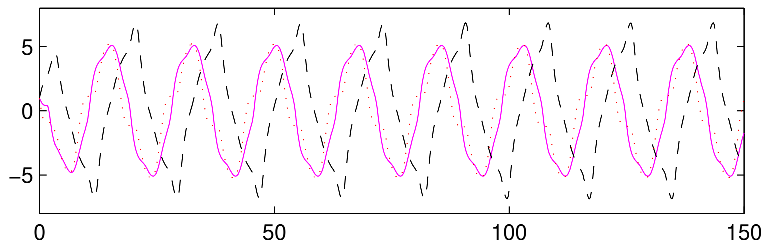


(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.

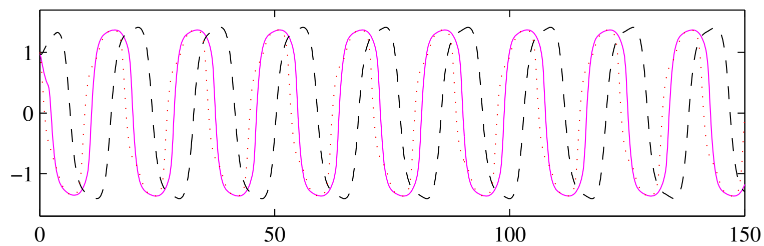
Figure 4.3: Periodic oscillation of the solutions, delays:

$$[1.5, 1.8, 1.6].$$

$r = 0.5$, activation function: $\arctan(x)$.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.



(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.

Figure 4.4: Periodic oscillation of the solutions, delays:

$$[1.5, 1.8, 1.6].$$

$r = 1.5$, activation function: $\arctan(x)$.

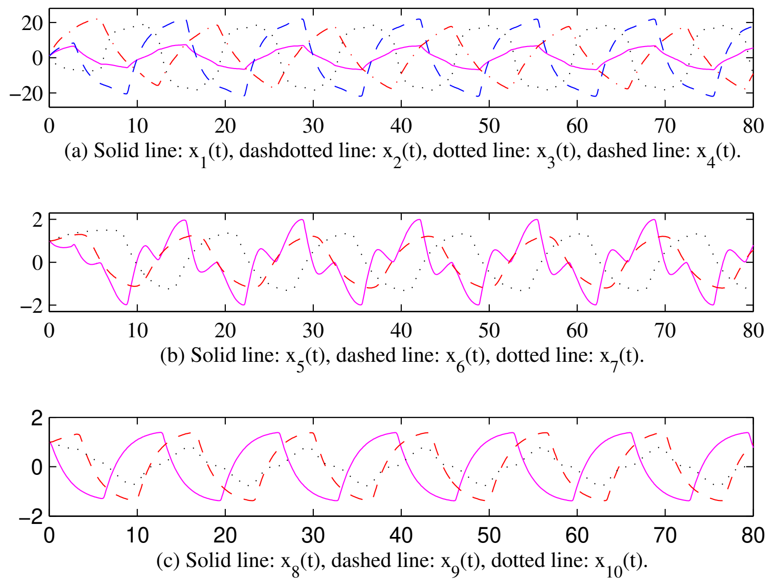


Figure 4.5: Periodic oscillation of the solutions, delays:

$$[2.5, 2.6, 2.7, 2.4, 2.5].$$

$r = 0.5$, activation function: $\arctan(x)$.

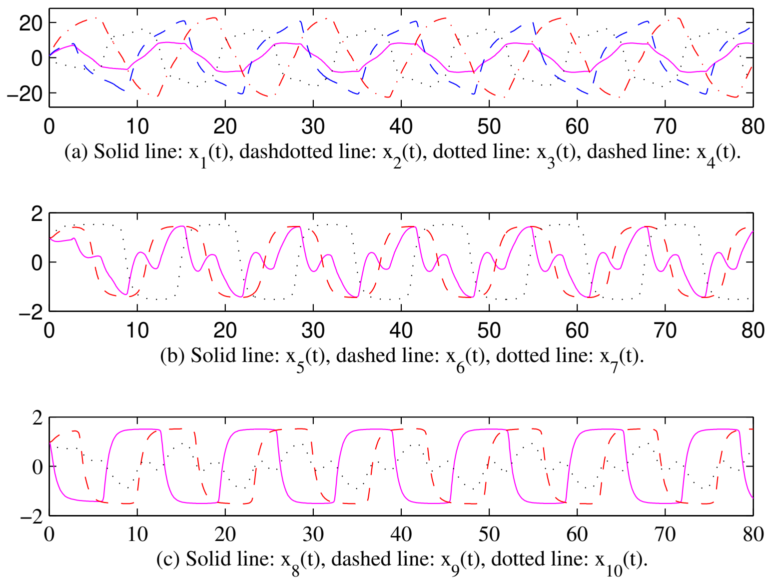
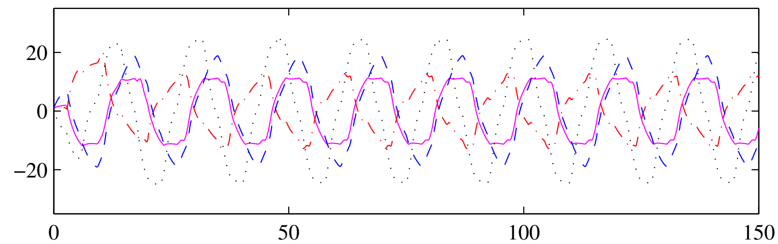


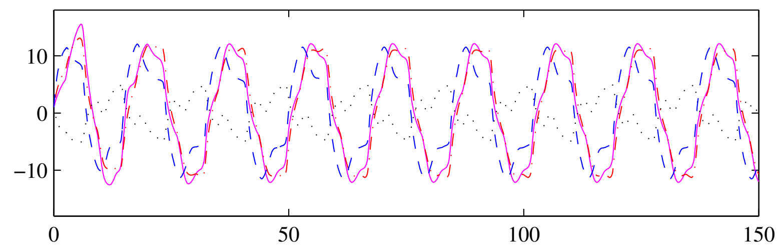
Figure 4.6: Periodic oscillation of the solutions, delays:

$$[2.5, 2.6, 2.7, 2.4, 2.5].$$

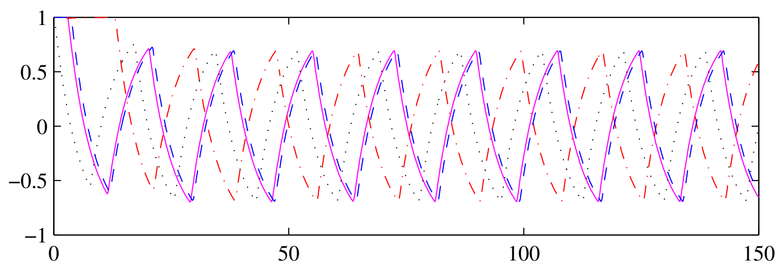
$r = 1.8$, activation function: $\arctan(x)$.



(a) Solid line: $x_1(t)$, dashdotted line: $x_2(t)$, dotted line: $x_3(t)$, dashed line: $x_4(t)$.



(b) Solid line: $x_5(t)$, dashdotted line: $x_6(t)$, dotted line: $x_7(t)$, dashed line: $x_8(t)$.



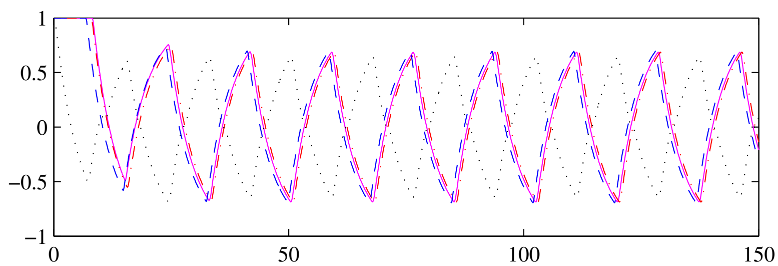


Figure 4.7: Periodic oscillation of the solutions, delays:

$$[2.5, 2.6, 2.7, 2.4, 2.5, 2.8, 2.3, 2.4].$$

$r = 0.2$, activation functions: $\arctan(x)$, $\tanh(x)$, $\dfrac{|x+1| - |x-1|}{2}$.

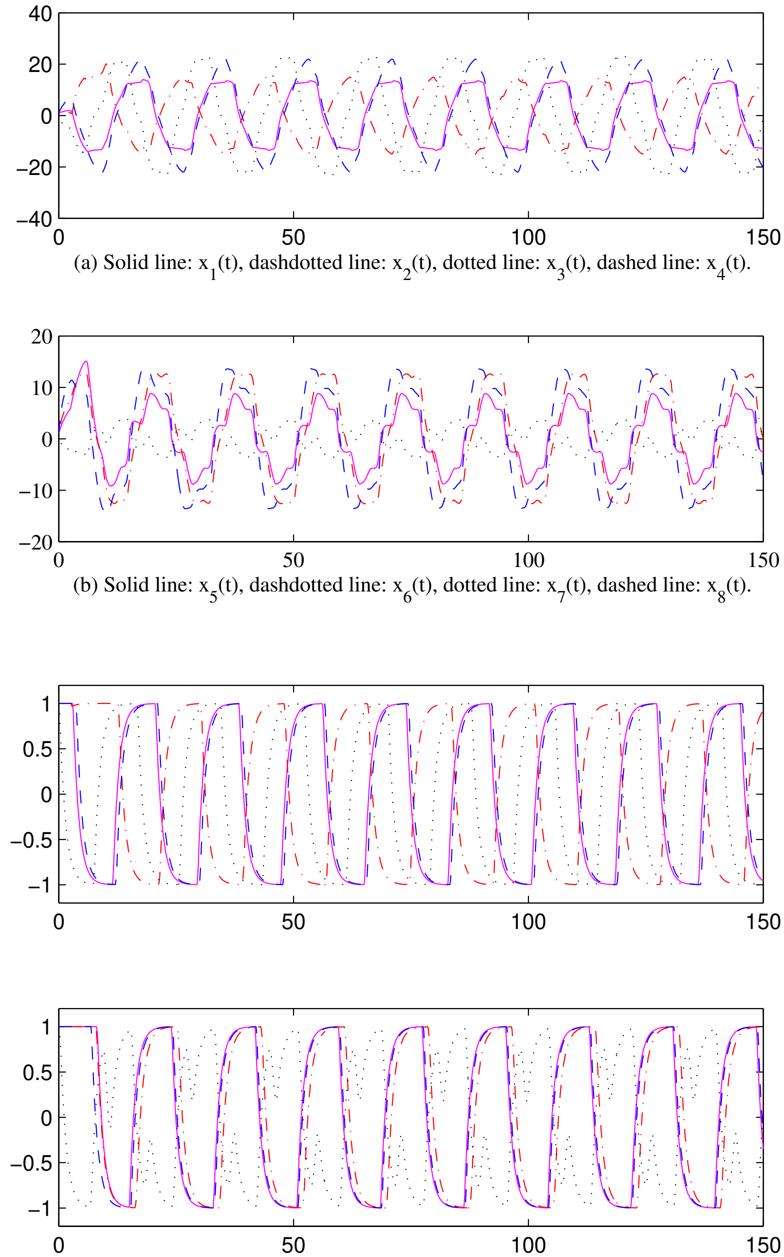
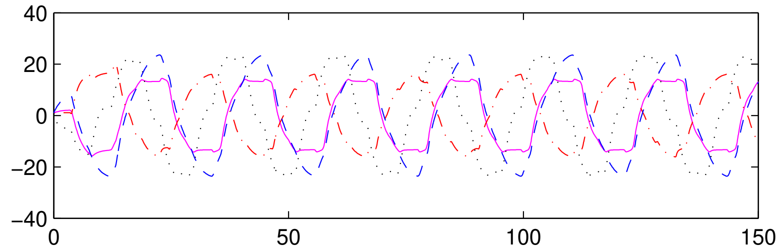


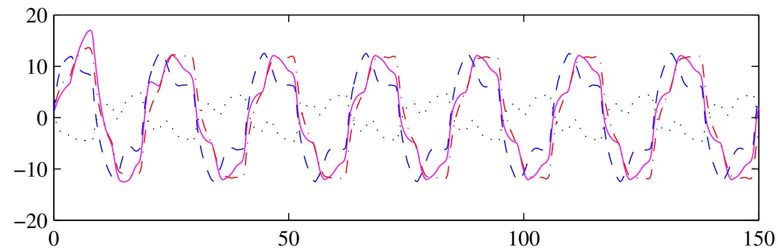
Figure 4.8: Periodic oscillation of the solutions, delays:

$$[2.5, 2.6, 2.7, 2.4, 2.5, 2.8, 2.3, 2.4].$$

$r = 0.8$, activation functions: $\arctan(x)$, $\tanh(x)$, $\dfrac{|x+1| - |x-1|}{2}$.



(a) Solid line: $x_1(t)$, dashdotted line: $x_2(t)$, dotted line: $x_3(t)$, dashed line: $x_4(t)$.

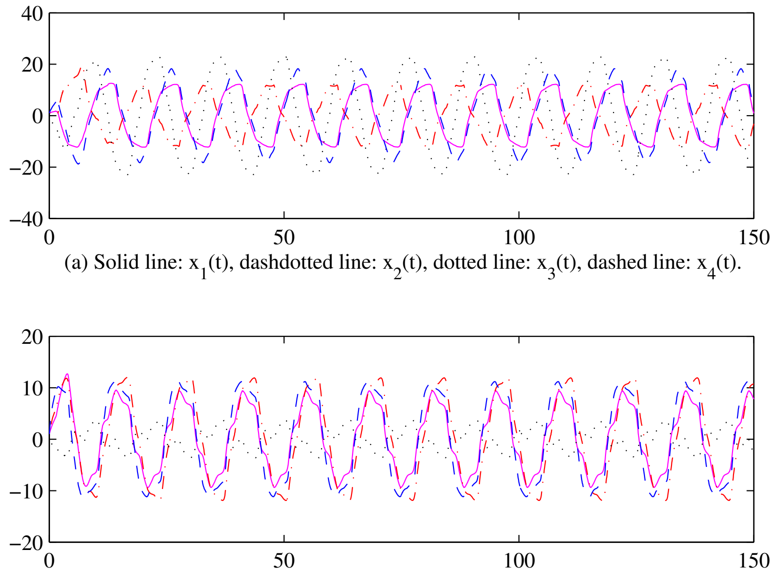


(b) Solid line: $x_5(t)$, dashdotted line: $x_6(t)$, dotted line: $x_7(t)$, dashed line: $x_8(t)$.

Figure 4.9: Periodic oscillation of the solutions, delays:

$$[3.5, 3.6, 3.7, 3.4, 3.5, 3.8, 3.3, 3.4].$$

$r = 0.8$, activation functions: $\arctan(x)$, $\tanh(x)$, $\dfrac{|x+1| - |x-1|}{2}$.



(a) Solid line: $x_1(t)$, dashdotted line: $x_2(t)$, dotted line: $x_3(t)$, dashed line: $x_4(t)$.

(b) Solid line: $x_5(t)$, dashdotted line: $x_6(t)$, dotted line: $x_7(t)$, dashed line: $x_8(t)$.

Figure 4.10: Periodic oscillation of the solutions, delays:

$$[1.5, 1.6, 1.7, 1.4, 1.5, 1.8, 1.3, 1.4].$$

$r = 0.8$, activation functions: $\arctan(x)$, $\tanh(x)$, $\dfrac{|x+1| - |x-1|}{2}$.

5 Conclusion

This paper considers a network model which includes discrete delays and distributed delay. Four theorems guarantee the existence of the permanent oscillations for the model are obtained. The selection of the parameters is provided to demonstrate the proposed results. Our simple criterion to guarantee the existence of permanent oscillations is easy to check. The simulation suggests that our criteria are only sufficient conditions.

References

[1] A. Abdurahman, H. Jiang. The existence and stability of the anti-periodic solution for delayed Cohen-Grossberg neural networks with impulsive effects, *Neurocomputing*, 149:22–28, 2015.

[2] M. Syed Ali, P. Balasubramaniam. Global asymptotic stability of stochastic fuzzy cellular neural networks with multiple discrete and distributed time-varying delays, *Commun Nonlinear Sci Numer Simulat*, 16:2907–2916, 2011.

[3] N. Chafee. A bifurcation problem for a functional differential equation of finitely retarded type, *Journal of Mathematical Analysis and Applications*, 35:312–348, 1971.

[4] Z. Cheng, D. Li, J. Cao. Stability and Hopf bifurcation of a three-layer neural network model with delays, *Neurocomputing*, 175:355–370, 2016.

[5] Y. Ding, W. Jiang, P. Yu. Bifurcation analysis in a recurrent neural network model with delays, *Commun Nonlinear Sci Numer Simulat*, 18:351–372, 2013.

[6] Y. Ding, W. Jiang, P. Yu. Bifurcation analysis in a recurrent neural network model with delays, *Commun Nonlinear Sci Numer Simulat*, 18:351–372, 2013.

[7] L. Duan, L. Huang, Z. Guo, X. Fang. Periodic attractor for reaction-diffusion high-order Hopfield neural networks with time-varying delays, *Computers and Mathematics with Applications*, 73:233–245, 2017.

[8] C. Feng, R. Plamondon. An oscillatory criterion for a time delayed neural ring network model, *Neural Networks*, 29:70–79, 2012.

[9] A. Hajihosseini, G. R. Lamooki, B. Beheshti, F. Maleki. The Hopf bifurcation analysis on a time-delayed recurrent neural network in the frequency domain, *Neurocomputing*, 73:991–1005, 2001.

[10] K. Gopalsamy. Stability and oscillations in delay different equations of population dynamics, Kluwer Academic, Dordrecht, Norwell, MA, 1992.

[11] J.K. Hale, S.M. Verduyn. Introduction to functional differential equations, New York, Springer, 1993.

[12] X. He, C.D. Li, T.W. Huang, C.J. Li. Codimension two bifurcation in a delayed neural network with unidirectional coupling, *Nonlinear Analysis: RWA*, 14:1191–1202, 2013.

[13] Y. Horikawa, H. Kitajima. Bifurcation and stabilization of oscillations in ring neural networks with inertia, *Physica D: Nonlinear Phenomena*, 238:2409–2418, 2009.

[14] E. Karaoglu, E. Yilmaz, H. Merdan. Stability and bifurcation analysis of two-neuron network with discrete and distributed delays, *Neurocomputing*, 182:102–110, 2016.

[15] X. Liao, S. Li, G. Chen. Bifurcation analysis on a two-neuron system with distributed delays in the frequency domain, *Neural Networks*, 17:545–561, 2004.

[16] X. Liu, N. Jiang. Robust stability analysis of generalized neural networks with multiple discrete delays and multiple distributed delays, *Neurocomputing*, 72:1789–1796, 2009.

[17] J. Liu, X. Liu, W. Xie. Global convergence of neural networks with mixed time-varying delays and discontinuous neuron activations, *Information Sciences*, 183:92–105, 2012.

[18] I. Ncube. Stability switching and Hopf bifurcation in a multiple-delayed neural network with distributed delay, *Journal of Mathematical Analysis and Applications*, 407:141–146, 2013.

[19] K. Ratnavelu, M. Manikandan, P. Balasubramaniam. Synchronization of fuzzy bidirectional associative memory neural networks with various time delays, *Applied Mathematics and Computation*, 270:582–605, 2015.

[20] R. Sebdani, S. Farjami. Bifircations and chaos in a discrete-time-delayed Hopfield neural network with ring structures and different internal delays, *Neurocomputing*, 99:154–162, 2013.

[21] R.M. Sebdani, S. Farjami. On a discrete-time-delayed Hopfield neural network with ring structures and different internal decays: Bifurcations analysis and chaotic behavior, *Neurocomputing*, 151:188–195, 2015.

[22] Y. Shan, S. Zhong, J. Cui, L. Li. Improved criteria of delay-dependent stability for discrete-time neural networks with leakage delay, *Neurocomputing*, 266:409–419, 2017.

[23] Q. Song, H. Shu, Z. Zhao, Y. Liu, F.E. Alsaadi. Lagrange stability analysis for complex-valued neural networks with leakage delay and mixed time-varying delays, *Neurocomputing*, 244:33–41, 2017.

[24] W. Su, Y. Chen. Global robust stability criteria of stochastic Cohen-Grossberg neural networks with discrete and distributed time-varying delays, *Commun Nonlinear Sci Numer Simulat*, 14:520–528, 2009.

[25] X. Tian, R. Xu, Q. Gan. Hopf bifurcation analysis of a BAM neural network with multiple time delays and diffusion *Applied Mathematics and Computation*, 266:909–926, 2015.

[26] C. Wang. Piecewise pseudo almost periodic solution for impulsive nonautonomous high-order Hopfield neural networks with variable delays, *Neurocomputing*, 171:1291–1301, 2016.

[27] M. Xiao, W. Zheng, J. Cao. Bifurcation and control in a neural network with small and large delays, *Neural Networks*, 44:132–142, 2013.

[28] C. Xu, Q. Zhang. Existence and global exponential stability of anti-periodic solutions of high-order bidirectional associative memory (BAM) networks with time-varying delays on time scales, *Journal of Computational Science*, 8:48–61, 2015.

[29] B. Yang, J. Wang, M. Hao, H. Zeng. Further results on passivity analysis for uncertain neural networks with discrete and distributed delays, *Information Science*, 430:77–86, 2018.

[30] X. Zeng, Z. Xiong, C. Wang. Hopf bifurcation for neutral-type neural network model with two delays *Applied Mathematics and Computation*, 282:17–31, 2016.

[31] G. Zhang, Z. Zeng, J. Hu. New results on global exponential dissipativity analysis of memristive inertial neural networks with distributed time-varying delays, *Neural Networks*, 97:183–191, 2018.

[32] H. Zhao, L. Wang, C. Ma. Hopf bifurcation and stability analysis on discrete-time Hopfield neural network with delay, *Nonlinear Analysis: RWA*, 9:103–113, 2008.

[33] W. Zhao. New results of existence and stability of periodic solution for a delay multispecies Logarithmic population model, *Nonlinear Analysis: RWA*, 10:544–533, 2009.

[34] B. Zhou, Q. Song. Stability and Hopf bifurcation analysis of a tri-neuron BAM neural network with distributed delay, *Neurocomputing*, 82:69–83, 2012.

[35] Q. Zhou. Global exponential stability of BAM neural networks with distributed delays and impulses, *Nonlinear Analysis: RWA*, 10:144–153, 2007.

[36] X. Zhou, Y. Wu, Y. Li, X. Yao. Stability and Hopf bifurcation analysis on a two-neuron network with discrete and distributed delays, *Chaos, Solitons and Fractals*, 40:1493–1505, 2009.