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Local and Global Dynamics of Certain Second-Order Rational Difference Equations Containing Quadratic Terms

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Abstract

We investigate local and global dynamics of certain second-order rational difference equation with six nonnegative parameters and nonnegative initial conditions. Second order rational difference equations with quadratic terms in their numerators and denominators exhibit a rich variety of dynamic behaviors which depends on the parameters and initial conditions.

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1 Introduction and Preliminaries

In this paper, we investigate local and global dynamics of difference equation

$$x_{n+1} = \frac{Bx_n x_{n-1} + Cx_{n-1}^2 + F}{bx_n x_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, \dots \tag{1.1}$$

with positive parameters and nonnegative initial conditions x_{-1}, x_0. Equation (1.1) is the special case of a general second-order quadratic fractional equation of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \;\; n = 0, 1, 2, \dots \tag{1.2}$$

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with 12 positive parameters and nonnegative initial conditions. Some special cases of (1.2) have been considered in the series of papers [3, 4, 10, 11, 19], but systematic study of rational equations containing quadratic terms has not been conducted. Some special second order quadratic fractional difference equations have appeared in analysis of competitive and anti-competitive systems of linear fractional difference equations in the plane, see [7, 8, 18]. Since (1.2) contains as a special cases many equations with complicated dynamics, determining the global dynamics of (1.2) is a formidable task. For example, even the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, \ldots \tag{1.3}$$

which dynamics was investigated in great details in [12] and in many papers which solved some conjectures and open problems posed in [12], until today has some unsolved cases.

In this paper, we will determine the local stability analysis of all three equilibrium points of (1.1) and we will give the necessary and sufficient conditions for the equilibrium to be locally asymptotically stable, a saddle point, a repeller, or a nonhyperbolic equilibrium. All analysis which will be done in order to determine a number of equilibrium points and their local stability when $F > 0$ will be implemented in the (f, F) plane. The local stability analysis indicates that some possible dynamics scenarios for (1.1) include Naimark–Sacker bifurcations, as in the case of equation

$$x_{n+1} = \frac{F}{bx_nx_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, 2, \ldots$$

considered in [16]. Another possible behavior, which is a consequence of the presence of quadratic terms in (1.1), is appearance of very interesting phenomena, known as Allee effect, and coexistence of three minimal period solutions as in the case of equation

$$x_{n+1} = \frac{x_{n-1}^2}{bx_nx_{n-1} + cx_{n-1}^2 + f}, \quad n = 0, 1, 2, \ldots$$

considered in [15].

The paper is organized as follows. The rest of this section contains some global attractivity results which will be used in Section 4 to obtain global asymptotic stability results for some special cases of (1.1). Section 2 gives local stability analysis of all three equilibrium points when $F > 0$ and Section 3 gives local stability analysis of all equilibrium points when $F = 0$. Finally, Section 4 gives some global attractivity results in some special cases.

The global attractivity results obtained specifically for complicated cases of (1.2), when many terms in numerator and denominator are present, are the following theorems [20].

Theorem 1.1. *Assume that* (1.2) *has the unique equilibrium* $\overline{x}$*. If the condition*

$$\frac{(|A - a\overline{x}| + |B - b\overline{x}| + |C - c\overline{x}|)(U + \overline{x}) + |D - d\overline{x}| + |E - e\overline{x}|}{(a + b + c)L^2 + (d + e)L + f} < 1, \tag{1.4}$$

holds, where L *and* U *are lower and upper bounds of all solutions of* (1.2)*, then* $\overline{x}$ *is globally asymptotically stable.*

Theorem 1.2. *Assume that* (1.2) *has the unique equilibrium* $\overline{x} \in [m, M]$ *where* $m = \min\{\overline{x}, x_{-1}, x_0\}$ *and* $M = \max\{\overline{x}, x_{-1}, x_0\}$ *are lower and upper bounds of specific solution of* (1.2)*. If the condition*

$$(|A - a\overline{x}| + |B - b\overline{x}| + |C - c\overline{x}|)(M + \overline{x}) + |D - d\overline{x}| + |E - e\overline{x}| <$$

$$(a + b + c)m^2 + (d + e)m + f,$$

holds, then $\overline{x}$ *is globally asymptotically stable on the interval* $[m, M]$.

In the case of (1.1), Theorems 1.1 and 1.2 give the following special results.

Corollary 1.3. *If the condition*

$$\frac{(|B - b\overline{x}| + |C - c\overline{x}|)\,(U + \overline{x})}{(b + c)\,L^2 + f} < 1, \tag{1.5}$$

holds, where L *and* U *are lower and upper bounds of all solutions of* (1.1)*, then* $\overline{x}$ *is globally asymptotically stable.*

Corollary 1.4. *If the condition*

$$(|B - b\overline{x}| + |C - c\overline{x}|)\,(M + \overline{x}) < (b + c)m^2 + f, \tag{1.6}$$

holds, where $m = \min\{\overline{x}, x_{-1}, x_0\}$ *and* $M = max\{\overline{x}, x_{-1}, x_0\}$ *are lower and upper bounds of specific solution of* (1.1) *which depends of initial condition, then the unique equilibrium* $\overline{x}$ *is globally asymptotically stable on the interval* $[m, M]$.

2 Local Stability Analysis

First we will determine the number of equilibrium points. The equilibrium point $\overline{x}$ satisfies the following equation

$$\overline{x} = \frac{(B + C)\overline{x}^2 + F}{(b + c)\overline{x}^2 + f}$$

i.e.,

$$(b + c)\overline{x}^3 - (B + C)\overline{x}^2 + f\overline{x} - F = 0. \tag{2.1}$$

If we denote the left side of previous relation by

$$G\left(x\right)=\left(b+c\right)x^{3}-\left(B+C\right)x^{2}+fx-F, \tag{2.2}$$

it is easy to see

$$G(-\infty)=-\infty, G(\infty)=\infty, G(0)=-F \tag{2.3}$$

and

$$G'\left(x\right)=3\left(b+c\right)x^{2}-2\left(B+C\right)x+f.$$

Let $x_{1,2}^{(1)}$ denote the roots of the first derivative and D_1 the discriminant of first derivative. Hence,

$$G'\left(x\right)=0\Leftrightarrow x_{1,2}^{(1)}=\tfrac{(B+C)\pm\sqrt{D_1}}{3(b+c)},\ D_1=\left(B+C\right)^{2}-3f\left(b+c\right).$$

We can see that $G'\left(x\right)>0$ for $x\leq 0$, so for those x function $G(x)$ is obviously increasing. Furthermore, it holds

$$G_{\max}=G(x_1^{(1)}), G_{\min}=G(x_2^{(1)}) \text{ and } x_1^{(1)}<x_2^{(1)}.$$

If $\left(B+C\right)^{2}-3f\left(b+c\right)\leq 0$, i.e., $f\geq\frac{(B+C)^2}{3(b+c)}=f_D^{(1)}$, then $G'(x)\geq 0$ for every x, which implies that $G(x)$ is increasing function for every x, and considering relations (2.3) it follows that equilibrium point is unique. If $\left(B+C\right)^{2}-3f\left(b+c\right)>0$, that is $f<f_D^{(1)}$, then equation $G'\left(x\right)=0$ has two real and distinct solutions and the number of equilibrium points will depend of the sign of product $G(x_1^{(1)})G(x_2^{(1)})$, i.e.,

$$\Omega_1\left(f,F\right)=G(x_1^{(1)})G(x_2^{(1)}).$$

It is easy to see that holds:

$$\begin{aligned}
\Omega_1\left(f,F\right) &> 0\Leftrightarrow \text{there exists unique equilibrium point},\\
\Omega_1\left(f,F\right) &= 0\Leftrightarrow \text{there exist two equilibrium points},\\
\Omega_1\left(f,F\right) &< 0\Leftrightarrow \text{there exist three equilibrium points}.
\end{aligned}$$

After calculation, we get

$$\begin{aligned}
\Omega_1 = 27\left(b+c\right)^{2}F^{2}+2\left(B+C\right)\left(2\left(B+C\right)^{2}-9f\left(b+c\right)\right)F\\
-f^{2}\left(\left(B+C\right)^{2}-4\left(b+c\right)f\right).
\end{aligned}$$

Hence, $\Omega_1\left(f,F\right)=0$ is an implicit function which graph in the first quadrant is represented by union of graphs of two explicit functions $F_1^{(1)}\left(f\right)$ and $F_2^{(1)}\left(f\right)$ inside the area $D_1>0$, where

$$\left.\begin{aligned}
F_1^{(1)}\left(f\right) &= \tfrac{-(B+C)\left(2(B+C)^2-9f(b+c)\right)-2\left(\sqrt{(B+C)^2-3f(b+c)}\right)^3}{27(b+c)^2}\\
F_2^{(1)}\left(f\right) &= \tfrac{-(B+C)\left(2(B+C)^2-9f(b+c)\right)+2\left(\sqrt{(B+C)^2-3f(b+c)}\right)^3}{27(b+c)^2}
\end{aligned}\right\} \tag{2.4}$$

we get as a solutions of quadratic equation, if the implicit function $\Omega_1(f,F) = 0$ consider as quadratic equation in variable F, where its discriminant is of the form

$$D_{F_{1,2}^{(1)}} = 16D_1^3.$$

One can see that $F_1^{(1)}(f)$ and $F_2^{(1)}(f)$ are increasing and continuous functions and that it holds

$$\begin{array}{rcl} F_1^{(1)}(f) & = & 0 \Leftrightarrow f = \frac{(B+C)^2}{4(b+c)} = f_N^{(1)}, \\ F_2^{(1)}(f) & = & 0 \Leftrightarrow f = 0, \\ F_2^{(1)}\left(f_D^{(1)}\right) & = & F_1^{(1)}\left(f_D^{(1)}\right) = \frac{(B+C)^3}{27(b+c)^2} \end{array}$$

and

$$F_2^{(1)}(f) - F_1^{(1)}(f) = \frac{4\left(\sqrt{(B+C)^2 - 3f(b+c)}\right)^3}{27(b+c)^2} > 0.$$

Let Ω_1^0 represent the union of graphs of functions $F_1^{(1)}(f)$ and $F_2^{(1)}(f)$ as it follows

$$\Omega_1^0 = \left\{(f,F) : f \in \left(0, f_D^{(1)}\right), F = F_2^{(1)}(f) \vee F = F_1^{(1)}(f)\right\}. \tag{2.5}$$

Let Ω_1^- be the area between curve Ω_1^0 and f- axis, i.e.,

$$\Omega_1^- = \left\{(f,F) : 0 < f < f_D^{(1)}, F \in \left(\max\left\{0, F_1^{(1)}(f)\right\}, F_2^{(1)}(f)\right)\right\}, \tag{2.6}$$

and Ω_1^+ the complement of the union of sets Ω_1^0 and Ω_1^- inside the first quadrant

$$Q_1 = \{(f,F) : f > 0 \text{ and } F > 0\},$$

i.e., $\Omega_1^+ = Q_1 \backslash \left(\Omega_1^0 \cup \Omega_1^-\right)$, hence

$$\left.\begin{array}{c} \Omega_1^+ = \left\{(f,F) : 0 < f \leq f_N^{(1)}, F > F_2^{(1)}(f)\right\} \cup \\ \left\{(f,F) : f_N^{(1)} < f < f_D^{(1)}, \left(F > F_2^{(1)}(f) \vee F < F_1^{(1)}(f)\right)\right\} \cup \\ \left\{\left(f_D^{(1)}, F_{1,2}^{(1)}\left(f_D^{(1)}\right)\right)\right\}. \end{array}\right\} \tag{2.7}$$

It is easy to check that

$$\begin{array}{rcl} \Omega_1(f,F) & > & 0 \Leftrightarrow (f,F) \in \Omega_1^+, \\ \Omega_1(f,F) & < & 0 \Leftrightarrow (f,F) \in \Omega_1^-, \\ \Omega_1(f,F) & = & 0 \Leftrightarrow (f,F) \in \Omega_1^0. \end{array}$$

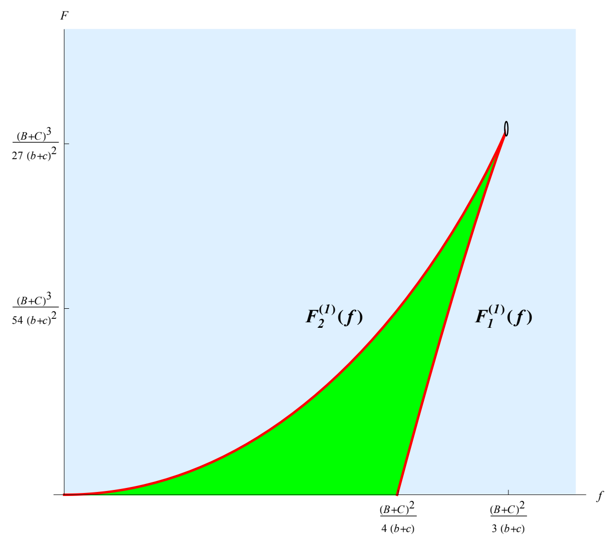

Figure 2.1: Visual representation of the number of equilibrium points in the (f, F) plane: blue - unique equilibrium point, red - two equilibrium points, green - three equilibrium points.

One can notice that $\Omega_1(f,F)=0$ implies $G(x_1^{(1)})=0$ or $G(x_2^{(1)})=0$, and therefore, the equation (1.1) has two equilibrium points of the form

$$\begin{gathered}\overline{x}_1=\frac{(B+C)-\sqrt{(B+C)^2-3(b+c)f}}{3(b+c)} \text{ and } \overline{x}_2=\frac{(B+C)+2\sqrt{(B+C)^2-3(b+c)f}}{3(b+c)} \\ \left(\overline{x}_1=x_1^{(1)}<\overline{x}_2\right),\end{gathered} \tag{2.8}$$

or

$$\begin{gathered}\overline{x}_1=\frac{(B+C)-2\sqrt{(B+C)^2-3(b+c)f}}{3(b+c)} \text{ and } \overline{x}_2=\frac{(B+C)+\sqrt{(B+C)^2-3(b+c)f}}{3(b+c)} \\ \left(\overline{x}_1<\overline{x}_2=x_2^{(1)}\right),\end{gathered} \tag{2.9}$$

respectively. Specially, for $f=f_N^{(1)}$ equilibrium points $\overline{x}_1=\frac{(B+C)}{6(b+c)}$ and $\overline{x}_2=\frac{2(B+C)}{3(b+c)}$.

With the previous consideration, we have proved the following theorem.

Theorem 2.1. *Equation* (1.1) *has:*

(i) unique equilibrium point if $(f,F)\in\Omega_1^+$,

(ii) two equilibrium points if $(f,F)\in\Omega_1^0$,

(iii) three equilibrium points if $(f,F)\in\Omega_1^-$,

where Ω_1^+,Ω_1^0 *and* Ω_1^- *are given with* (2.7), (2.5) *and* (2.6) *respectively.*

See Figure 2.1.

3 Case $F > 0$

In this section, we present the local stability of equilibrium points of (1.1) in the case $F > 0$. Set

$$h(u,v) = \frac{Buv + Cv^2 + F}{buv + cv^2 + f}.$$

The linearized equation associated with (1.1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial h}{\partial u}(\overline{x},\overline{x}) = \overline{x}\frac{Bf - Fb + (Bc - Cb)\overline{x}^2}{\left(f + (c+b)\overline{x}^2\right)^2},$$
$$q = \frac{\partial h}{\partial v}(\overline{x},\overline{x}) = \frac{(Bf - Fb + 2Cf - 2Fc)\overline{x} + (Cb - Bc)\overline{x}^3}{\left(f + (c+b)\overline{x}^2\right)^2}.$$

For $\overline{x} \neq 0$, we have $\overline{x} = \frac{(B+C)\overline{x}^2 + F}{(b+c)\overline{x}^2 + f}$, so

$$p = \frac{(B - \overline{x}b)\overline{x}}{(b+c)\overline{x}^2 + f}, \quad q = \frac{\overline{x}(B + 2C - (b+2c)\overline{x})}{f + (b+c)\overline{x}^2}.$$

In order to examine the local stability, it is necessary to determine the sign of the following expressions:

$$\begin{aligned}
p + q - 1 &= \frac{-3(b+c)\overline{x}^2 + 2(B+C)\overline{x} - f}{f + (b+c)\overline{x}^2} = \frac{-G'(\overline{x})}{f + (b+c)\overline{x}^2} = \frac{G_1(\overline{x})}{f + (b+c)\overline{x}^2},\\
p - q + 1 &= \frac{(b+3c)\overline{x}^2 - 2C\overline{x} + f}{f + (b+c)\overline{x}^2} = \frac{G_2(\overline{x})}{f + (b+c)\overline{x}^2},\\
q + 1 &= \frac{-c\overline{x}^2 + (B+2C)\overline{x} + f}{f + (b+c)\overline{x}^2} = \frac{G_3(\overline{x})}{f + (b+c)\overline{x}^2},\\
q - 1 &= \frac{-(2b+3c)\overline{x}^2 + (B+2C)\overline{x} - f}{f + (b+c)\overline{x}^2} = \frac{G_4(\overline{x})}{f + (b+c)\overline{x}^2}.
\end{aligned}$$

Since $f + (b+c)\overline{x}^2$ is always positive, the sign of above mentioned expressions depends only on the functions in the numerator. Set D_1, D_2, D_3 and D_4 for discriminants, and $x_{1,2}^{(1)}$, $x_{1,2}^{(2)}$, $x_{1,2}^{(3)}$ and $x_{1,2}^{(4)}$ for the roots of quadratic functions in the numerator, i.e., functions $G_1(x)$, $G_2(x)$, $G_3(x)$ and $G_4(x)$ respectively. Since $G'(x) > 0$ for every x in the case of unique equilibrium point, that means $p + q - 1 < 0$.

Let us determine the sign of $p - q + 1$. The zeros of the function $G_2(x) = (b+3c)x^2 - 2Cx + f$ are

$$x_{1,2}^{(2)} = \frac{C \pm \sqrt{D_2}}{b + 3c}, \quad D_2 = C^2 - f(b+3c).$$

If $D_2 = C^2 - f(b+3c) < 0$, then $G_2(x) > 0$ for every x, and consequently, for every equilibrium point. The same conclusion holds if $B \geq C$. Indeed, at the equilibrium point, we have $(b+c)\overline{x}^3 - (B+C)\overline{x}^2 + f\overline{x} - F = 0$, so we get

$$G_2(\overline{x}) = (b+3c)\overline{x}^2 - 2C\overline{x} - \frac{(b+c)\overline{x}^3 - (B+C)\overline{x}^2 - F}{\overline{x}} = \frac{F}{\overline{x}} + (B - C + 2c\overline{x})\overline{x} > 0$$

for $B \geq C$. In the further work, we will consider that $D_2 \geq 0$ and $B < C$. Let us formulate a product

$$\Omega_2(f,F) = G\left(x_1^{(2)}\right) G\left(x_2^{(2)}\right).$$

After calculation, we get

$$\begin{aligned}\Omega_2 = F^2 + 2\tfrac{2C^2(Bb+3Bc-Cb+Cc)-f(b+3c)(Bb+3Bc-Cb+3Cc)}{(b+3c)^3}F \\ +\tfrac{f^2\left((B-C)(Bb+3Bc-Cb+Cc)+4c^2f\right)}{(b+3c)^3}.\end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned}\Omega_2(f,F) > 0 \Leftrightarrow (p-q+1)(\overline{x}) > 0, \\ \Omega_2(f,F) = 0 \Leftrightarrow (p-q+1)(\overline{x}) = 0, \\ \Omega_2(f,F) < 0 \Leftrightarrow (p-q+1)(\overline{x}) < 0.\end{aligned}$$

Hence, $\Omega_2(f,F) = 0$ is an implicit function which graph in the first quadrant is represented by union of graphs of two explicit functions $F_1^{(2)}(f)$ and $F_2^{(2)}(f)$ inside the area $D_2 \geq 0$ where

$$\left.\begin{aligned}F_1^{(2)}(f) = \tfrac{f(b+3c)(Bb+3Bc-Cb+3Cc)-2C^2(Bb+3Bc-Cb+Cc)-2|C(Bb+3Bc-Cb)+cD_2|\sqrt{D_2}}{(b+3c)^3}, \\ F_2^{(2)}(f) = \tfrac{f(b+3c)(Bb+3Bc-Cb+3Cc)-2C^2(Bb+3Bc-Cb+Cc)+2|C(Bb+3Bc-Cb)+cD_2|\sqrt{D_2}}{(b+3c)^3},\end{aligned}\right\} \tag{3.2}$$

we get as a solutions of quadratic equation, if the implicit function $\Omega_2(f,F) = 0$ consider as quadratic equation in variable F, where its discriminant is of the form

$$D_{F_{1,2}^{(2)}} = \tfrac{16D_2(C(Bb+3Bc-Cb)+cD_2)^2}{(b+3c)^6}.$$

The graph of the functions (3.2) exists in the first quadrant only if $f \in \left(0, f_D^{(2)}\right]$. Further,

$$\begin{aligned}D_{F_{1,2}^{(2)}} = 0 \quad &\Leftrightarrow \quad \frac{16D_2\left(C\left(Bb+3Bc-Cb\right)+cD_2\right)^2}{\left(b+3c\right)^6} = 0 \\ &\Leftrightarrow \quad D_2 = 0 \ \vee \ BCb + 3BCc - C^2b + cD_2 = 0 \\ &\Leftrightarrow \quad f_D^{(2)} = \frac{C^2}{b+3c} \ \vee \ f_P^{(2)} = \frac{C\left(Bb+3Bc-Cb+Cc\right)}{c\left(b+3c\right)}.\end{aligned}$$

Since

$$F_{1,2}^{(2)}\left(f_D^{(2)}\right) = \tfrac{-C^2(Bb+3Bc-Cb-Cc)}{(b+3c)^3}$$

and

$$F_{1,2}^{(2)}\left(f_P^{(2)}\right) = \frac{C\left(Bb+3Bc-Cb+Cc\right)^2}{c\left(b+3c\right)^3},$$

the functions (3.2) are united into the endpoint of the domain $f_D^{(2)}$ and intersect i.e., touch themselves at $f_P^{(2)}$ if $0 < f_P^{(2)} < f_D^{(2)}$. The point $(0,0)$ lies in $\Omega_2(f,F) = 0$ so, at least one of the functions (3.2) contains that point. Let $f_N^{(2)}$ denote the zero of the free term of quadratic function by variable F given with (3.1). Then we get

$$f_N^{(2)} = \frac{(C-B)(Bb+3Bc-Cb+Cc)}{4c^2}.$$

Since $B < C$, it implies

$$f_N^{(2)} = \tfrac{(C-B)(Bb+3Bc-Cb+Cc)}{4c^2} \geq 0 \Leftrightarrow Bb+3Bc-Cb+Cc \geq 0.$$

Hence, $\Omega_2\left(f_N^{(2)}, F\right) = 0$ for

$$F_1^{(2)}\left(f_N^{(2)}\right) = \min\left\{0, -\tfrac{(Bb+3Bc-Cb-Cc)(Bb+3Bc-Cb+Cc)^2}{2c^2(b+3c)^3}\right\}$$

and

$$F_2^{(2)}\left(f_N^{(2)}\right) = \max\left\{0, -\tfrac{(Bb+3Bc-Cb-Cc)(Bb+3Bc-Cb+Cc)^2}{2c^2(b+3c)^3}\right\},$$

if $0 < f_N^{(2)} < f_D^{(2)}$. Since the position of points $f_N^{(2)}$, $f_D^{(2)}$ and $f_P^{(2)}$ is very important for the discussion that follows, first we will find the conditions determining their mutual position.Thus,

$$\begin{array}{rcl}
f_N^{(2)} - f_D^{(2)} & = & \frac{(C-B)(Bb+3Bc-Cb+Cc)}{4c^2} - \frac{C^2}{b+3c} = -\frac{(Bb+3Bc-Cb-Cc)^2}{4c^2(b+3c)}, \\
f_N^{(2)} - f_D^{(2)} & < & 0 \Leftrightarrow Bb+3Bc-Cb-Cc \neq 0, \\
f_N^{(2)} - f_P^{(2)} & = & \frac{(C-B)(Bb+3Bc-Cb+Cc)}{4c^2} - \frac{C(Bb+3Bc-Cb+Cc)}{c(b+3c)} = -\frac{(Bb+3Bc-Cb+Cc)^2}{4c^2(b+3c)}, \\
f_N^{(2)} - f_P^{(2)} & < & 0 \Leftrightarrow Bb+3Bc-Cb+Cc \neq 0, \\
f_P^{(2)} - f_D^{(2)} & = & \frac{C(Bb+3Bc-Cb+Cc)}{c(b+3c)} - \frac{C^2}{b+3c} = C\frac{Bb+3Bc-Cb}{c(b+3c)}, \\
f_P^{(2)} - f_D^{(2)} & < & 0 \Leftrightarrow Bb+3Bc-Cb < 0.
\end{array}$$

Now we can conclude

$$\begin{array}{l}
\Omega_2(f,F) = 0 \text{ if } (f,F) \in P_2^0, \\
\Omega_2(f,F) < 0 \text{ if } (f,F) \in P_2^-, \\
\Omega_2(f,F) > 0 \text{ if } (f,F) \in P_2^+,
\end{array}$$

where sets

$$P_2^0,\ P_2^- \text{ and } P_2^+ = Q_1 \setminus \left(P_2^- \cup P_2^0\right),$$

are defined as follows:

1. If $Bb+3Bc-Cb+Cc = 0$, then $f_N^{(2)} = 0$ and $f_P^{(2)} = 0$, which implies

$$F_{1,2}^{(2)}\left(f_N^{(2)}\right) = 0 \text{ and } F_{1,2}^{(2)}\left(f_D^{(2)}\right) = \frac{2cC^3}{(b+3c)^2} > 0$$

so:

$$\left.\begin{array}{l} P_2^0 = A_2 \cup B_2, \\ P_2^- = \left\{(f,F) : f \in \left(0, f_D^{(2)}\right), F \in \left(F_1^{(2)}(f), F_2^{(2)}(f)\right)\right\}, \end{array}\right\} \tag{3.3}$$

where

$$\begin{aligned} A_2 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = F_1^{(2)}(f)\right\}, \\ B_2 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = F_2^{(2)}(f)\right\}. \end{aligned}$$

2. If $Bb + 3Bc - Cb + Cc > 0$ and $Bb + 3Bc - Cb - Cc \geq 0$, then

$$0 < f_N^{(2)} \leq f_D^{(2)} < f_P^{(2)} \text{ and } F_{1,2}^{(2)}\left(f_D^{(2)}\right) \leq 0$$

so we have

$$\left.\begin{array}{l} P_2^0 = \left\{(f,F) : f \in \left(0, f_N^{(2)}\right), F = F_2^{(2)}(f)\right\}, \\ P_2^- = \left\{(f,F) : f \in \left(0, f_N^{(2)}\right), F \in \left(0, F_2^{(2)}(f)\right)\right\}. \end{array}\right\} \tag{3.4}$$

3. If $Bb+3Bc-Cb+Cc > 0$, $Bb+3Bc-Cb-Cc < 0$ and $Bb+3Bc-Cb \geq 0$, then

$$0 < f_N^{(2)} \leq f_D^{(2)} \leq f_P^{(2)} \text{ and } F_{1,2}^{(2)}\left(f_D^{(2)}\right) > 0,$$

so it holds

$$P_2^0 = A_2 \cup B_2, P_2^- = C_2 \cup D_2, \tag{3.5}$$

where

$$\begin{aligned} A_2 &= \left\{(f,F) : f \in \left(f_N^{(2)}, f_D^{(2)}\right], F = F_1^{(2)}(f)\right\}, \\ B_2 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = F_2^{(2)}(f)\right\}, \\ C_2 &= \left\{(f,F) : f \in \left(0, f_N^{(2)}\right], F \in \left(0, F_2^{(2)}(f)\right)\right\}, \\ D_2 &= \left\{(f,F) : f \in \left(f_N^{(2)}, f_D^{(2)}\right), F \in \left(F_1^{(2)}(f), F_2^{(2)}(f)\right)\right\}. \end{aligned}$$

4. If $Bb+3Bc-Cb+Cc > 0$, $Bb+3Bc-Cb-Cc < 0$ and $Bb+3Bc-Cb < 0$, then

$$0 < f_N^{(2)} < f_P^{(2)} < f_D^{(2)} \text{ and } F_{1,2}^{(2)}\left(f_D^{(2)}\right) > 0,$$

so

$$P_2^0 = A_2 \cup B_2,\ P_2^- = C_2 \cup D_2, \tag{3.6}$$

where

$$\begin{aligned} A_2 &= \left\{(f,F) : f \in \left(f_N^{(2)}, f_D^{(2)}\right], F = F_1^{(2)}(f)\right\}, \\ B_2 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = F_2^{(2)}(f)\right\}, \\ C_2 &= \left\{(f,F) : f \in \left(0, f_N^{(2)}\right], F \in \left(0, F_2^{(2)}(f)\right)\right\}, \\ D_2 &= \left\{(f,F) : f \in \left(f_N^{(2)}, f_D^{(2)}\right], F \in \left(F_1^{(2)}(f), F_2^{(2)}(f)\right)\right\}. \end{aligned}$$

5. If $Bb+3Bc-Cb+Cc<0$, then $Bb+3Bc-Cb-Cc<0$ and $Bb+3Bc-Cb<0$, so
$$f_N^{(2)} < f_P^{(2)} < 0 < f_D^{(2)},$$
that implies
$$\left.\begin{array}{l} P_2^0 = A_2 \cup B_2, \\ P_2^- = \left\{(f,F) : f \in \left(0, f_D^{(2)}\right), F \in \left(F_1^{(2)}(f), F_2^{(2)}(f)\right)\right\}, \end{array}\right\} \tag{3.7}$$
where
$$\begin{aligned} A_2 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = F_1^{(2)}(f)\right\}, \\ B_2 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = F_2^{(2)}(f)\right\}. \end{aligned}$$

All sets denoted by P_2^0, P_2^- and P_2^+ in (3.3), (3.4), (3.5), (3.6) and (3.7) can be unified with Ω_2^0, Ω_2^- and Ω_2^+ given as:

$$\Omega_2^0 = D_2^0 \cup G_2^0, \tag{3.8}$$

$$\begin{aligned} \Omega_2^- = \Big\{(f,F) : f \in \left(0, f_D^{(2)}\right), \\ F \in \left(\max\left\{0, F_1^{(2)}(f)\right\}, \max\left\{0, F_2^{(2)}(f)\right\}\right)\Big\}, \end{aligned} \tag{3.9}$$

$$\Omega_2^+ = Q_1 \setminus \left(\Omega_2^- \cup \Omega_2^0\right), \tag{3.10}$$

where

$$\begin{aligned} D_2^0 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = \max\left\{0, F_1^{(2)}(f)\right\}, F \neq 0\right\}, \\ G_2^0 &= \left\{(f,F) : f \in \left(0, f_D^{(2)}\right], F = \max\left\{0, F_2^{(2)}(f)\right\}, F \neq 0\right\}. \end{aligned}$$

This proves the following lemma.

Lemma 3.1. *Let*

$$B \geq C. \tag{3.11}$$

Then $p - q + 1 > 0$ *for every* $(f, F) \in Q_1$ *in each equilibrium point of equation* (1.1) *whether there is one, two or three equilibrium points. In case that holds*

$$B < C \quad \textit{and} \quad C^2 - f(b + 3c) < 0, \tag{3.12}$$

we have $p - q + 1 > 0$ *for every* $(f, F) \in Q_f^{(2)} = \left\{(f, F) : f > \frac{C^2}{b+3c}\right\}$ *in each equilibrium point of equation* (1.1) *whether there is one, two or three equilibrium points. If any of the following conditions holds:*

(a) $Bb + 3Bc - Cb + Cc = 0$,

(b) $C > B$ *and* $Bb + 3Bc - Cb - Cc \geq 0$,

(c) $Bb + 3Bc - Cb - Cc < 0$ *and* $Bb + 3Bc - Cb \geq 0$,

(d) $Bb + 3Bc - Cb + Cc > 0$ *and* $Bb + 3Bc - Cb < 0$,

(e) $Bb + 3Bc - Cb + Cc < 0$,

then for unique equilibrium point of equation (1.1) *holds:*

(i) $p - q + 1 = 0$ *if* $(f, F) \in \Omega_2^0$,

(ii) $p - q + 1 < 0$ *if* $(f, F) \in \Omega_2^-$,

(iii) $p - q + 1 > 0$ *if* $(f, F) \in \Omega_2^+$.

Now, we will determine the sign of $q + 1$. The zero points of the function

$$G_3(x) = -cx^2 + (B + 2C)x + f$$

are given as

$$x_{1,2}^{(3)} = \frac{B + 2C \pm \sqrt{D_3}}{2c}, \; D_3 = (B + 2C)^2 + 4fc.$$

It is obviously $x_1^{(3)} < 0$, $x_2^{(3)} > 0$ for $D_3 > 0$. Let us formulate a product

$$\Omega_3(f, F) = G\left(x_1^{(3)}\right) G\left(x_2^{(3)}\right).$$

After very complicated calculation, we obtain

$$\Omega_3 = F^2 - \frac{\left((B+2C)^2(Bb+2Cb+Cc)+cf(3Bb+2Bc+6Cb+6Cc)\right)}{c^3}F - \frac{\left((2B+3C)(Bb+2Cb+Cc)+f(b+2c)^2\right)f^2}{c^3}. \tag{3.13}$$

One can note

$$\begin{aligned}\Omega_3(f,F)>0 &\Leftrightarrow (q+1)(\overline{x})<0,\\ \Omega_3(f,F)=0 &\Leftrightarrow (q+1)(\overline{x})=0,\\ \Omega_3(f,F)<0 &\Leftrightarrow (q+1)(\overline{x})>0.\end{aligned}$$

If $\Omega_3(f,F)$ consider as quadratic function by parameter F which discriminant is

$$D_{F_{1,2}^{(3)}}=\frac{D_3(cf(b+2c)+(B+2C)(Bb+2Cb+Cc))^2}{c^6}>0,$$

then $F_1^{(3)}(f)<0$ and $F_2^{(3)}(f)>0$ where positive solution is given as

$$F_2^{(3)}(f)=\frac{\left((B+2C)^2(Bb+2Cb+Cc)+cf(3Bb+2Bc+6Cb+6Cc)\right)+(cf(b+2c)+(B+2C)(Bb+2Cb+Cc))\sqrt{D_3}}{2c^3}. \tag{3.14}$$

Function $F_2^{(3)}(f)$ separates the first quadrant in the two areas Ω_3^+ and Ω_3^- in the following way:

$$\begin{aligned}\Omega_3^0 &= \left\{(f,F): f\in(0,+\infty), F=F_2^{(3)}(f)\right\},\\ \Omega_3^- &= \left\{(f,F): f\in(0,+\infty), F\in\left(0,F_2^{(3)}(f)\right)\right\},\\ \Omega_3^+ &= Q_1\setminus\left(\Omega_3^-\cup\Omega_0^0\right).\end{aligned}$$

Now, it holds

$$\begin{aligned}\Omega_3(f,F) &= 0 \text{ if } (f,F)\in\Omega_3^0,\\ \Omega_3(f,F) &< 0 \text{ if } (f,F)\in\Omega_3^-,\\ \Omega_3(f,F) &> 0 \text{ if } (f,F)\in\Omega_3^+,\end{aligned}$$

i.e.,

$$(q+1)(\overline{x})<0\Leftrightarrow(f,F)\in\Omega_3^+, \tag{3.15}$$

$$(q+1)(\overline{x})=0\Leftrightarrow(f,F)\in\Omega_3^0, \tag{3.16}$$

$$(q+1)(\overline{x})>0\Leftrightarrow(f,F)\in\Omega_3^-. \tag{3.17}$$

Intersection with F-axis is in $\left(0,F_N^{(3)}\right)=\left(0,\frac{(B+2C)^2(Bb+2Cb+Cc)}{c^3}\right)$.

This proves the following lemma.

Lemma 3.2. *For unique equilibrium of the equation* (1.1) *holds:*

(i) $q+1=0$ *if* $(f,F)\in\Omega_3^0$,

(ii) $q+1<0$ *if* $(f,F)\in\Omega_3^+$,

(iii) $q+1>0$ *if* $(f,F)\in\Omega_3^-$.

Now, we will determine the sign of $q-1$. The zero points of function

$$G_4(x)=-(2b+3c)x^2+(B+2C)x-f$$

are of the form

$$x_{1,2}^{(4)}=\frac{B+2C\pm\sqrt{(B+2C)^2-4f(2b+3c)}}{2(2b+3c)},\ D_4=(B+2C)^2-4f(2b+3c).$$

If $D_4=(B+2C)^2-4f(2b+3c)<0$, then $G_4(x)<0$ for every x, and consequently, in every equilibrium point of the equation (1.1). The product

$$\Omega_4(f,F)=G\left(x_1^{(4)}\right)G\left(x_2^{(4)}\right)$$

is given as

$$\Omega_4=F^2+\frac{\left((B+2C)^2(Bb+2Bc+Cc)-f(2b+3c)(3Bb+6Bc+2Cb+6Cc)\right)}{(2b+3c)^3}F$$
$$-\frac{f^2\left(C(Bb+2Bc+Cc)-f(b+2c)^2\right)}{(2b+3c)^3}. \tag{3.18}$$

The discriminant of this quadratic equation by F is of the form

$$D_{F_{1,2}^{(4)}}=\frac{((B+2C)(Bb+2Bc+Cc)-f(2b+3c)(b+2c))^2D_4}{(2b+3c)^6},$$

while its zero points are given as

$$\left.\begin{aligned}F_1^{(4)}(f)&=\frac{\Gamma_1-\Gamma_2}{2(2b+3c)^3},\\ F_2^{(4)}(f)&=\frac{\Gamma_1+\Gamma_2}{2(2b+3c)^3},\end{aligned}\right\} \tag{3.19}$$

where

$$\Gamma_1=f(2b+3c)(3Bb+6Bc+2Cb+6Cc)-(B+2C)^2(Bb+2Bc+Cc),$$
$$\Gamma_2=|(B+2C)(Bb+2Bc+Cc)-f(2b+3c)(b+2c)|\sqrt{D_4}.$$

Notice that the sign of the discriminant $D_{F_{1,2}^{(4)}}$ is the same as the sign of the discriminant D_4 as well, and that holds:

$$\Omega_4(f,F)<0\Leftrightarrow(q-1)(\overline{x})>0,$$
$$\Omega_4(f,F)=0\Leftrightarrow(q-1)(\overline{x})=0,$$
$$\Omega_4(f,F)>0\Leftrightarrow(q-1)(\overline{x})<0.$$

Further,

$$\begin{aligned}
D_{F_{1,2}^{(4)}} = 0 \quad &\Leftrightarrow \quad \frac{\left((B+2C)(Bb+2Bc+Cc) - f(2b+3c)(b+2c)\right)^2 D_4}{(2b+3c)^6} = 0 \\
&\Leftrightarrow \quad D_4 = 0 \ \vee \ (B+2C)(Bb+2Bc+Cc) - f(2b+3c)(b+2c) = 0 \\
&\Leftrightarrow \quad f_D^{(4)} = \frac{(B+2C)^2}{4(2b+3c)} \ \vee \ f_P^{(4)} = \frac{(B+2C)(Bb+2Bc+Cc)}{(2b+3c)(b+2c)}.
\end{aligned}$$

Since

$$F_{1,2}^{(4)}\left(f_D^{(4)}\right) = \frac{-f(Bb+2Bc-2Cb-2Cc)}{2(2b+3c)^2}$$

and

$$F_{1,2}^{(4)}\left(f_P^{(4)}\right) = \frac{-(B+2C)(Bb+2Bc+Cc)(Bb+2Bc-2Cb-2Cc)}{2(b+2c)(2b+3c)^3},$$

the endpoint of the domain $f_D^{(4)}$ is a common point of the functions (3.19) and they touch themselves at $f_P^{(4)}$ if $0 < f_P^{(4)} < f_D^{(4)}$. The point $(0,0)$ lies in $\Omega_4(f,F) = 0$, so at least one of the functions (3.19) contains that point. Let $f_N^{(4)}$ denote the zero of the free term of quadratic function by variable F given with (3.18). We get $f_N^{(4)} = \frac{C(Bb+2Bc+Cc)}{(b+2c)^2} > 0$. First, we will find the conditions determining mutual position of the points $f_N^{(4)}$, $f_D^{(4)}$ and $f_P^{(4)}$:

$$\begin{aligned}
f_D^{(4)} - f_P^{(4)} &= \frac{(B+2C)^2}{4(2b+3c)} - \frac{(B+2C)(Bb+2Bc+Cc)}{(2b+3c)(b+2c)} = -\frac{(B+2C)(3Bb+6Bc-2Cb)}{4(2b+3c)(b+2c)}, \\
f_D^{(4)} - f_P^{(4)} &\leq 0 \Leftrightarrow 3Bb+6Bc-2Cb \geq 0, \\
f_N^{(4)} - f_P^{(4)} &= \frac{C(Bb+2Bc+Cc)}{(b+2c)^2} - \frac{(B+2C)(Bb+2Bc+Cc)}{(2b+3c)(b+2c)} = -\frac{(Bb+2Bc+Cc)^2}{(2b+3c)(b+2c)^2} < 0, \\
f_N^{(4)} - f_D^{(4)} &= \frac{C(Bb+2Bc+Cc)}{(b+2c)^2} - \frac{(B+2C)^2}{4(2b+3c)} = -\frac{(Bb+2Bc-2Cb-2Cc)^2}{4(2b+3c)(b+2c)^2} \leq 0.
\end{aligned}$$

Now, it is possible to conclude the following

$$\begin{aligned}
\Omega_4(f,F) = 0 \text{ if } (f,F) \in P_4^0, \\
\Omega_4(f,F) < 0 \text{ if } (f,F) \in P_4^-, \\
\Omega_4(f,F) > 0 \text{ if } (f,F) \in P_4^+,
\end{aligned}$$

where the sets

$$P_4^0,\ P_4^- \ \text{ and } \ P_4^+ = Q_1 \setminus \left(P_4^- \cup P_4^0\right)$$

are defined as follows:

1. If $Bb + 2Bc - 2Cb - 2Cc = 0$, then

$$\left.\begin{aligned}
P_4^0 &= A_4 \cup B_4, \\
P_4^- &= \left\{(f,F) : f \in \left(0, f_D^{(4)}\right), F \in \left(0, F_2^{(4)}(f)\right)\right\},
\end{aligned}\right\} \tag{3.20}$$

where

$$\begin{aligned} A_4 &= \left\{(f,F) : f \in \left(0, f_D^{(4)}\right], F = F_2^{(4)}(f)\right\}, \\ B_4 &= \left\{(f,F) : f \in \left(f_N^{(4)}, f_D^{(4)}\right], F = F_1^{(4)}(f)\right\}. \end{aligned}$$

2. If $Bb + 2Bc - 2Cb - 2Cc > 0$, then

$$\left.\begin{array}{l} P_4^0 = \left\{(f,F) : f \in \left(0, f_N^{(4)}\right], F = F_2^{(4)}(f)\right\}, \\ P_4^- = \left\{(f,F) : f \in \left(0, f_N^{(4)}\right), F \in \left(0, F_2^{(4)}(f)\right)\right\}. \end{array}\right\} \tag{3.21}$$

3. If $Bb + 2Bc - 2Cb - 2Cc < 0$ and $3Bb + 6Bc - 2Cb \geq 0$, then

$$P_4^0 = A_4 \cup B_4,\ P_4^- = C_4 \cup D_4, \tag{3.22}$$

where

$$\begin{aligned} A_4 &= \left\{(f,F) : f \in \left(0, f_D^{(4)}\right], F = F_2^{(4)}(f)\right\}, \\ B_4 &= \left\{(f,F) : f \in \left(f_N^{(4)}, f_D^{(4)}\right], F = F_1^{(4)}(f)\right\}, \\ C_4 &= \left\{(f,F) : f \in \left(0, f_N^{(4)}\right), F \in \left(0, F_2^{(4)}(f)\right)\right\}, \\ D_4 &= \left\{(f,F) : f \in \left(f_N^{(4)}, f_D^{(4)}\right), F \in \left(F_1^{(4)}(f), F_2^{(4)}(f)\right)\right\}. \end{aligned}$$

4. If $Bb + 2Bc - 2Cb - 2Cc < 0$ and $3Bb + 6Bc - 2Cb < 0$, then

$$P_4^0 = A_4 \cup B_4,\ P_4^- = C_4 \cup D_4, \tag{3.23}$$

where

$$\begin{aligned} A_4 &= \left\{(f,F) : f \in \left(0, f_D^{(4)}\right], F = F_2^{(4)}(f)\right\}, \\ B_4 &= \left\{(f,F) : f \in \left(f_N^{(4)}, f_D^{(4)}\right], F = F_1^{(4)}(f)\right\}, \\ C_4 &= \left\{(f,F) : f \in \left(0, f_N^{(4)}\right), F \in \left(0, F_2^{(4)}(f)\right)\right\}, \\ D_4 &= \left\{(f,F) : f \in \left(f_N^{(4)}, f_D^{(4)}\right), F \in \left(F_1^{(4)}(f), F_2^{(4)}(f)\right)\right\}. \end{aligned}$$

All sets denoted by P_4^0, P_4^- and P_4^+ in (3.20), (3.21), (3.22) and (3.23) can be unified with the following marks Ω_4^0, Ω_4^- and Ω_4^+ given as:

$$\Omega_4^0 = D_4^0 \cup G_4^0, \tag{3.24}$$

where

$$\begin{aligned} D_4^0 &= \left\{(f,F) : f \in \left(0, f_D^{(4)}\right), F = \max\left\{0, F_1^{(4)}(f)\right\}, F \neq 0\right\}, \\ G_4^0 &= \left\{(f,F) : f \in \left(0, f_D^{(4)}\right), F = \max\left\{0, F_2^{(4)}(f)\right\}, F \neq 0\right\}, \end{aligned}$$

$$\Omega_4^- = \left\{(f,F) : f \in \left(0, f_D^{(4)}\right),\right.$$

$$\left. F \in \left(\max\left\{0, F_1^{(4)}(f)\right\}, \max\left\{0, F_2^{(4)}(f)\right\}\right)\right\}, \tag{3.25}$$

$$\Omega_4^+ = Q_1 \setminus \left(\Omega_4^- \cup \Omega_4^0\right). \tag{3.26}$$

This proves the following lemma.

Lemma 3.3. *Suppose that the following condition is satisfied*

$$(B+2C)^2 - 4f(2b+3c) < 0, \tag{3.27}$$

then we have $q-1<0$ *for every* $(f,F) \in Q_f^{(4)} = \left\{(f,F) : f > \frac{(B+2C)^2}{4(2b+3c)}\right\}$ *in every equilibrium point of the equation* (1.1) *whether there is one, two or three equilibrium points. If any of the following conditions holds:*

(a) $Bb + 2Bc - 2Cb - 2Cc = 0$*,*

(b) $Bb + 2Bc - 2Cb - 2Cc > 0$*,*

(c) $Bb + 2Bc - 2Cb - 2Cc < 0$ *and* $3Bb + 6Bc - 2Cb \geq 0$*,*

(d) $Bb + 2Bc - 2Cb - 2Cc < 0$ *and* $3Bb + 6Bc - 2Cb < 0$*,*

then for unique equilibrium point of equation (1.1) *holds*

(i) $q-1=0$ *if* $(f,F) \in \Omega_4^0$*,*

(ii) $q-1>0$ *if* $(f,F) \in \Omega_4^-$*,*

(iii) $q-1<0$ *if* $(f,F) \in \Omega_4^+$*.*

Lemma 3.4. *Suppose that* $(f,F) \in \Omega_3^0$*. Then holds* $p \leq 2$*.*

Proof. Let $(f,F) \in \Omega_3^0$. Then, by using Lemma 3.2, we have $q=-1$. Inequality $p \leq 2$ i.e., $\frac{(B-\overline{x}b)\overline{x}}{(b+c)\overline{x}^2+f} \leq 2$, can be written as $-(3b+2c)\overline{x}^2 + B\overline{x} - 2f \leq 0$. Since $q=-1$, that implies $-c\overline{x}^2 + (B+2C)\overline{x} + f = 0$, i.e., $B\overline{x} = c\overline{x}^2 - 2C\overline{x} - f$, and after substitution, inequality $p \leq 2$ becomes $-(3b+c)\overline{x}^2 - 2C\overline{x} - 3f \leq 0$ which is always true. □

Specially, in the following lemma, local stability analysis of unique equilibrium point which is also stationary and inflection point of the function $G(x)$ has been done.

Lemma 3.5. *Set* $f = f_D^{(1)} = \frac{(B+C)^2}{3(b+c)}$ *and* $F_{1,2}^{(1)}\left(f_D^{(1)}\right) = \frac{(B+C)^3}{27(b+c)^2}$. *Then equation* (1.1) *has unique equilibrium point* $\overline{x}_1 = \overline{x}_2 = \overline{x}_3 = \frac{(B+C)}{3(b+c)}$ *which is nonhyperbolic point.*

Proof. For $f = f_D^{(1)}$ and $F_{1,2}^{(1)} = \frac{(B+C)^3}{27(b+c)^2}$, we have

$$G(\overline{x}) = (b+c)\,\overline{x}^3 - (B+C)\,\overline{x}^2 + \left(\frac{(B+C)^2}{3(b+c)}\right)\overline{x} - \frac{(B+C)^3}{27(b+c)^2} = -\frac{(B+C-3(b+c)\overline{x})^3}{27(b+c)^2} = 0$$

$$\Leftrightarrow \overline{x} = \frac{(B+C)}{3(b+c)}.$$

Further,

$$G_1\left(\frac{(B+C)}{3(b+c)}\right) = -3\,(b+c)\left(\frac{(B+C)}{3(b+c)}\right)^2 + 2\,(B+C)\left(\frac{(B+C)}{3(b+c)}\right) - \frac{(B+C)^2}{3(b+c)} = 0,$$

hence $p+q-1=0$ so the statement follows, i.e., the equilibrium point is nonhyperbolic point. □

Now, it is easy to formulate the theorem for local stability of the unique equilibrium point.

Theorem 3.6. *Suppose* $(f,F) \in \Omega_1^+$. *Then the equation* (1.1) *has unique equilibrium point which is:*

(i) locally asymptotically stable if $(f,F) \in \Omega_2^+ \cap \Omega_3^- \cap \Omega_4^+$,

(ii) repeller if $(f,F) \in \Omega_2^+ \cap \left(\Omega_3^+ \cup \Omega_4^-\right)$,

(iii) a saddle point if $(f,F) \in \Omega_2^-$,

(iv) a nonhyperbolic point if $(f,F) \in \Omega_2^0 \cup \Omega_3^0 \cup \left\{f_D^{(1)}, F_{1,2}^{(1)}\left(f_D^{(1)}\right)\right\}$.

Proof. The proof follows by using Lemmas 3.1–3.5. □

In order to determine the local stability in the case when two or three equilibrium points exist, we need the following lemma.

Lemma 3.7. *Let* $(f,F) \in \Omega_1^0 \cup \Omega_1^-$. *Then the equation* (1.1) *has two or three equilibrium points for which holds* $q+1>0$.

Proof. It is enough to show that $F_2^{(1)}(f) < F_2^{(3)}(f)$ holds for every $f \in \left(0, f_D^{(1)}\right)$. Before we do that, notice the following

$$\begin{aligned} F_2^{(3)}(f) &= \frac{\left((B+2C)^2(Bb+2Cb+Cc)+cf(3Bb+2Bc+6Cb+6Cc)\right)}{c^3} \\ &\quad + \frac{(cf(b+2c)(B+2C)(Bb+2Cb+Cc))\sqrt{\left((B+2C)^2+4fc\right)}}{c^3} \\ &> \frac{\left((B+2C)^2(Bb+2Cb+Cc)+cf(3Bb+2Bc+6Cb+6Cc)\right)}{c^3} \\ &\quad + \frac{(cf(b+2c)(B+2C)(Bb+2Cb+Cc))(B+2C)}{c^3} \\ &= \frac{2(B+2C)^2(Bb+2Cb+Cc)}{c^3} + \frac{2f(2Bb+2Bc+4Cb+5Cc)}{c^2}, \end{aligned}$$

and

$$\begin{aligned} F_2^{(1)}(f) &= \frac{-(B+C)\left(2(B+C)^2-9f(b+c)\right)+2\left((B+C)^2-3f(b+c)\right)\sqrt{(B+C)^2-3f(b+c)}}{27(b+c)^2} \\ &< \frac{-(B+C)\left(2(B+C)^2-9f(b+c)\right)+2\left((B+C)^2-3f(b+c)\right)(B+C)}{27(b+c)^2} = \frac{(B+C)f}{9(b+c)}, \end{aligned}$$

as well, i.e.,

$$F_2^{(3)}(f) - F_2^{(1)}(f) > \frac{\left(36Bb^2+35Bc^2+72Cb^2+89Cc^2+72Bbc+162Cbc\right)f}{9c^2(b+c)} > 0,$$

so the statement follows. □

Now we can formulate the theorem in the case, we have two equilibrium points.

Theorem 3.8. *Let* $(f, F) \in \Omega_1^0$. *Then the equation* (1.1) *has two equilibrium points of the form* (2.8) *where the equilibrium point* $\overline{x}_1 = x_1^{(1)}$ *is nonhyperbolic point, and the equilibrium point* $\overline{x}_2$ *is:*

(i) *locally asymptotically stable if* $(f, F) \in \Omega_2^+ \cap \Omega_3^- \cap \Omega_4^+$,

(ii) *repeller if* $(f, F) \in \Omega_2^+ \cap \Omega_4^-$,

(iii) *a saddle point if* $(f, F) \in \Omega_2^-$,

(iv) *a nonhyperbolic point if* $(f, F) \in \Omega_2^0 \cup \Omega_3^0$,

i.e., two equilibrium points of the form (2.9) *where now the equilibrium point* $\overline{x}_2 = x_2^{(1)}$ *is nonhyperbolic point, and the equilibrium point* $\overline{x}_1$ *has the same character as the equilibrium point* $\overline{x}_2$ *had in the previous case.*

Proof. If the equilibrium points are of the form (2.8), then, $\overline{x}_1 = x_1^{(1)}$ so $G(\overline{x}_1) = 0$, and since $x_1^{(1)}$ is stationary point also, that is $G'\left(x_1^{(1)}\right) = 0 \Leftrightarrow G_1\left(x_1^{(1)}\right) = 0$, i.e., $p + q - 1 = 0$. Hence, $|p| = |1 - q|$ which means $\overline{x}_1$ is nonhyperbolic point. Local stability analysis of the second equilibrium point follows from the proof of the Theorem 3.6 and Lemma 3.7. Similarly, if the equilibrium points are of the form (2.9), the same conclusion follows, only the equilibrium points $\overline{x}_1$ and $\overline{x}_2$ will switch their roles. □

Now we will demonstrate how the conditions for local stability looks like for specifically chosen the value of parameters f and F (see Figure 2.1).

Theorem 3.9. *Assume that* $f = \frac{(B+C)^2}{4(b+c)}$ *and* $F = \frac{(B+C)^3}{54(b+c)^2}$. *Then the equation* (1.1) *has two equilibrium points* $\overline{x}_1 = \frac{(B+C)}{6(b+c)}$ *which is nonhyperbolic point and* $\overline{x}_2 = \frac{2(B+C)}{3(b+c)}$ *which is:*

a) locally asymptotically stable if $25Bb + 57Bc - 23Cb + 9Cc > 0$, *b) a saddle point if* $25Bb+57Bc-23Cb+9Cc<0$,

c) a nonhyperbolic point if $25Bb+57Bc-23Cb+9Cc=0$.

Proof. For $f=\frac{(B+C)^2}{4(b+c)}$ and $F=\frac{(B+C)^3}{54(b+c)^2}$, we have

$$\begin{aligned} G(x) &= (b+c)x^3-(B+C)x^2+\tfrac{(B+C)^2}{4(b+c)}x-\tfrac{(B+C)^3}{54(b+c)^2} \\ &= -\tfrac{(2(B+C)-3(b+c)x)(B+C-6(b+c)x)^2}{108(b+c)^2}, \\ G(x) &= 0 \Leftrightarrow \overline{x}_1=\tfrac{(B+C)}{6(b+c)} \text{ and } \overline{x}_2=\tfrac{2(B+C)}{3(b+c)}. \end{aligned}$$

Since $G_1(\overline{x}_1)=0$, the equilibrium point $\overline{x}_1$ is nonhyperbolic point. For equilibrium point $\overline{x}_2$ holds

$$G_1(\overline{x}_2)=-\tfrac{(B+C)^2}{4(b+c)}<0,$$

which implies

$$(p+q-1)(\overline{x}_2)<0.$$

Further,

$$G_2(\overline{x}_2)=\tfrac{(B+C)(25Bb+57Bc-23Cb+9Cc)}{36(b+c)^2},$$

so it holds

$$\begin{aligned} (p-q+1)(\overline{x}_2) &< 0 \Leftrightarrow 25Bb+57Bc-23Cb+9Cc<0, \\ (p-q+1)(\overline{x}_2) &= 0 \Leftrightarrow 25Bb+57Bc-23Cb+9Cc=0, \\ (p-q+1)(\overline{x}_2) &> 0 \Leftrightarrow 25Bb+57Bc-23Cb+9Cc>0. \end{aligned}$$

Moreover, we have

$$G_3(\overline{x}_2)=\tfrac{(B+C)(33Bb+17Bc+57Cb+41Cc)}{36(b+c)^2}>0,$$

so

$$(q+1)(\overline{x}_2)>0.$$

For the function G_4

$$G_4(\overline{x}_2)=-\tfrac{(B+C)(17Bb+33Bc-7Cb+9Cc)}{36(b+c)^2},$$

so

$$\begin{aligned} (q-1)(\overline{x}_2) &< 0 \Leftrightarrow 17Bb+33Bc-7Cb+9Cc>0, \\ (q-1)(\overline{x}_2) &= 0 \Leftrightarrow 17Bb+33Bc-7Cb+9Cc=0, \\ (q-1)(\overline{x}_2) &> 0 \Leftrightarrow 17Bb+33Bc-7Cb+9Cc<0. \end{aligned}$$

One can see that if $D_4 < 0$ i.e., if $B^2b + 2B^2c - 2C^2b - C^2c + 2BCc > 0$, then is $(q-1)(\overline{x}_3) < 0$ so it must be $17Bb + 33Bc - 7Cb + 9Cc > 0$. Indeed

$$\begin{aligned} 17Bb + 33Bc - 7Cb + 9Cc &> 17Bb + 33Bc - 7Cb - 9\tfrac{B^2b+2B^2c-2C^2b-C^2c+2BCc}{2B} \\ &= \tfrac{25B^2b+48B^2c+18C^2b+9C^2c-14BCb-18BCc}{2B} \\ &= \tfrac{18B^2b+39B^2c+11C^2b+\left(7b(B-C)^2+9c(B-C)^2\right)}{2B} > 0. \end{aligned}$$

Further, if $25Bb + 57Bc - 23Cb + 9Cc > 0$, then $-Cb > -\dfrac{25Bb + 57Bc + 9Cc}{23}$, so

$$\begin{aligned} 17Bb + 33Bc - 7Cb + 9Cc &> 17Bb + 33Bc - 7\left(\tfrac{25Bb+57Bc+9Cc}{23}\right) + 9Cc \\ &= \tfrac{216Bb+360Bc+144Cc}{23} > 0, \end{aligned}$$

hence, if we assume that $25Bb+57Bc-23Cb+9Cc > 0$, then $17Bb+33Bc-7Cb+9Cc < 0$ is not possible, so we get the conclusion that equilibrium point is locally asymptotically stable, a saddle point or a nonhyperbolic point. $\square$

Local stability analysis is the most complicated in the case when three equilibrium points exist. From the previous consideration, we know the following: since the function $G(x)$ is increasing as it passes through the first equilibrium point $\overline{x}_1$, decreasing as it passes through the second equilibrium point $\overline{x}_2$ and increasing as it passes through the third equilibrium point $\overline{x}_3$, it holds

$$(p+q-1)(\overline{x}_1) < 0, (p+q-1)(\overline{x}_2) > 0, (p+q-1)(\overline{x}_3) < 0.$$

Furthermore, by using Lemma 3.7, we obtain $(q+1)(\overline{x}_i) > 0$ for every $i \in \{1,2,3\}$. Next, if $B \geq C$ or $\left(B < C \text{ and } f > f_D^{(2)}\right)$, then $(p-q+1)(\overline{x}_i) > 0$ for every $i \in \{1,2,3\}$, and if $f > f_D^{(4)}$, then $(q-1)(\overline{x}_i) < 0$ for every $i \in \{1,2,3\}$. The detailed analysis implemented in the manner as was done in the case of the unique equilibrium point could be done, in a similar way, here also, but it would be extremely complicated due to the large number of parameters. Therefore, we will below to formulate the theorem in the case of specially selected values of the parameters f and F.

Theorem 3.10. *Let* $f = \frac{(B+C)^2}{4(b+c)}$ *and* $F = \frac{(B+C)^3}{108(b+c)^2} < \frac{(B+C)^3}{54(b+c)^2}$. *Then holds:*

(a) equilibrium point $\overline{x}_1 = \frac{\left(2-\sqrt{3}\right)(B+C)}{6(b+c)}$ *is locally asymptotically stable,*

(b) equilibrium point $\overline{x}_2 = \frac{(B+C)}{3(b+c)}$ *is:*

i) repeller if $13Bb + 21Bc - 11Cb - 3Cc < 0$,

ii) nonhyperbolic if $13Bb + 21Bc - 11Cb - 3Cc = 0$,

iii) a saddle point if $B \geq C$ *or* $D_2 < 0$ *or* $13Bb + 21Bc - 11Cb - 3Cc > 0$, *(c) equilibrium point* $\overline{x}_3 = \frac{\left(2+\sqrt{3}\right)(B+C)}{6(b+c)}$ *is:*

i) locally asymptotically stable if

$$B \geq C \text{ or } D_2 < 0 \text{ or}$$
$$\tfrac{3}{2}Cc - Cb\left(2\sqrt{3}+2\right) + Bc\left(3\sqrt{3}+\tfrac{15}{2}\right) + Bb\left(\sqrt{3}+4\right) > 0,$$

ii) nonhyperbolic if

$$\frac{3}{2}Cc - Cb\left(2\sqrt{3}+2\right) + Bc\left(3\sqrt{3}+\frac{15}{2}\right) + Bb\left(\sqrt{3}+4\right) = 0,$$

iii) a saddle point if

$$\frac{3}{2}Cc - Cb\left(2\sqrt{3}+2\right) + Bc\left(3\sqrt{3}+\frac{15}{2}\right) + Bb\left(\sqrt{3}+4\right) < 0.$$

Proof. a) For equilibrium $\overline{x}_1$ is $(p+q-1)(\overline{x}_1) < 0$ and $(q+1)(\overline{x}_1) > 0$. More,

$$G_2(\overline{x}_1) = \frac{(B+C)\left(\frac{3}{2}Cc + Cb\left(2\sqrt{3}-2\right) + Bc\left(\frac{15}{2}-3\sqrt{3}\right) + Bb\left(4-\sqrt{3}\right)\right)}{9(b+c)^2} > 0$$

for every values of parameters so it holds also $(p-q+1)(\overline{x}_1) > 0$.
Since $(q-1)(\overline{x}_1) < 0$ is a consequence of

$$(p+q-1)(\overline{x}_1) < 0 \text{ and } (p-q+1)(\overline{x}_1) > 0,$$

the conclusion follows.

b) For equilibrium $\overline{x}_2$ holds $x_1^{(1)} < \overline{x}_2 < \frac{2(B+C)}{3(b+c)} = x_2^{(1)}$ so $(p+q-1)(\overline{x}_2) > 0$ and $(q+1)(\overline{x}_2) > 0$. Further,

$$G_2(\overline{x}_2) = \frac{(B+C)(13Bb+21Bc-11Cb-3Cc)}{36(b+c)^2}$$

which is positive for $B \geq C$ or if

$$D_2 = C^2 - f(b+3c) = -\frac{B^2b+3B^2c-3C^2b-C^2c+2BCb+6BCc}{4(b+c)} < 0,$$

and that is certainly true if $B^2b+3B^2c-3C^2b-C^2c+2BCb+6BCc > 0$. If

$$B^2b+3B^2c-3C^2b-C^2c+2BCb+6BCc \leq 0,$$

then

$$\begin{aligned}
(p-q+1)(\overline{x}_2) &< 0 \Leftrightarrow 13Bb+21Bc-11Cb-3Cc < 0,\\
(p-q+1)(\overline{x}_2) &= 0 \Leftrightarrow 13Bb+21Bc-11Cb-3Cc = 0,\\
(p-q+1)(\overline{x}_2) &> 0 \Leftrightarrow 13Bb+21Bc-11Cb-3Cc > 0.
\end{aligned}$$

Next,

$$G_4(\overline{x}_2) = G_4\left(\frac{(B+C)}{3(b+c)}\right) = \frac{(B+C)(-5Bb-9Bc+7Cb+3Cc)}{36(b+c)^2},$$

which means $G_4(\overline{x}_2) < 0$ if

$$D_4 = (B+2C)^2 - 4f(2b+3c) = -\frac{B^2b+2B^2c-2C^2b-C^2c+2BCc}{b+c} < 0$$

i.e., if

$$B^2b + 2B^2c - 2C^2b - C^2c + 2BCc > 0.$$

If $B^2b + 2B^2c - 2C^2b - C^2c + 2BCc \leq 0$, then holds

$$(q-1)(\overline{x}_2) < 0 \Leftrightarrow -5Bb - 9Bc + 7Cb + 3Cc < 0,$$
$$(q-1)(\overline{x}_2) = 0 \Leftrightarrow -5Bb - 9Bc + 7Cb + 3Cc = 0,$$
$$(q-1)(\overline{x}_2) > 0 \Leftrightarrow -5Bb - 9Bc + 7Cb + 3Cc > 0.$$

c) For third equilibrium point holds $x_2^{(1)} < \overline{x}_3 < \frac{2(B+C)}{3(b+c)}$, so we obtain

$$(p+q-1)(\overline{x}_3) < 0 \text{ and } (q+1)(\overline{x}_3) > 0$$

and $(q+1)(\overline{x}_3) > 0$. Beside that,

$$G_2(\overline{x}_3) = \frac{(B+C)\left(\frac{3}{2}Cc - Cb\left(2\sqrt{3}+2\right) + Bc\left(3\sqrt{3}+\frac{15}{2}\right) + Bb\left(\sqrt{3}+4\right)\right)}{9(b+c)^2},$$

so for $B \geq C$, we have $G_2(\overline{x}_3) > 0$, or if

$$D_2 = C^2 - f(b+3c) = -\frac{B^2b+3B^2c-3C^2b-C^2c+2BCb+6BCc}{4(b+c)} < 0,$$

which is surely true if $B^2b + 3B^2c - 3C^2b - C^2c + 2BCb + 6BCc > 0$. If

$$B^2b + 3B^2c - 3C^2b - C^2c + 2BCb + 6BCc \leq 0,$$

then

$$(p-q+1)(\overline{x}_2)\begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \Longleftrightarrow$$

$$\frac{3}{2}Cc - Cb\left(2\sqrt{3}+2\right) + Bc\left(3\sqrt{3}+\frac{15}{2}\right) + Bb\left(\sqrt{3}+4\right)\begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$

Next,

$$G_4(\overline{x}_3) = \frac{(B+C)\left(-6Cc + Cb\left(4\sqrt{3}+1\right) - Bb\left(2\sqrt{3}+11\right) - Bc\left(6\sqrt{3}+18\right)\right)}{36(b+c)^2},$$

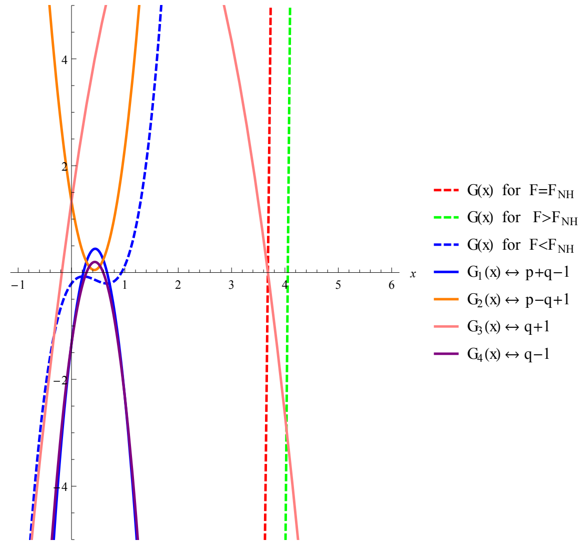

Figure 3.1: The position of the function $G(x)$ (in the case of the unique equilibrium point when the equilibrium point is smaller, equal or bigger then the positive zero of the function $G_3(x)$) relative to the functions $G_1(x)$, $G_2(x)$, $G_3(x)$ and $G_4(x)$ for the following values of the parameters:
$B = 1$, $C = 3$, $b = 1$, $c = 2$, $f = \dfrac{4}{3}$, $F_{NH} = \frac{313\sqrt{537}}{144} + \frac{2399}{48}$.

so it holds $G_4(\overline{x}_3) < 0$ if

$$D_4 = (B + 2C)^2 - 4f(2b + 3c) = -\frac{B^2b+2B^2c-2C^2b-C^2c+2BCc}{b+c} < 0$$

i.e., if $B^2b+2B^2c-2C^2b-C^2c+2BCc > 0$. If $B^2b+2B^2c-2C^2b-C^2c+2BCc \leq 0$, then

$$(q-1)(\overline{x}_2) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \iff$$

$$-6Cc + Cb\left(4\sqrt{3}+1\right) - Bb\left(2\sqrt{3}+11\right) - Bc\left(6\sqrt{3}+18\right) \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$

and this concludes the proof. □

4 Case $F = 0$

This section gives complete local stability analysis for all equilibrium points (up to three) of (1.1) when $F = 0$. In this case (1.1) has always the zero equilibrium.

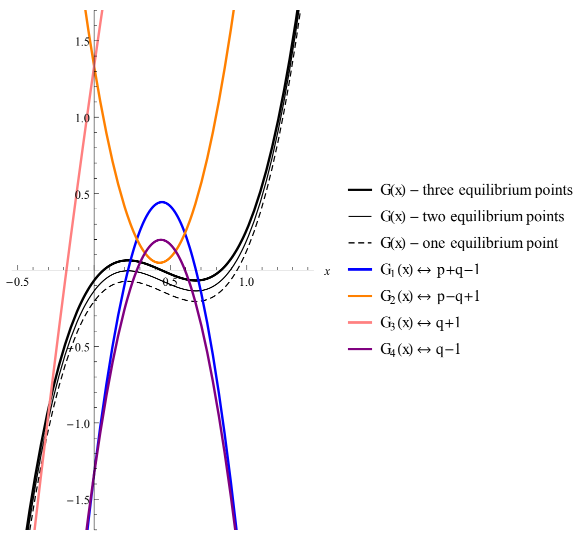

Figure 3.2: The position of the function $G(x)$ (in the case when we have one, two or three equilibrium points) relative to the functions $G_1(x)$, $G_2(x)$, $G_3(x)$ and $G_4(x)$ for $B = 1$, $C = 3$, $b = 1$, $c = 2$, and f and F in a specifically way calculated.

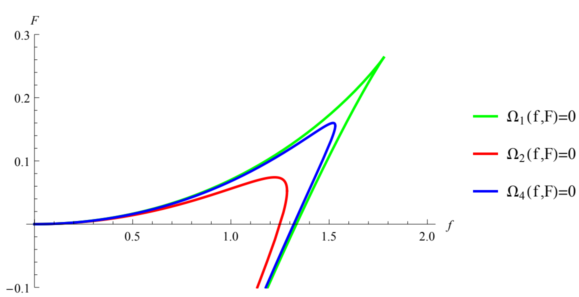

Figure 3.3: Mutual position of the function $\Omega_1(f, F)$, $\Omega_2(f, F)$, $\Omega_3(f, F)$ and $\Omega_4(f, F)$ for $B = 1$, $C = 3$, $b = 1$ i $c = 2$. Notice that for specifically chosen values of the parameters, the function $\Omega_3(f, F)$ is significantly above all other functions.

4.1 Equilibrium Points

Equilibrium points of (1.1) are the positive solutions of the equation

$$\overline{x} = \frac{(B+C)\,\overline{x}^2}{(b+c)\,\overline{x}^2 + f},$$

or equivalently

$$(b+c)\,\overline{x}^3 - (B+C)\,\overline{x}^2 + f\overline{x} = 0.$$

One can see that for $F = 0$, the equation (1.1) has:

1. unique equilibrium point $\overline{x} = 0$ if $f > \frac{(B+C)^2}{4(b+c)}$,
2. two equilibrium points $\overline{x} = 0$ and $\overline{x} = \frac{(B+C)}{2(b+c)}$ if $f = \frac{(B+C)^2}{4(b+c)}$,
3. three equilibrium points $x = 0$ and $x_{\pm} = \frac{(B+C) \pm \sqrt{(B+C)^2 - 4(b+c)f}}{2(b+c)}$ if $f < \frac{(B+C)^2}{4(b+c)}$.

4.2 Local Stability Analysis

Denote as

$$g(u,v) = \frac{Buv + Cv^2}{buv + cv^2 + f}.$$

The linearized equation of (1.1) for $F = 0$ is of the form

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial g}{\partial u}(\overline{x},\overline{x}) = \overline{x}\frac{Bf - Fb + (Bc - Cb)\overline{x}^2}{\left(f + (c+b)\overline{x}^2\right)^2},$$
$$q = \frac{\partial g}{\partial v}(\overline{x},\overline{x}) = \frac{(Bf - Fb + 2Cf - 2Fc)\overline{x} + (Cb - Bc)\overline{x}^3}{\left(f + (c+b)\overline{x}^2\right)^2}.$$

Now we have the following result on local stability of the equilibrium points.

Theorem 4.1. *(a) The equilibrium point $\overline{x} = 0$ is locally asymptotically stable for every values of parameters.*

(b) If $f = \frac{(B+C)^2}{4(b+c)}$, then the equilibrium point $\overline{x} = \frac{(B+C)}{2(b+c)}$ is nonhyperbolic point.

(c) If $f < \frac{(B+C)^2}{4(b+c)}$, then the equilibrium point $\overline{x}_-$ is:

(i) repeller if any of the following condition is satisfied:

1. $B = 0$,

2. $Bb + 2Bc - Cb \leq 0$ *and* $B < C$,
3. $Bb + 2Bc - Cb > 0$ *and* $3cB^2 + bB^2 - 2cBC - 2bBC - cC^2 + bC^2 + 4fc^2 < 0$,

(ii) a saddle point if any of the following condition is satisfied:

1. $B \geq C$,
2. $Bb + 2Bc - Cb > 0$ *and* $3cB^2 + bB^2 - 2cBC - 2bBC - cC^2 + bC^2 + 4fc^2 > 0$,

(iii) a nonhyperbolic point if holds

$$Bb+2Bc-Cb > 0 \text{ and } 3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2 = 0,$$

and the equilibrium point $\overline{x}_+$ *is:*

(i) locally asymptotically stable if any of the following condition is satisfied:

1. $B \geq C$,
2. $Bb + 2Bc - Cb \geq 0$,
3. $Bb+2Bc-Cb < 0$ *and* $3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2 < 0$,

(ii) a saddle point if holds

$$Bb + 2Bc - Cb < 0 \text{ and } 3cB^2 + bB^2 - 2cBC - 2bBC - cC^2 + bC^2 + 4fc^2 > 0,$$

(iii) a nonhyperbolic point if holds

$$Bb + 2Bc - Cb < 0 \text{ and } 3cB^2 + bB^2 - 2cBC - 2bBC - cC^2 + bC^2 + 4fc^2 = 0.$$

Proof. a) For $\overline{x} = 0$, we have $p = q = 0$ so the zero equilibrium is always locally asymptotically stable.

b) For $\overline{x} \neq 0$, we have $\overline{x} = \frac{(B+C)\overline{x}^2}{(b+c)\overline{x}^2+f}$ i.e., $\frac{1}{B+C} = \frac{\overline{x}}{(b+c)\overline{x}^2+f}$, which implies

$$p = g_u(\overline{x}, \overline{x}) = \frac{(B-\overline{x}b)\overline{x}}{(b+c)\overline{x}^2+f} = \frac{B-\overline{x}b}{B+C},$$
$$q = g_v(\overline{x}, \overline{x}) = \frac{\overline{x}(B+2C-(b+2c)\overline{x})}{f+(b+c)\overline{x}^2} = \frac{B+2C-(b+2c)\overline{x}}{B+C}.$$

Now,

$$\begin{aligned} p+q-1 &= \frac{B-\overline{x}b}{B+C} + \frac{B+2C-(b+2c)\overline{x}}{B+C} - 1 = \frac{B+C-2(b+c)\overline{x}}{B+C}, \\ p-q+1 &= \frac{B-\overline{x}b}{B+C} - \frac{B+2C-(b+2c)\overline{x}}{B+C} + 1 = \frac{B-C+2c\overline{x}}{B+C}, \end{aligned}$$

$$\begin{aligned} q+1 &= \tfrac{B+2C-(b+2c)\overline{x}}{B+C}+1=\tfrac{2B+3C-(b+2c)\overline{x}}{B+C}, \\ q-1 &= \tfrac{B+2C-(b+2c)\overline{x}}{B+C}-1=\tfrac{C-(b+2c)\overline{x}}{B+C}. \end{aligned}$$

Since $(p+q-1)\left(\frac{(B+C)}{2(b+c)}\right)=\frac{B+C-2(b+c)\frac{(B+C)}{2(b+c)}}{B+C}=0$, that is $|p|=|1-q|$ which means that the equilibrium point is nonhyperbolic.

c) One can see that the function $G(x)$ is decreasing as it passes through the equilibrium point $\overline{x}_-$, which implies $p+q-1>0$, and $p-q+1>0$ for $B\geq C$. It holds:

$(p+q-1)(\overline{x}_-)=\frac{\sqrt{(B+C)^2-4(b+c)f}}{B+C}>0$,

$(p-q+1)(\overline{x}_-)=\frac{Bb+2Bc-Cb-c\sqrt{(B+C)^2-4(b+c)f}}{(B+C)(b+c)}$,

$(q+1)(\overline{x}_-)=\frac{(b+2c)\sqrt{(B+C)^2-4(b+c)f}+3Bb+2Bc+5Cb+4Cc}{2(B+C)(b+c)}>0$,

$(q-1)(\overline{x}_-)=\frac{-(Bb+2Bc-Cb)+(b+2c)\sqrt{(B+C)^2-4(b+c)f}}{2(B+C)(b+c)}$.

i) If $B=0$, or $Bb+2Bc-Cb\leq 0$ and $B<C$, or $Bb+2Bc-Cb>0$ and $3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2<0$, then $(p-q+1)(\overline{x}_-)<0$ and $(q-1)(\overline{x}_-)>0$, hence $|p|<|1-q|$ and $q>1$ so the equilibrium point $\overline{x}_-$ is repeller.

ii) If $B\geq C$, or $Bb+2Bc-Cb>0$ and $3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2>0$, then $(p-q+1)(\overline{x}_-)>0$ so it holds $|p|>|1-q|$ and the equilibrium point $\overline{x}_-$ in that case is a saddle point.

iii) If $Bb+2Bc-Cb>0$ and $3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2=0$, then $(p-q+1)(\overline{x}_-)=0$ so the equilibrium point is nonhyperbolic.

Notice that the function $G(x)$ is increasing as it passes through the equilibrium point $\overline{x}_+$, which implies $p+q-1<0$, and in this case also is $p-q+1>0$ for $B\geq C$. Indeed,

$$\begin{aligned} p+q-1 &= -\tfrac{\sqrt{(B+C)^2-4(b+c)f}}{B+C}<0, \\ p-q+1 &= \tfrac{Bb+2Bc-Cb+c\sqrt{(B+C)^2-4(b+c)f}}{(B+C)(b+c)}, \\ q+1 &= \tfrac{-(b+2c)\sqrt{(B+C)^2-4(b+c)f}+3Bb+2Bc+5Cb+4Cc}{2(B+C)(b+c)}>\tfrac{Bb+2Cb+Cc}{(B+C)(b+c)}>0, \\ q-1 &= -\tfrac{Bb+2Bc-Cb+(b+2c)\sqrt{(B+C)^2-4(b+c)f}}{2(B+C)(b+c)}. \end{aligned}$$

Now,

i) if one of the conditions holds $B \geq C$, or $Bb + 2Bc - Cb \geq 0$, or $Bb + 2Bc - Cb < 0$ and $3cB^2 + bB^2 - 2cBC - 2bBC - cC^2 + bC^2 + 4fc^2 < 0$, then $(p - q + 1)(\overline{x}_+) > 0$ and $(q - 1)(\overline{x}_+) < 0$ so the equilibrium point is locally asymptotically stable,

ii) if $Bb+2Bc-Cb < 0$ and $3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2 > 0$, then $(p - q + 1)(\overline{x}_+) < 0$ which implies $|p| > |1 - q|$, so the equilibrium point $\overline{x}_+$ is a saddle point,

iii) if $Bb+2Bc-Cb < 0$ and $3cB^2+bB^2-2cBC-2bBC-cC^2+bC^2+4fc^2 = 0$, then $(p - q + 1)(\overline{x}_+) = 0$ which means $|p| = |1 - q|$, and the equilibrium point $\overline{x}_+$ is nonhyperbolic point.

□

5 Global Asymptotic Stability Results

In this section, we give some global asymptotic stability results for some special cases of (1.1).

Theorem 5.1. *Consider* (1.1)*, where all coefficients are positive, subject to the condition*

$$\frac{(|B - b\overline{x}| + |C - c\overline{x}|)(U + \overline{x})}{(b + c)L^2 + f} < 1, \tag{5.1}$$

where $L = \frac{\min\{B,C,F\}}{\max\{b,c,f\}}$, $U = \frac{\max\{B,C,F\}}{\min\{b,c,f\}}$ *and assume the hypotheses of Theorem 2.1* i). *Then the equilibrium* $\overline{x}$ *is globally asymptotically stable.*

Proof. In view of Corollary 1.3, we need to find the lower and upper bounds for all solutions of (1.1), for $n \geq 1$. In this case the lower and upper bounds for all solutions of (1.1) for $n \geq 1$ are derived as:

$$x_{n+1} \geq \frac{\min\{B, C, F\}(x_n x_{n-1} + x_{n-1}^2 + 1)}{\max\{b, c, f\}(x_n x_{n-1} + x_{n-1}^2 + 1)} \geq \frac{\min\{B, C, F\}}{\max\{b, c, f\}} = L.$$

$$x_{n+1} \leq \frac{\max\{B, C, F\}(x_n x_{n-1} + x_{n-1}^2 + 1)}{\min\{b, c, f\}(x_n x_{n-1} + x_{n-1}^2 + 1)} \leq \frac{\max\{B, C, F\}}{\min\{b, c, f\}} = U.$$

□

Theorem 5.2. *Consider* (1.1)*, where* $F = 0$ *and all other coefficients are positive, subject to the condition* $f > \frac{\max\{B,C\}(B+C)}{\min\{b,c\}}$. *Then the unique equilibrium* $\overline{x} = 0$ *is globally asymptotically stable.*

Proof. For $f > \frac{\max\{B,C\}(B+C)}{\min\{b,c\}} > \frac{(B+C)^2}{4(b+c)}$ the equation (1.1) has unique equilibrium point $\overline{x} = 0$. The lower and upper bounds for all solutions of (1.1) for $n \geq 1$ are derived as:

$$\begin{aligned} L = 0 \leq x_{n+1} &= \frac{Bx_n x_{n-1} + Cx_{n-1}^2}{bx_n x_{n-1} + cx_{n-1}^2 + f} \leq \frac{Bx_n x_{n-1} + Cx_{n-1}^2}{bx_n x_{n-1} + cx_{n-1}^2} \\ &\leq \frac{\max\{B,C\}(x_n x_{n-1} + x_{n-1}^2)}{\min\{b,c\}(x_n x_{n-1} + x_{n-1}^2)} = \frac{\max\{B,C\}}{\min\{b,c\}} = U, \end{aligned}$$

an application of Corollary 1.3 completes the proof. □

Theorem 5.3. *Consider* (1.1)*, where $f = 0$ and all other coefficients are positive, subject to the condition*

$$\left(|B - b\overline{x}| + |C - c\overline{x}|\right)\left(U + \overline{x}\right) < \left(b + c\right) L^2, \tag{5.2}$$

where $L = \frac{\min\{B,C\}}{\max\{B,C\}}$ and $U = \frac{\max\{B,C\}}{\min\{b,c\}} + \frac{F}{(b+c)L^2}$. Then the equilibrium $\overline{x}$ is globally asymptotically stable.

Proof. In this case, by using Theorem 2.1 i) equation (1.1) has unique equilibrium point. The lower and upper bounds for all solutions of (1.1) for $n \geq 1$ are derived as

$$\begin{aligned} x_{n+1} &= \frac{Bx_n x_{n-1} + Cx_{n-1}^2 + F}{bx_n x_{n-1} + cx_{n-1}^2} \geq \frac{Bx_n x_{n-1} + Cx_{n-1}^2}{bx_n x_{n-1} + cx_{n-1}^2} \\ &\geq \frac{\min\{B,C\}(x_n x_{n-1} + x_{n-1}^2)}{\max\{b,c\}(x_n x_{n-1} + x_{n-1}^2)} = \frac{\min\{B,C\}}{\max\{b,c\}} = L. \end{aligned}$$

$$\begin{aligned} x_{n+1} &= \frac{Bx_n x_{n-1} + Cx_{n-1}^2 + F}{bx_n x_{n-1} + cx_{n-1}^2} = \frac{Bx_n x_{n-1} + Cx_{n-1}^2}{bx_n x_{n-1} + cx_{n-1}^2} + \frac{F}{bx_n x_{n-1} + cx_{n-1}^2} \\ &\leq \frac{\max\{B,C\}}{\min\{b,c\}} + \frac{F}{(b+c)\,L^2} = U. \end{aligned}$$

Now, an application of Corollary 1.3 completes the proof. □

Theorem 5.4. *Consider the equation* (1.1)*, where $B = 0$ and all other coefficients are positive, subject to the condition*

$$\frac{\left(b\overline{x} + |C - c\overline{x}|\right)\left(U + \overline{x}\right)}{\left(b + c\right) L^2 + f} < 1, \tag{5.3}$$

where $L = 0$, $U = \frac{\max\{C,F\}}{\min\{c,f\}}$ and one of the conditions holds

1. $f > \frac{C^2}{3(b+c)}$,

2. $f = \frac{C^2}{3(b+c)}$,
3. $f < \frac{C^2}{3(b+c)}$ and $4FC^3 - f^2C^2 - 18FCf\,(b+c) + 27F^2\,(b+c)^2 + 4\,(b+c)\,f^3 > 0$.

Then the equilibrium $\overline{x}$ is globally asymptotically stable.

Proof. The lower and upper bounds for all solutions of (1.1) for $n \geq 1$ are derived as:

$$L = 0 \leq x_{n+1} = \frac{Cx_{n-1}^2 + F}{bx_nx_{n-1} + cx_{n-1}^2 + f} \leq \frac{Cx_{n-1}^2 + F}{cx_{n-1}^2 + f} \leq \frac{\max\{C, F\}}{\min\{c, f\}} = U,$$

and by using Theorem 2.1 and Corollary 1.3 the conclusion follows. □

Theorem 5.5. *Consider the equation* (1.1)*, where* $C = 0$ *and all other coefficients are positive, subject to the condition*

$$\frac{(|B - b\overline{x}| + c\overline{x})\,(U + \overline{x})}{(b+c)\,L^2 + f} < 1, \tag{5.4}$$

where $L = 0$, $U = \frac{\max\{C,F\}}{\min\{c,f\}}$ *and one of the conditions holds*

1. $f > \frac{B^2}{3(b+c)}$,
2. $f = \frac{B^2}{3(b+c)}$,
3. $f < \frac{B^2}{3(b+c)}$ and $4FB^3 - f^2B^2 - 18FBf\,(b+c) + 27F^2\,(b+c)^2 + 4\,(b+c)\,f^3 > 0$.

Then the equilibrium $\overline{x}$ is globally asymptotically stable.

Proof. The lower and upper bounds for all solutions of (1.1) for $n \geq 1$ are derived as:

$$L = 0 \leq x_{n+1} = \frac{Bx_nx_{n-1} + F}{bx_nx_{n-1} + cx_{n-1}^2 + f} \leq \frac{Bx_nx_{n-1} + F}{bx_nx_{n-1} + f} \leq \frac{\max\{B, F\}}{\min\{b, f\}} = U,$$

and by using Theorem 2.1 and Corollary 1.3 the conclusion follows. □

Theorem 5.6. *Consider the equation* (1.1)*, where* $C = F = 0$ *and all other coefficients are positive, subject to the condition* $f > \frac{B^2}{b}$*. Then the unique equilibrium* $\overline{x} = 0$ *is globally asymptotically stable.*

Proof. The lower and upper bounds for all solutions of (1.1) are derived as:

$$L = 0 \leq x_{n+1} = \frac{Bx_nx_{n-1}}{bx_nx_{n-1} + cx_{n-1}^2 + f} \leq \frac{Bx_nx_{n-1}}{bx_nx_{n-1}} = \frac{B}{b} = U,$$

$f > \frac{B^2}{b} > \frac{B^2}{3(b+c)}$ and an application of Corollary 1.3 completes the proof. □

Theorem 5.7. *Consider the equation* (1.1)*, where* $B = F = 0$ *and all other coefficients are positive, subject to the condition* $f > \frac{C^2}{b}$*. Then the unique equilibrium* $\overline{x} = 0$ *is globally asymptotically stable.*

Proof. The lower and upper bounds for all solutions of (1.1) are derived as:

$$L = 0 \leq x_{n+1} = \frac{Cx_{n-1}^2}{bx_n x_{n-1} + cx_{n-1}^2 + f} \leq \frac{Cx_{n-1}^2}{cx_{n-1}^2} = \frac{C}{c} = U,$$

$f > \frac{C^2}{b} > \frac{C^2}{3(b+c)}$ and an application of Corollary 1.3 completes the proof. □

Theorem 5.8. *Consider the equation* (1.1)*, where* $B = C = 0$ *and all other coefficients are positive, subject to the condition*

$$\frac{(b+c)(U+\overline{x})\overline{x}}{(b+c)\,L^2 + f} < 1, \tag{5.5}$$

where $L = 0$ *and* $U = \frac{F}{f}$*. Then the equilibrium* $\overline{x}$ *is globally asymptotically stable.*

Proof. The lower and upper bounds for all solutions of (1.1) are derived as:

$$L = 0 \leq x_{n+1} = \frac{F}{bx_n x_{n-1} + cx_{n-1}^2 + f} \leq \frac{F}{f} = U,$$

and an application of Corollary 1.3 completes the proof. □

Theorem 5.9. *Consider the equation* (1.1)*, where* $b = B = 0$ *and all other coefficients are positive, subject to the condition*

$$\frac{|C - c\overline{x}|(U+\overline{x})}{cL^2 + f} < 1, \tag{5.6}$$

where $L = \frac{\min\{C,F\}}{\max\{c,f\}}$*,* $U = \frac{\max\{C,F\}}{\min\{c,f\}}$ *and one of the following conditions holds:*

1. $f > \frac{C^2}{3c}$,
2. $f = \frac{C^2}{3c}$,
3. $f < \frac{C^2}{3c}$ and $4FC^3 - f^2C^2 - 18FfcC + 27F^2c^2 + 4cf^3 > 0$.

Then the equilibrium $\overline{x}$ is globally asymptotically stable.

Proof. The lower and upper bounds for all solutions of (1.1) in this case are derived as:

$$x_{n+1} = \frac{Cx_{n-1}^2 + F}{cx_{n-1}^2 + f} \geq \frac{\min\{C,F\}}{\max\{c,f\}} = L,$$

$$x_{n+1} = \frac{Cx_{n-1}^2 + F}{cx_{n-1}^2 + f} \leq \frac{\max\{C,F\}}{\min\{c,f\}} = U,$$

and by using Theorem 2.1 and Corollary 1.3 the conclusion follows. □

Theorem 5.10. *Consider the equation* (1.1)*, where* $c = C = 0$ *and all other coefficients are positive, subject to the condition*

$$\frac{|B - b\overline{x}|(U + \overline{x})}{bL^2 + f} < 1, \tag{5.7}$$

where $L = \frac{\min\{B,F\}}{\max\{b,f\}}$, $U = \frac{\max\{B,F\}}{\min\{b,f\}}$ *and:*

1. $f > \frac{B^2}{3b}$,
2. $f = \frac{B^2}{3b}$ or
3. $f < \frac{B^2}{3b}$ and $4FB^3 - f^2B^2 - 18FfbB + 27F^2b^2 + 4bf^3 > 0$.

Then the equilibrium $\overline{x}$ is globally asymptotically stable.

Proof. In this case, the lower and upper bounds for all solutions of (1.1) are derived as:

$$x_{n+1} = \frac{Bx_nx_{n-1} + F}{bx_nx_{n-1} + f} \geq \frac{\min\{B, F\}}{\max\{b, f\}} = L,$$

$$x_{n+1} = \frac{Bx_nx_{n-1} + F}{bx_nx_{n-1} + f} \leq \frac{\max\{B, F\}}{\min\{b, f\}} = U,$$

and by using Theorem 2.1 and Corollary 1.3 the conclusion follows. □

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