Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 13, Number 2, pp. 91–99 (2018)
http://campus.mst.edu/adsa

On Monotonic Solutions of a Cubic Urysohn Integral Equation with Linear Modification of the Argument

Hamed Kamal Awad
Damanhour University
Faculty of Science
Department of Mathematics
Damanhour, Egypt
hamedk66@sci.dmu.edu.eg

Mohamed Abdalla Darwish
Damanhour University
Faculty of Science
Department of Mathematics
Damanhour, Egypt
madarwish@sci.dmu.edu.eg

Abstract

In the space of real and continuous functions on the interval $[0, 1]$, we establish the existence of nondecreasing solutions to a cubic Urysohn integral equation with linear modification of the argument. To achieve our goal, we use the concept of a measure of noncompactness related to monotonicity which introduced by J. Banaś and L. Olszowy, and Darbo's fixed point theorem.

AMS Subject Classifications: 45G10, 47H09, 45M99.
Keywords: Cubic integral equation, Urysohn, measure of noncompactness.

1 Introduction

Integral equations are encountered in a variety of applications in many disciplines of engineering and sciences applications. Several real world phenomena in different areas of science for example mechanics, physics, biology, electricity, etc. are studied and modeled with nonlinear integral equations. Therefore, the study of existence of solutions to

Received August 8, 2018; Accepted October 11, 2018
Communicated by Mehmet Ünal

nonlinear integral equations is an essential area of scientific inquiry [8, 11, 17]. On the other hand, the study of differential and integral equations with a modified argument is arise in the modeling of many problems from the natural and social sciences such as biology, physics and economics, see [1, 4–6, 10, 12–16] and references therein.

In this paper, we will study the cubic integral equation of Urysohn type with linear modification of the argument

$$x(t) = f(t, x(t)) + x^2(t) \int_0^1 u(t, s, x(s), x(\lambda s))\, ds, \tag{1.1}$$

where $t \in I = [0, 1]$ and $0 < \lambda < 1$.

We establish the existence of solutions of Eq.(1.1) in the space of real functions, defined and continuous on a bounded interval. Our proof depends on suitable combination of the technique of measures of noncompactness and the Darbo fixed point principle. Mainely, our result extend the result obtained by J. Caballero et al [7]. It is worthwhile mentioning that up to now the work of J. Caballero et al [7] is the only paper concerning with the study of cubic integral equation of Urysohn type in the space of real functions, defined and continuous on a bounded interval.

2 Auxiliary Facts and Results

First, let us denote by $(E, \|\cdot\|)$ a real Banach space with a zero element 0 and we denote by $B(x, r)$ the closed ball of radius r and center x. Also, we denote by B_r the closed ball $B(0, r)$. Second, let $X \neq \emptyset$ be a subset of E and denote by $\overline{X}$ and $\mathrm{Conv}X$ the closure and convex closure of the set X, respectively. Let $X + Y$ and λX, $\lambda \in \mathbb{R}$, denote the usual algebraic operations on X and Y. Moreover, let the symbol $\mathfrak{M}_E$ stands for the family of all nonempty and bounded subsets of E and let the symbol $\mathfrak{N}_E$ stands for the subfamily of $\mathfrak{M}_E$ consisting of all relatively compact subsets of E.

Now, we state the definition a measure of noncompactness [2].

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is called a measure of noncompactness in E if the following conditions is verified.

1° $\emptyset \neq \ker\mu \subset \mathfrak{N}_E$, where $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ stands for the kernel of a measure of noncompactness μ.

2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)\ \forall X, Y \in \mathfrak{M}_E$.

3° $\mu(\overline{X}) = \mu(X) = \mu(\mathrm{Conv}X)$.

4° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)\ \forall X, Y \in \mathfrak{M}_E;\ \lambda \in [0, 1]$.

5° If (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ such that $\lim\limits_{n\to\infty} \mu(X_n) = 0$ and $X_{n+1} \subset X_n$ then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

Further, we denote by $C(I)$ the Banach space consisting of all real functions defined and continuous on $I = [0,1]$, which is furnished with the standard norm $\|x\| = \max\{|x(t)| : x \in I\}$, $x \in C(I)$.

In the following, we define the measure of noncompactness in $C(I)$, which will used throughout this paper [3]. Let us fix $\emptyset \neq X \subset C(I)$ be bounded. For $x \in X$ and $\varepsilon \geq 0$ let the symbol $\omega(x,\varepsilon)$ stands for the modulus of continuity of the function x, i.e.,

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I,\ |t-s| \leq \varepsilon\}.$$

Further, let us put

$$\omega(X,\varepsilon) = \sup_{x\in X} \omega(x,\varepsilon)$$

and

$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon).$$

Now, let us define the quantities

$$\sup\{|x(t) - x(s)| - (x(t) - x(s)) : t, s \in I, s \leq t\}$$

and

$$i(X) = \sup_{x\in X} i(x).$$

Notice that $i(X) = 0$ if and only if all functions belonging to X are nondecreasing on I.

Next, let

$$\mu(X) = \omega_0(X) + i(X).$$

In [3], Banaś and Olszowy proved that the function μ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $\ker\mu$ consists of all subsets X of the set $\mathfrak{M}_{C(I)}$ such that all functions from X are nondecreasing and equicontinuous on I.

Finally, in [9], Darbo proved the following theorem.

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of E and let $\mathcal{H} : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ satisfying*

$$\mu(\mathcal{H}X) \leq k\mu(X),$$

for any nonempty subset X of Ω, where μ is a measure of noncompactness. Then $\mathcal{H}$ has a fixed point in the Ω.

Notice that under the assumptions of the above theorem the set $\mathrm{Fix}\mathcal{H}$ of all fixed points of $\mathcal{H}$ belonging to Ω belongs to the set $\ker\mu$ [2].

3 Existence Theorem

In this section, we investigate Eq.(1.1) under the following assumptions.

(A_1) The function $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$. Moreover, there exists a nonnegative constant c such that

$$|f(t,y) - f(t,x)| \leq c|y-x| \; \forall (t,x,y) \in I \times \mathbb{R}^2.$$

(A_2) The superposition operator F satisfies for any nonnegative function x the condition $i(Fx) \leq ci(x)$, where c is the same constant as in (A_1).

(A_3) The function $u : I^2 \times \mathbb{R}^2 \to \times\mathbb{R}$ is a continuous and $u : I^2 \times \mathbb{R}^2_+ \to \mathbb{R}_+$. Moreover, for arbitrary fixed $(s,x,y) \in I \times \mathbb{R}^2_+$ the function $t \to u(t,s,x,y)$ is nondecreasing on I.

(A_4) There exists a continuous nondecreasing function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t,s,x,y)| \leq \phi(|x|,|y|) \;\; \forall \; (t,s,x,y) \in I^2 \times \mathbb{R}^2.$$

(A_5) There exists r_0 such that

$$cr_0 + f^* + r_0^2 \phi(r_0, r_0) \leq r_0$$

and

$$c + 2r_0\phi(r_0, r_0) < 1,$$

where $f^* = \max\limits_{0 \leq t \leq 1} |f(t,0)|$.

Now, we can state and prove our main result in this paper.

Theorem 3.1. *Let the assumptions $(A_1) - (A_5)$ be verified. Then Eq.(1.1) has at least one solution $x \in C(I)$ and is nondecreasing on I.*

Proof. Let us consider the two operators $\mathcal{U}$ and $\mathcal{V}$ defined on the space $C(I)$ as follows

$$(\mathcal{V}x)(t) = f(t, x(t)) + x^2(t) \int_0^1 u(t,s,x(s),x(\lambda s)) \, ds$$

and

$$(\mathcal{U}x)(t) = \int_0^1 u(t,s,x(s),x(\lambda s)) \, ds.$$

For better readability, we divide our proof into several steps.

Step 1. We shall prove that if $x \in C(I)$ then $\mathcal{V}x \in C(I)$.
For this we have just to prove that if $x \in C(I)$ then $\mathcal{U}x \in C(I)$. $x \in C(I)$ then

$\mathcal{U}x \in C(I)$, thanks (a_1) and (a_2). Fix $\varepsilon > 0$ and let $x \in C(I)$ and $(t_1, t_2) \in I^2$ with $t_2 \geq t_1$ and $t_2 - t_1 \leq \varepsilon$. Then, we obtain

$$\begin{aligned}
&|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \\
&= \left| \int_0^1 u(t_2, s, x(s), x(\lambda s))\, ds - \int_0^1 u(t_1, s, x(s), x(\lambda s))\, ds \right| \\
&\leq \int_0^1 |u(t_2, s, x(s), x(\lambda s)) - u(t_1, s, x(s), x(\lambda s))|\ ds \\
&\leq \int_0^1 \omega_u(\varepsilon, \cdot, \|x\|, \|x\|)\, ds \\
&= \omega_u(\varepsilon, \cdot, \|x\|, \|x\|),
\end{aligned}$$

where $\omega_u(\varepsilon, \cdot, \|x\|, \|x\|)$ stands for

$$\sup\{|u(t,s,x,y) - u(\tau,s,x,y)| : (t,\tau,s) \in I^3,\ |t-\tau| \leq \varepsilon,\ (x,y) \in [-\|x\|, \|x\|]^2\}.$$

Since the function u is uniformly continuous on $I^2 \times [-\|x\|, \|x\|]^2$ then as $\varepsilon \to 0$, we have $\omega_u(\varepsilon, \cdot, \|x\|, \|x\|) \to 0$ and, consequently, $\mathcal{U}x \in C(I)$.

Step 2. We shall prove that $\mathcal{V}$ is continuous on $C(I)$.
To do this, let us fix $\varepsilon > 0$ and take $x \in C(I)$. Let $L = \|x\| + \varepsilon$ and take a function $y \in C(I)$ with $\|y - x\| \leq \varepsilon$. Then, for each $t \in I$ we have

$$\begin{aligned}
&|(\mathcal{V}y)(t) - (\mathcal{V}x)(t)| \\
&= \Bigg| f(t, y(t)) + y^2(t) \int_0^1 u(t, s, y(s), y(\lambda s)) ds \\
&\quad - f(t, x(t)) - x^2(t) \int_0^1 u(t, s, x(s), x(\lambda s))\, ds \Bigg| \\
&\leq c|y(t) - x(t)| \\
&\quad + \left| y^2(t) \int_0^1 u(t, s, y(s), y(\lambda s)) ds - x^2(t) \int_0^1 u(t, s, y(s), y(\lambda s))\, ds \right| \\
&\quad + \left| x^2(t) \int_0^1 u(t, s, y(s), y(\lambda s)) ds - x^2(t) \int_0^1 u(t, s, x(s), x(\lambda s))\, ds \right| \\
&\leq c|y(t) - x(t)| + \left| y^2(t) - x^2(t) \right| \int_0^1 |u(t, s, y(s), y(\lambda s))|\ ds \\
&\quad + |x^2(t)| \int_0^1 |u(t, s, y(s), y(\lambda s)) - u(t, s, x(s), x(\lambda s))| ds \\
&\leq c\|y - x\| + |y(t) - x(t)|\ |y(t) + x(t)| \int_0^1 \phi(|y(s)|, |y(\lambda s)|)\, ds \\
&\quad + \|x\|^2 \int_0^1 \beta_u(\varepsilon)\, ds
\end{aligned}$$

$$\leq c\|y-x\| + 2\|y-x\|(\|y\|+\varepsilon)\phi(\|x\|,\|x\|) + \|x\|^2\beta_u(\varepsilon),$$

where

$$\beta_u(\varepsilon) = \sup\{|u(t,s,x_2,y_2) - u(t,s,x_1,y_1)| : t,\ s \in I,\ x_1,x_2,y_1,y_2 \in [-L,L],\\ |x_2-x_1| \leq \varepsilon \text{ and } |y_2-y_1| \leq \varepsilon\}.$$

From the previous estimate we infer that

$$\|(\mathcal{V}y) - (\mathcal{V}x)\| \leq c\varepsilon + 2\varepsilon(\|y\|+\varepsilon)\phi(\|x\|,\|x\|) + \|x\|^2\beta_u(\varepsilon).$$

By the uniform continuity of the function u on $I^2 \times [-L,L]^2$ we have that $\beta_u(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus $\mathcal{V}$ is continuous on $C(I)$.

Step 3. $\mathcal{V}$ transforms the ball B_{r_0} into itself.
For each $t \in I$ we have

$$\begin{aligned} |(\mathcal{V}x)(t)| &= \left| f(t,x(t)) + x^2(t)\int_0^1 u(t,s,x(s),x(\lambda s))\,ds \right| \\ &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \|x\|^2 \int_0^1 |u(t,s,x(s),x(\lambda s))|\,ds \\ &\leq c\|x\| + f^* + \|x\|^2\phi(\|x\|,\|x\|). \end{aligned}$$

Hence

$$\|\mathcal{V}x\| \leq c\|x\| + f^* + \|x\|^2\phi(\|x\|,\|x\|).$$

Thus, if $\|x\| \leq r_0$ from assumption (A_5) we deduce that

$$\|\mathcal{V}x\| \leq cr_0 + f^* + r_0^2\phi(r_0,r_0) \leq r_0$$

and, consequently, the operator $\mathcal{V}$ maps B_{r_0} into itself.

Step 4. $\mathcal{V}$ transforms the ball $B_{r_0}^+$ into itself and is continuous.
Here, we consider the operator $\mathcal{V}$ on the set $B_{r_0}^+$ defined by

$$B_{r_0} \supset B_{r_0}^+ = \{x \in B_{r_0} : x(t) \geq 0,\ \text{for } t \in I\}.$$

It is obvious that the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. Notice that in view of this facts and our assumptions (A_1), (A_2) and (A_3), we infer that if $x(t) \geq 0$ for $t \in I$ then $(\mathcal{V}x)(t) \geq 0$ for $t \in I$. Therefore, $\mathcal{V}$ maps the set $B_{r_0}^+$ into itself and is continuous.

Step 5. An estimate of $\mathcal{V}$ with respect to ω_0 in $B_{r_0}^+$.
We take $\emptyset \neq X \subset B_{r_0}^+$ and $x \in X$. Then, for a fixed $\varepsilon > 0$ and $(t_1,t_2) \in I^2$ such that $t_2 \geq t_1$ and $|t_2 - t_1| \leq \varepsilon$, we get

$$|(\mathcal{V}x)(t_2) - (\mathcal{V}x)(t_1)|$$

$$
\begin{aligned}
&\leq |f(t_2,x(t_2))-f(t_1,x(t_1))| + \left|x^2(t_2)(\mathcal{U}x)(t_2)-x^2(t_2)(\mathcal{U}x)(t_1)\right| \\
&\quad + \left|x^2(t_2)(\mathcal{U}x)(t_1)-x^2(t_1)(\mathcal{U}x)(t_1)\right| \\
&\leq |f(t_2,x(t_2))-f(t_1,x(t_2))| + |f(t_1,x(t_2))-f(t_1,x(t_1))| \\
&\quad +|x^2(t_2)|\ \ |(\mathcal{U}x)(t_2)-(\mathcal{U}x)(t_1)| + |x^2(t_2)-x^2(t_1)|\ \ |(\mathcal{U}x)(t_1)| \\
&\leq \gamma_{r_0}(f,\varepsilon)+c\omega(x,\varepsilon)+\|x\|^2\omega_u(\varepsilon,\cdot,r_0,r_0)+2\omega(\varepsilon,x)\|x\|\phi(\|x\|,\|x\|),
\end{aligned}
$$

where

$$\gamma_{r_0}(f,\varepsilon)=\sup\{|f(t_2,x)-f(t_1,x)|:(t_1,t_2)\in I^2,\ x\in[0,r_0],\ |t_2-t_1|\leq\varepsilon\}$$

and

$$
\begin{aligned}
\omega_u(\varepsilon,\cdot,r_0,r_0) \quad = \quad & \sup\left\{|u(t_2,s,x,y)-f(t_1,s,x,y)|:(t_1,t_2,s)\in I^3,\right. \\
& \left.(x,y)\in[0,r_0]^2,\ |t_2-t_1|\leq\varepsilon\right\}.
\end{aligned}
$$

Therefore,

$$\omega(\mathcal{V}x,\varepsilon)\leq\gamma_{r_0}(f,\varepsilon)+c\omega(x,\varepsilon)+r_0^2\omega_u(\varepsilon,\cdot,r_0,r_0)+2\omega(\varepsilon,x)r_0\phi(r_0,r_0),$$

and consequently

$$\omega(\mathcal{V}X,\varepsilon)\leq\gamma_{r_0}(f,\varepsilon)+c\omega(X,\varepsilon)+r_0^2\omega_u(\varepsilon,\cdot,r_0,r_0)+2\omega(\varepsilon,X)r_0\phi(r_0,r_0).$$

Taking into account the uniform continuity of the function f on the set $I\times[0,r_0]$ and the uniform continuity of the function u on $I^2\times[0,r_0]^2$, the last inequality implies

$$\omega_0(\mathcal{V}X)\leq(c+2r_0\phi(r_0,r_0))\omega_0(X). \tag{3.1}$$

Step 6. An estimate of $\mathcal{V}$ with respect to i in $B_{r_0}^+$.
We fix $x\in X$ and $t_1,t_2\in I$ with $t_2\geq t_1$. Then, we have

$$
\begin{aligned}
&|(\mathcal{V}x)(t_2)-(\mathcal{V}x)(t_1)|-((\mathcal{V}x)(t_2)-(\mathcal{V}x)(t_1)) \\
&\leq |f(t_2,x(t_2))-f(t_1,x(t_1))|-(f(t_2,x(t_2))-f(t_1,x(t_1))) \\
&\quad + \left|x^2(t_2)(\mathcal{U}x)(t_2)-x^2(t_1)(\mathcal{U}x)(t_2)\right| + \left|x^2(t_1)(\mathcal{U}x)(t_2)-x^2(t_1)(\mathcal{U}x)(t_1)\right| \\
&- \left(x^2(t_2)(\mathcal{U}x)(t_2)-x^2(t_1)(\mathcal{U}x)(t_2)\right) - \left(x^2(t_1)(\mathcal{U}x)(t_2)-x^2(t_1)(\mathcal{U}x)(t_1)\right) \\
&\leq i(Fx)+\left[|x^2(t_2)-x^2(t_1)|-\left(x^2(t_2)-x^2(t_1)\right)\right]\int_0^1 u(t_2,s,x(s),x(\lambda(s)))\,ds \\
&\quad +|x^2(t_1)|\int_0^1 |u(t_2,s,x(s),x(\lambda s))-u(t_1,s,x(s),x(\lambda s))|\,ds \\
&\quad -x^2(t_1)\int_0^1 (u(t_2,s,x(s),x(\lambda s))-u(t_1,s,x(s),x(\lambda s)))\ ds \\
&\leq i(Fx)+[|x(t_2)-x(t_1)|\ |x(t_2)+x(t_1)|-(x(t_2)-x(t_1))\,(x(t_2)+x(t_1))]
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^1 \phi(|x(s)|, |x(\lambda s)|)\, ds \\
&\leq i(Fx) + [|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))]\,(x(t_2) + x(t_1))\,\phi(\|x\|, \|x\|) \\
&\leq i(Fx) + 2\|x\|\phi(\|x\|, \|x\|)i(x) \\
&\leq i(Fx) + 2r_0\phi(r_0, r_0)i(x).
\end{aligned}
$$

Therefore,

$$i(\mathcal{V}x) \leq ci(x) + 2r_0\phi(r_0, r_0)i(x)$$

and consequently,

$$i(\mathcal{V}X) \leq (c + 2r_0\phi(r_0, r_0))i(X). \tag{3.2}$$

Step 7. $\mathcal{V}$ is contraction with respect to the measure of noncompactness μ. From (3.1) and (3.2), we obtain

$$\mu(\mathcal{V}X) \leq (c + 2r_0\phi(r_0, r_0))\mu(X).$$

The above obtained inequality together with the fact that $c + 2r_0\phi(r_0, r_0) < 1$ allow us to apply Theorem 2.2. Therefore Eq.(1.1) has at least one solution $x \in C(I)$ which is nondecreasing on I. This completes the proof. □

References

[1] D. Bainov and D. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Hilger, Bristol, 1991.

[2] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics **60**, Marcel Dekker, New York, 1980.

[3] J. Banaś and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math.* **41** (2001), 13–23.

[4] M. Benchohra and M.A. Darwish, On Unique Solvability of Quadratic Integral Equations with Linear Modification of the Argument, *Miskolc Math. Notes* **10** (2009), 3–10.

[5] J. Caballero, B. López and K. Sadarangani, Existence of nondecreasing and continuous solutions of an integral equation with linear modification of the argument, *Acta Math. Sin. (Engl. Ser.)* **2** (2007), 1719–1728.

[6] J. Carr and J. Dyson, The functional differential equation $y'(x) = ay(\lambda x) + by(x)$, *Proc. Roy. Soc. Edinburgh Sect. A* **74 (1974/75)** (1976), 165–174.

[7] J. Caballero, D. O'Regan and K. Sadarangani, On nondecreasing solutions of cubic integral equations of Urysohn type, *Comment. Math. (Prace Mat.)* **44** (2004), 39–53.

[8] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.

[9] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Sem. Mat. Univ. Padova* **24** (1955), 84–92.

[10] M.A. Darwish, On a quadratic fractional integral equation with linear modification of the argument, *Can. Appl. Math. Q.* **16** (2008), 45–58.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[12] G.M. Dunkel, *Function Differential Equations: Examples and Problems*, Lecture Notes in Math. **144**, Springer, Berlin, 1970.

[13] M.R.S. Kulenović, Oscillation of the Euler differential equation with delay, *Czechoslovak Math. J.* **45** (1995), 1–6.

[14] H. Melvin, A family of solutions of the IVP for the equation $x'(t) = ax(\lambda t)$, $\lambda > 1$, *Aequationes Math.* **9** (1973), 273–280.

[15] V. Mureşan, Volterra integral equations ith iterations of linear modification of the argument, *Novi Sad J. Math.* **33** (2003), 1–10.

[16] V. Mureşan, On a class of Volterra integral equations with deviating argument, *Studia Univ. Babeş-Bolyai Math.* **44** (1999), 47–54.

[17] D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equation*, Kluwer Academic Publishers, Dordrechet, 1998.