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On a Sturm–Liouville Type Functional Differential Inclusion with "Maxima"

Aurelian Cernea
University of Bucharest
Faculty of Mathematics and Computer Science
Bucharest, 010014, Romania
Academy of Romanian Scientists
Bucharest, 050094, Romania
acernea@fmi.unibuc.ro

Abstract

We study a Sturm–Liouville type functional differential inclusion with "maxima" and with boundary conditions of mixed type. Some existence results are obtained for this problem by using suitable fixed point theorems.

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1 Introduction

Differential equations with maximum have proved to be strong tools in the modelling of many physical problems: systems with automatic regulation, problems in control theory that correspond to the maximal deviation of the regulated quantity etc.. As a consequence there was an intensive development of the theory of differential equations with "maxima" [1, 8, 9, 12–15, 18] etc..

A classical example is the one of an electric generator ([1]). In this case the mechanism becomes active when the maximum voltage variation is reached in an interval of time. The equation describing the action of the regulator has the form

$$x'(t) = ax(t) + b \max_{s\in[t-h,t]} x(s) + f(t),$$

where a, b are constants given by the system, $x(\cdot)$ is the voltage and $f(\cdot)$ is a perturbation given by the change of voltage.

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In the theory of ordinary differential equations it is well-known that any linear real second-order differential equation may be written in the self adjoint form $-(r(t)x')' + q(t)x = 0$. This equation together with boundary conditions of the form $a_1x(0) - a_2x'(0) = 0$, $b_1x(T) - b_2x'(T) = 0$ is called the Sturm–Liouville problem. This is the reason why differential inclusions of the form $(r(t)x')' \in F(t,x)$ are usually called Sturm-Liouville type differential inclusions, even if the boundary value problems associated are not as at the original Sturm–Liouville problem. Recent results on Sturm–Liouville differential inclusions may be found in [11].

This paper is devoted to the study of second-order functional differential inclusions of the form

$$(p(t)x'(t))' \in F(t, x(t), \max_{s\in[t-h_1,t]} x(s), \max_{s\in[t,t+h_2]} x(s)) \quad a.e.\ ([a,b]), \tag{1.1}$$

with “boundary conditions” of mixed type

$$x(t) = \alpha(t), \quad t \in [a-h_1, a], \quad x(t) = \beta(t), \quad t \in [b, b+h_2], \tag{1.2}$$

where $h_1, h_2 > 0$ are given, $p(.) : [a,b] \to \mathbb{R}$, $\alpha(.) : [a-h_1, a] \to \mathbb{R}$, $\beta(.) : [b, b+h_2] \to \mathbb{R}$ are continuous mappings and $F : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map.

The aim of the present paper is to present several existence results for problem (1.1)–(1.2) when the right hand side has convex or non convex values. Our results are essentially based on a nonlinear alternative of Leray–Schauder type, on Bressan–Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. The methods used are known in the theory of differential inclusions, however their exposition in the framework of problem (1.1)–(1.2) is new. The results of the present paper may be regarded as extensions of our previous results obtained in [5] for second-order differential inclusions (i.e., $p(t) \equiv 1$) to the more general problem (1.1). Similar results for fractional differential inclusions are established in [4].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space with the corresponding norm $|.|$ and let $I \subset \mathbb{R}$ be a compact interval. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_I |x(t)|\mathrm{d}t$.

A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u,v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for T if $x \in T(x)$. T is said to be bounded on bounded sets if $T(B) := \cup_{x\in B}T(x)$ is a bounded subset of X for all bounded sets B in X. T is said to be compact if $\overline{T(B)}$ is relatively compact for any bounded sets B in X. T is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. T is said to be upper semicontinuous if for any $x_0 \in X, T(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $T(x_0)$ there exists an open neighborhood V_0 of x_0 such that $T(V_0) \subset D$. Let E a Banach space, $Y \subset E$ a nonempty closed subset and $T : Y \to \mathcal{P}(E)$ a multifunction with nonempty closed values. T is said to be lower semicontinuous if for any open subset $D \subset E$, the set $\{y \in Y; T(y) \cap D \neq \emptyset\}$ is open. T is called completely continuous if it is upper semicontinuous and totally compact on X.

It is well known that a compact set-valued map T with nonempty compact values is upper semicontinuous if and only if T has a closed graph.

We recall the following nonlinear alternative of Leray–Schauder type and its consequences (e.g., [16]).

Theorem 2.1. *Let D and $\overline{D}$ be open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.3. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

i) the equation $x = T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

We recall that a multifunction $T : X \to \mathcal{P}(X)$ is said to be lower semicontinuous if for any closed subset $C \subset X$, the subset $\{s \in X : T(s) \subset C\}$ is closed.

If $F : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map with compact values and $x \in C([a-h_1, b+h_2], \mathbb{R})$ we define

$$S_F(x) = \{f \in L^1([a,b],\mathbb{R}); f(t) \in F(t, x(t), \max_{s\in[t-h_1,t]} x(s), \max_{s\in[t,t+h_2]} x(s))\, a.e.([a,b])\}.$$

We say that F is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with closed and decomposable values. The next result is proved in [2].

Theorem 2.4. *Let S be a separable metric space and $G : S \to \mathcal{P}(L^1(I,\mathbb{R}))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then G has a continuous selection (i.e., there exists a continuous mapping $g : S \to L^1(I,\mathbb{R})$ such that $g(s) \in G(s) \quad \forall s \in S$).

A set-valued map $G : I \to \mathcal{P}(\mathbb{R}^n)$ with nonempty compact convex values is said to be *measurable* if for any $x \in \mathbb{R}^n$ the function $t \to d(x, G(t))$ is measurable. A set-valued map $F : I \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ is said to be *Carathéodory* if $t \to F(t,x)$ is measurable for all $x \in \mathbb{R}^n$ and $x \to F(t,x)$ is upper semicontinuous for almost all $t \in I$. F is said to be L^1*–Carathéodory* if for any $l > 0$ there exists $h_l \in L^1(I,\mathbb{R})$ such that $\sup\{|v| : v \in F(t,x)\} \leq h_l(t)$ a.e. I, $\forall x \in \overline{B_l(0)}$. The proof of the next result may be found in [10].

Theorem 2.5. *Let X be a Banach space, let $F : I \times X \to \mathcal{P}(X)$ be a L^1–Carathéodory set-valued map with $S_F \neq \emptyset$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I,X) \to \mathcal{P}(C(I,X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.

Note that if $dimX < \infty$, and F is as in Theorem 2.5, then $S_F(x) \neq \emptyset$ for any $x \in C(I,X)$ (e.g., [10]).

Consider a set valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that

$$d_H(T(x), T(y)) \leq \lambda d(x,y) \quad \forall x, y \in X.$$

The set-valued contraction principle [6] states that if X is complete, and $T : X \to \mathcal{P}(X)$ is a set valued contraction with nonempty closed values, then T has a fixed point, i.e., a point $z \in X$ such that $z \in T(z)$.

Let $I(\cdot) : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ a set-valued map with compact convex values defined by $I(t) = [a(t), b(t)]$, where $a(\cdot), b(\cdot) : \mathbb{R} \to \mathbb{R}$ are continuous functions with $a(t) \leq b(t)$ $\forall t \in \mathbb{R}$. For $x(.) : \mathbb{R} \to \mathbb{R}$ continuous we define $(\max_I)(t) = \max_{s\in I(t)} x(s)$. $\max_I : C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ is an operator whose properties are summarized in the next lemma proved in [15].

Lemma 2.6. *If* $x(\cdot), y(\cdot) \in C(\mathbb{R}, \mathbb{R})$*, then one has*

i) $|\max_{s \in I(t)} x(s) - \max_{s \in I(t)} y(s)| \leq \max_{s \in I(t)} |x(s) - y(s)| \ \forall t \in \mathbb{R}.$

ii) $\max_{t \in K} |\max_{s \in I(t)} x(s) - \max_{s \in I(t)} y(s)| \leq \max_{s \in \cup_{t \in K} I(t)} |x(s) - y(s)| \ \forall t \in \mathbb{R}.$

Remark 2.7. We recall that if $f \in L^1([a, b], \mathbb{R})$ then the solution $x \in C([a - h_1, b + h_2], \mathbb{R}) \cap C^2([a, b], \mathbb{R})$ of problem $(p(t)x'(t))' = f(t), \ t \in [a, b]$ with boundary conditions (1.2) is given by

$$x(t) = \begin{cases} \alpha(t), \quad t \in [a - h_1, a], \\ P(t) - \displaystyle\int_a^b G(t, s) f(s) ds, \quad t \in [a, b], \\ \beta(t), \quad t \in [b, b + h_2], \end{cases}$$

where $S(t, \sigma) := \displaystyle\int_\sigma^t \frac{1}{p(s)} ds$, $t, \sigma \in [a, b]$, $P(t) = \dfrac{S(t, a)}{S(b, a)}(\beta(b) - \alpha(a))$, $G(t, \sigma) = \dfrac{S(t, a)S(b, \sigma) - S(t, \sigma)S(b, a)\chi_{[0,t]}(\sigma)}{S(b, a)}$ and $\chi_U(\cdot)$ is the characteristic function of the set U.

In what follows we assume that $p : [a, b] \to (0, \infty)$ is a continuous function such that $|S(t, \sigma)| \leq m_0 \ \forall t, \sigma \in [a, b]$. Denote $m_1 := \sup_{t \in [a,b]} |P(t)|$ and $M_1 := \sup_{t, \sigma \in [a,b]} |G(t, \sigma)|$.

3 The main results

In what follows $I = [a, b]$ and the Banach space $C([a - h_1, b + h_2], \mathbb{R})$ is endowed with Chebyshev norm $||x(\cdot)|| = \sup_{t \in [a - h_1, b + h_2]} |x(t)|$.

We are able now to present the existence results for problem (1.1)-(1.2). We consider first the case when F is convex valued.

Hypothesis H1. i) $F : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact convex values and is Carathéodory.*

ii) *There exist* $\varphi \in L^1(I, \mathbb{R})$ *with* $\varphi(t) > 0$ *a.e.* I *and there exists a nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *such that*

$$\sup\{|v|, \ \ v \in F(t, x, y, z)\} \leq \varphi(t)\psi(\max\{|x|, |y|, |z|\}) \quad a.e. \ I, \ \forall x, y, z \in \mathbb{R}.$$

Theorem 3.1. *Assume that Hypothesis H1 is satisfied and there exists* $r > 0$ *such that*

$$r > m_1 + M_1 |\varphi|_1 \psi(r). \tag{3.1}$$

Then problem (1.1)–(1.2) *has at least one solution* x *such that* $||x|| < r$.

Proof. Let $X = C([a-h_1, b+h_2], \mathbb{R})$ and consider $r > 0$ as in (3.1) . It is obvious that the existence of solutions to problem (1.1)–(1.2) reduces to the existence of the solutions of the integral inclusion

$$\begin{aligned} &x(t) = \alpha(t), \quad t \in [a-h_1, a], \\ &x(t) \in P(t) - \int_a^b G(t,s)F(s, x(s), \max_{\sigma\in[s-h_1,s]} x(\sigma), \max_{\sigma\in[s,s+h_2]} x(\sigma))ds, \\ &x(t) = \beta(t), \quad t \in [b, b+h_2] \end{aligned} \tag{3.2}$$

Consider the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, \mathbb{R}))$ defined by

$$T(x) := \{v \in C(I, \mathbb{R});\ v(t) = P(t) - \int_a^b G(t,s)f(s)ds, \quad f \in S_F(x)\}. \tag{3.3}$$

We show that T satisfies the hypotheses of Corollary 2.2.

First, we show that $T(x) \subset C(I, \mathbb{R})$ is convex for any $x \in C(I, \mathbb{R})$. If $v_1, v_2 \in T(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has

$$v_i(t) = P(t) - \int_a^b G(t,s)f_i(s)ds, \quad i = 1, 2.$$

Let $0 \le \alpha \le 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1-\alpha)v_2)(t) = P(t) - \int_a^t G(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds.$$

The values of F are convex, thus $S_F(x)$ is a convex set and hence $\alpha v_1 + (1-\alpha)v_2 \in T(x)$.

Secondly, we show that T is bounded on bounded sets of $C(I, \mathbb{R})$. Let $B \subset C(I, \mathbb{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \le m\ \forall x \in B$. If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = P(t) - \int_a^b G(t,s)f(s)ds$. One may write for any $t \in I$

$$\begin{aligned} |v(t)| &\le m_1 + M_1 \int_a^b |f(s)|ds \\ &\le m_1 + M_1 \int_a^b \varphi(s)\psi(\max\{|x(s)|, |\max_{\sigma\in[s-h_1,s]} x(\sigma)|, |\max_{\sigma\in[s,s+h_2]} x(\sigma)|\})ds \\ &\le m_1 + M_1 \int_a^b \varphi(s)\psi(\max\{|x(s)|, \max_{\sigma\in[s-h_1,s]} |x(\sigma)|, \max_{\sigma\in[s,s+h_2]} |x(\sigma)|\})ds \\ &\le m_1 + M_1 \int_a^b \varphi(s)\psi(|x|_C))ds \le m_1 + M_1|\varphi|_1\psi(m). \end{aligned}$$

and therefore

$$|v|_C \le m_1 + M_1|\varphi|_1\psi(m)$$

$\forall v \in T(x)$, i.e., $T(B)$ is bounded.

We show next that T maps bounded sets into equi-continuous sets. Let $B \subset C(I, \mathbb{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = P(t) - \int_a^b G(t,s)f(s)ds$. Then for any $t, \tau \in I$ we have

$$\begin{aligned} |v(t) - v(\tau)| &\leq |\int_a^b G(t,s)f(s)ds - \int_a^b G(\tau,s)f(s)ds| \\ &\leq \int_a^b |G(t,s) - G(\tau,s)|\varphi(s)\psi(m)ds. \end{aligned}$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equi-continuous set in $C(I, \mathbb{R})$. We apply now Arzela–Ascoli's theorem we deduce that T is completely continuous on $C(I, \mathbb{R})$.

In the next step of the proof we prove that T has a closed graph. Let $x_n \in C(I, \mathbb{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)$ $\forall n \in \mathbb{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$. Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = P(t) - \int_a^b G(t,s)f_n(s)ds$. Define $\Gamma : L^1(I, \mathbb{R}) \to C(I, \mathbb{R})$ by $(\Gamma(f))(t) := -\int_a^b G(t,s)f(s)ds$. One has $|v_n(\cdot) - P(\cdot) - (v^*(\cdot) - P(\cdot))|_C = |v_n - v^*|_C \to 0$ as $n \to \infty$.

We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) - P(t) = -\int_a^b G(t,s)f^*(s)ds$. Therefore, T is upper semicontinuous and compact on $\overline{B_r(0)}$.

We apply Corollary 2.2 to deduce that either i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x|_C \leq m_1 + M_1|\varphi|_1\psi(r)$ which contradicts (3.1). Hence only i) is valid and theorem is proved. □

We consider now the case when F is not necessarily convex valued. Our first existence result in this case is based on the Leray–Schauder alternative for single valued maps and on Bressan–Colombo selection theorem.

Hypothesis H2. i) $F : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has compact values,* F *is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ *measurable and* $(x, y, z) \to F(t, x, y, z)$ *is lower semicontinuous for almost all* $t \in I$.

ii) *There exist* $\varphi \in L^1(I, \mathbb{R})$ *with* $\varphi(t) > 0$ *a.e.* I *and there exists a nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *such that*

$$\sup\{|v|, \ v \in F(t, x, y, z)\} \leq \varphi(t)\psi(\max\{|x|, |y|, |z|\}) \quad a.e.\ I, \quad \forall x, y \in \mathbb{R}.$$

Theorem 3.2. *Assume that Hypothesis H2 is satisfied and there exists* $r > 0$ *such that condition* (3.1) *is satisfied.*

Then problem (1.1)–(1.2) *has at least one solution.*

Proof. We note first that if Hypothesis H2 is satisfied then F is of lower semicontinuous type (e.g., [6]). Therefore, we apply Theorem 2.4 to deduce that there exists $f : C(I, \mathbb{R}) \to L^1(I, \mathbb{R})$ such that $f(x) \in S_F(x)\ \forall x \in C(I, \mathbb{R})$.

We consider the corresponding problem

$$\begin{aligned} &x(t) = \alpha(t), \quad t \in [a - h_1, a],\\ &x(t) \in P(t) - \int_a^b G(t,s)f(x(s))ds, \quad t \in I, \\ &x(t) = \beta(t), \quad t \in [b, b + h_2] \end{aligned} \tag{3.4}$$

It is clear that if $x \in C([a - h_1, b + h_2], \mathbb{R})$ is a solution of the problem (3.4) then x is a solution to problem (1.1)–(1.2) .

Let $r > 0$ that satisfies condition (3.1) and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(C(I, \mathbb{R}))$ by

$$(T(x))(t) := P(t) - \int_a^b G(t,s)f(x(s))ds.$$

Obviously, the integral equation (3.4) is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I. \tag{3.5}$$

It remains to show that T satisfies the hypotheses of Corollary 2.3.

We show that T is continuous on $\overline{B_r(0)}$. From Hypotheses 3.3. ii) we have

$$|f(x(t))| \le \varphi(t)\psi(\max\{|x(t)|, |\max_{s\in[t-h_1,t]} x(s)|, |\max_{s\in[t,t+h_2]} x(s)|\}) \quad a.e.\ (I)$$

for all $x \in C(I, \mathbb{R})$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \le \varphi(t)\psi(r) \quad a.e.\ (I).$$

From Lebesgue's dominated convergence theorem and the continuity of f we obtain, for all $t \in I$

$$\begin{aligned} \lim_{n\to\infty}(T(x_n))(t) &= P(t) - \lim_{n\to\infty}\int_a^b G(t,s)f(x_n(s))ds \\ &= P(t) - \int_a^b G(t,s)f(x(s))ds = (T(x))(t), \end{aligned}$$

i.e., T is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.1 with corresponding modifications it follows that T is compact on $\overline{B_r(0)}$. We apply Corollary 2.3 and we find that either i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.1 if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus only the statement i) is true and problem (1.1)–(1.2) has a solution with $|x|_C < r$. □

In order to obtain an existence result for problem (1.1)–(1.2) by using the set-valued contraction principle we introduce the following hypothesis on F.

Hypothesis H3. i) $F : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact values, is integrably bounded and for every* $x, y, z \in \mathbb{R}$, $F(\cdot, x, y, z)$ *is measurable.*

ii) *There exists* $l_1, l_2, l_3 \in L^1(I, \mathbb{R}_+)$ *such that for almost all* $t \in I$,

$$d_H(F(t, x_1, y_1, z_1), F(t, x_2, y_2, z_2)) \leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2| + l_3(t)|z_1 - z_2|$$

$\forall \ x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$.

iii) *There exists* $l \in L^1(I, \mathbb{R}_+)$ *such that for almost all* $t \in I$, $d(0, F(t, 0, 0, 0)) \leq l(t)$.

Theorem 3.3. *Assume that Hypothesis H3 is satisfied and* $M_1(|l_1|_1 + |l_2|_1 + |l_3|_1) < 1$. *Then problem* (1.1)–(1.2) *has a solution.*

Proof. We transform the problem (1.1)–(1.2) into a fixed point problem. Consider the set-valued map $T : C([a - h_1, b + h_2], \mathbb{R}) \to \mathcal{P}(C([a - h_1, b + h_2], \mathbb{R}))$ defined by

$$T(x) := \begin{cases} x(t), & \text{if } t \in [a - h_1, a], \\ x(t) \in P(t) - \displaystyle\int_a^b G(t, s) f(s) ds, \quad f \in S_F(x), & \text{if } t \in I, \\ x(t), & \text{if } t \in [b, b + h_2]. \end{cases}$$

Since the set-valued map $t \to F(t, x(t), \max\limits_{s \in [t - h_1, t]} x(s), \max\limits_{s \in [t, t + h_2]} x(s))$ is measurable with the measurable selection theorem (e.g., [3, Theorem III. 6]), it admits a measurable selection $f : I \to \mathbb{R}$. Moreover, since F is integrably bounded, $f \in L^1(I, \mathbb{R})$. Therefore, $S_F(x) \neq \emptyset$.

It is clear that the fixed points of T are solutions of problem (1.1)–(1.2). We shall prove that T fulfills the assumptions of Covitz–Nadler contraction principle.

First, we note that since $S_F(x) \neq \emptyset$, $T(x) \neq \emptyset$ for any $x \in C([a - h_1, b + h_2], \mathbb{R})$.

Secondly, we prove that $T(x)$ is closed for any $x \in C([a - h_1, b + h_2], \mathbb{R})$. Let $\{x_n\}_{n \geq 0} \in T(x)$ such that $x_n \to x^*$ in $C([a - h_1, b + h_2], \mathbb{R})$. Then $x^* \in C([a - h_1, b + h_2], \mathbb{R})$ and there exists $f_n \in S_F(x)$ such that

$$x_n(t) = P(t) - \int_a^b G(t, s) f_n(s) ds, \quad t \in I.$$

Since F has compact values and Hypothesis H3 is satisfied, we may pass to a subsequence (if necessary) to get that f_n converges to $f \in L^1(I, \mathbb{R})$ in $L^1(I, \mathbb{R})$. In particular, $f \in S_F(x)$ and for any $t \in I$ we have

$$x_n(t) \to x^*(t) = P(t) - \int_a^b G(t,s)f(s)ds,$$

i.e., $x^* \in T(x)$ and $T(x)$ is closed.

Finally, we show that T is a contraction on $C([a-h_1, b+h_2], \mathbb{R})$. Let $x_1, x_2 \in C([a-h_1, b+h_2], \mathbb{R})$ and $v_1 \in T(x_1)$. Then there exist $f_1 \in S_F(x_1)$ such that

$$v_1(t) = P(t) - \int_a^b G(t,s)f_1(s)ds, \quad t \in I.$$

For $t \in I$, we define the set-valued map

$$\begin{aligned} H(t) \quad &:= \quad F(t, x_2(t), \max_{s\in[t-h_1,t]} x_2(s), \max_{s\in[t,t+h_2]} x_2(s)) \cap \{x \in \mathbb{R}; |f_1(t) - x|; \\ &\leq \quad l_1(t)|x_1(t) - x_2(t)| + l_2(t)|\max_{s\in[t-h_1,t]} x_1(s) - \max_{s\in[t-h_1,t]} x_2(s)| + \\ &+ \quad l_3(t)|\max_{s\in[t,t+h_2]} x_1(s) - \max_{s\in[t,t+h_2]} x_2(s)|\}. \end{aligned}$$

From Hypothesis H3, one has

$$\begin{aligned} &d_H(F(t, x_1(t), \max_{s\in[t-h,t]} x_1(s), \max_{s\in[t,t+h_2]} x_1(s)), F(t, x_2(t), \max_{s\in[t-h,t]} x_2(s)|, \\ &\max_{s\in[t,t+h_2]} x_2(s))) \leq l_1(t)|x_1(t) - x_2(t)| + l_2(t)|\max_{s\in[t-h,t]} x_1(s) - \max_{s\in[t-h,t]} x_2(s)| + \\ &l_3(t)|\max_{s\in[t,t+h_2]} x_1(s) - \max_{s\in[t,t+h_2]} x_2(s)|, \end{aligned}$$

hence H has nonempty closed values. Moreover, since H is measurable, there exists f_2 a measurable selection of H. It follows that $f_2 \in S_F(x_2)$ and for any $t \in I$

$$\begin{aligned} |f_1(t) - f_2(t)| \quad &\leq \quad l_1(t)|x_1(t) - x_2(t)| + l_2(t)|\max_{s\in[t-h_1,t]} x_1(s) - \max_{s\in[t-h_1,t]} x_2(s)| \\ &+ \quad l_3(t)|\max_{s\in[t,t+h_2]} x_1(s) - \max_{s\in[t,t+h_2]} x_2(s)|. \end{aligned}$$

Define

$$v_2(t) = P(t) - \int_a^b G(t,s)f_2(s)ds, \quad t \in I$$

Using Lemma 2.6 we have

$$\begin{aligned} |v_1(t) - v_2(t)| \quad &\leq \quad M_1 \int_a^t |f_1(s) - f_2(s)|ds \leq \int_a^t [l_1(s)|x_1(s) - x_2(s)| \\ &+ \quad l_2(s)|\max_{s\in[t-h_1,t]} x_1(s) - \max_{s\in[t-h_1,t]} x_2(s)| + l_3(s)|\max_{s\in[t,t+h_2]} x_1(s) \end{aligned}$$

$$- \max_{s\in[t,t+h_2]} x_2(s)|]ds \leq M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)|x_1 - x_2|_C.$$

So, $|v_1 - v_2|_C \leq M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)|x_1 - x_2|_C$. From an analogous reasoning by interchanging the roles of x_1 and x_2, it follows

$$d_H(T(x_1), T(x_2)) \leq M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)|x_1 - x_2|_C.$$

Therefore, T admits a fixed point which is a solution to problem (1.1)-(1.2). □

Remark 3.4. If $p(t) \equiv 1$, then (1.1) reduces to

$$x''(t) \in F(t, x(t), \max_{s\in[t-h_1,t]} x(s), \max_{s\in[t,t+h_2]} x(s)) \quad a.e. \ ([a,b]). \tag{3.6}$$

Similar results, as the ones in Theorems 3.1, 3.2 and 3.3, for problem (3.6)–(1.2) may be found in [5].

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