Periodic Solutions to a Class of Mixed Max-Type Nonlinear Difference Equations

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Abstract

We investigate the closed form solutions of a certain system of nonlinear mixed max-type difference equations. Under certain conditions, we show that the solutions to the system are periodic. Furthermore, we give graphical evidence that verifies the periodicity of the system being analyzed.

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1 Introduction

Difference equations are pervasive in mathematics and understanding the behavior of such equations gives insight to many interesting problems, see [4, 7, 11, 16]. Studying the periodic nature of certain difference equations has attracted many authors, see [1–3, 5, 6, 8–10, 12–14].

In 2015, Nouressadat Touafek and Nabila Haddad studied the closed form periodic solutions in [15] to the following mixed max-type rational system of difference equations

\[ x_{n+1} = \frac{x_n y_n}{y_{n-1}}, \quad y_{n+1} = \max \left\{ \frac{A_n}{x_n}, y_{n-1} \right\}. \]

We study the periodic solutions of the system of difference equations

\[ x_{n+1} = \frac{f(y_n)}{x_n}, \quad y_{n+1} = \max \left\{ x_n^2, \frac{A}{x_n} \right\} \quad \text{for } n \in N_0, \tag{1.1} \]

where \( x_{-1} = \alpha, y_{-1} = \beta, x_0 = \lambda, \) and \( y_0 = \mu \) are positive numbers.

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2 Assumptions

The function $f$ will have one of the following forms:

\[ f(z) = 1, \quad (2.1) \]
\[ f(z) = \begin{cases} 
B, & \text{if } z > 0 \\
C, & \text{if } z < 0
\end{cases}, \quad (2.2) \]
\[ f(z) = \begin{cases} 
B, & \text{if } z > 0 \\
Cz, & \text{if } z < 0
\end{cases}, \quad (2.3) \]

where $B, C \in \mathbb{R}$ such that $B^2 + C^2 \neq 0$.

3 Main Results

**Theorem 3.1.** Assume that (2.1) holds with $0 < x_{-1}, y_{-1}, x_0, y_0 < A < 1$. Also, let \( \{x_n, y_n\} \) be a solution of the system of equations (1.1) with $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, and $y_0 = \mu$. Then all solutions of (1.1) are of the following:

\[
x_{4n-3} = \frac{1}{\alpha}, \quad y_{4n-3} = \frac{A}{\alpha} \\
x_{4n-2} = \frac{1}{\lambda}, \quad y_{4n-2} = \frac{A}{\lambda} \\
x_{4n-1} = \alpha, \quad y_{4n-1} = \frac{1}{\alpha^2} \\
x_{4n} = \lambda, \quad y_{4n} = \frac{1}{\lambda^2}.
\]

**Proof.** For $n = 1$, we have

\[
x_1 = \frac{f(y_0)}{x_{-1}} = \frac{1}{\alpha}, \quad y_1 = \max \left\{ x_{-1}^2, \frac{A}{x_{-1}} \right\} = \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \frac{A}{\alpha} \\
x_2 = \frac{f(y_0)}{x_0} = \frac{1}{\lambda}, \quad y_2 = \max \left\{ x_0^2, \frac{A}{x_0} \right\} = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \frac{A}{\lambda} \\
x_3 = \frac{f(y_2)}{x_1} = \frac{1}{1/\alpha} = \alpha, \quad y_3 = \max \left\{ x_1^2, \frac{A}{x_1} \right\} = \max \left\{ \frac{1}{\alpha^2}, A\alpha \right\} = \frac{1}{\alpha^2} \\
x_4 = \frac{f(y_3)}{x_2} = \frac{1}{1/\lambda} = \lambda, \quad y_4 = \max \left\{ x_2^2, \frac{A}{x_2} \right\} = \max \left\{ \frac{1}{\lambda^2}, A\lambda \right\} = \frac{1}{\lambda^2}.
\]
So the result holds for $n = 1$. Now suppose the result is true for some $k \in \mathbb{N}$, that is,

$$
\begin{align*}
  x_{4k-3} &= \frac{1}{\alpha}, & y_{4k-3} &= \frac{A}{\alpha} \\
  x_{4k-2} &= \frac{1}{\lambda}, & y_{4k-2} &= \frac{A}{\lambda} \\
  x_{4k-1} &= \alpha, & y_{4k-1} &= \frac{1}{\alpha^2} \\
  x_{4k} &= \lambda, & y_{4k} &= \frac{1}{\lambda^2}.
\end{align*}
$$

Then, for $k + 1$ we have the following:

$$
\begin{align*}
  x_{4k+1} &= f(y_{4k}) = \frac{1}{\alpha}, & y_{4k+1} &= \max \left\{ x_{4k+1}^2, \frac{A}{x_{4k+1}} \right\} = \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \frac{A}{\alpha} \\
  x_{4k+2} &= f(y_{4k+1}) = \frac{1}{\lambda}, & y_{4k+2} &= \max \left\{ x_{4k+2}^2, \frac{A}{x_{4k+2}} \right\} = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \frac{A}{\lambda} \\
  x_{4k+3} &= f(y_{4k+2}) = \frac{1}{1/\alpha} = \alpha \\
  y_{4k+3} &= \max \left\{ x_{4k+3}^2, \frac{A}{x_{4k+3}} \right\} = \max \left\{ \frac{1}{\alpha^2}, A\alpha \right\} = \frac{1}{\alpha^2} \\
  x_{4k+4} &= f(y_{4k+3}) = \frac{1}{1/\lambda} = \lambda \\
  y_{4k+4} &= \max \left\{ x_{4k+4}^2, \frac{A}{x_{4k+4}} \right\} = \max \left\{ \frac{1}{\lambda^2}, A\lambda \right\} = \frac{1}{\lambda^2}.
\end{align*}
$$

Therefore the result is true for every $k \in \mathbb{N}$. This concludes the proof. \qed

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.1 with $x_1 = 1/2$, $x_2 = 1/3$, $y_1 = 1/4$, $y_2 = 1/5$, and $A = 3/4$. 

Theorem 3.2. Assume that (2.2) holds with $B, C < 0$ and $0 < x_{-1}, y_{-1}, x_0, y_0 < A < 1$. Also, let $\{x_n, y_n\}$ be a solution of the system of equations (1.1) with $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, and $y_0 = \mu$. Then all solutions of (1.1) are of the following:

$$
\begin{align*}
    x_{4n-3} &= \frac{B}{\alpha}, \quad y_{4n-3} = \frac{A}{\alpha} \\
    x_{4n-2} &= \frac{B}{\lambda}, \quad y_{4n-2} = \frac{A}{\lambda} \\
    x_{4n-1} &= \alpha, \quad y_{4n-1} = \left(\frac{B}{\alpha}\right)^2 \\
    x_{4n} &= \lambda, \quad y_{4n} = \left(\frac{B}{\lambda}\right)^2.
\end{align*}
$$

Proof. For $n = 1$, we have

$$
\begin{align*}
    x_1 &= \frac{f(y_0)}{x_1} = \frac{B}{\alpha}, \quad y_1 = \max\left\{x_{-1}, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha, \frac{A}{\alpha}\right\} = \frac{A}{\alpha} \\
    x_2 &= \frac{f(y_0)}{x_1} = \frac{B}{\lambda}, \quad y_2 = \max\left\{x_{-1}, \frac{A}{x_{-1}}\right\} = \max\left\{\lambda, \frac{A}{\lambda}\right\} = \frac{A}{\lambda} \\
    x_3 &= \frac{f(y_2)}{x_1} = \frac{B}{B/\alpha} = \alpha, \quad y_3 = \max\left\{x_{-1}, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha, \frac{A}{\alpha}\right\} = \left(\frac{B}{\alpha}\right)^2 \\
    x_4 &= \frac{f(y_3)}{x_3} = \frac{B}{B/\lambda} = \lambda, \quad y_4 = \max\left\{x_{-1}, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha, \frac{A}{\alpha}\right\} = \left(\frac{B}{\lambda}\right)^2.
\end{align*}
$$

So the result holds for $n = 1$. Now suppose the result is true for some $k \in \mathbb{N}$, that is,

$$
\begin{align*}
    x_{4k-3} &= \frac{B}{\alpha}, \quad y_{4k-3} = \frac{A}{\alpha} \\
    x_{4k-2} &= \frac{B}{\lambda}, \quad y_{4k-2} = \frac{A}{\lambda} \\
    x_{4k-1} &= \alpha, \quad y_{4k-1} = \left(\frac{B}{\alpha}\right)^2 \\
    x_{4k} &= \lambda, \quad y_{4k} = \left(\frac{B}{\lambda}\right)^2.
\end{align*}
$$

Then, for $k + 1$ we have the following:

$$
\begin{align*}
    x_{4k+1} &= \frac{f(y_{4k})}{x_{4k}} = \frac{B}{\alpha}, \quad y_{4k+1} = \max\left\{x_{4k-1}, \frac{A}{x_{4k-1}}\right\} = \max\left\{\alpha, \frac{A}{\alpha}\right\} = \frac{A}{\alpha}
\end{align*}
$$
Therefore the result is true for every $k \in \mathbb{N}$. This concludes the proof. \hfill \Box

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.2 with $x_1 = 1/2$, $x_2 = 1/3$, $y_1 = 1/4$, $y_2 = 1/5$, $A = 3/4$, $B = -2$, and $C = -1$.

**Theorem 3.3.** Assume that (2.2) holds with $A, B, C > 0$ and $x_{-1}, y_{-1}, x_0, y_0 \leq 0$. Also, let $\{x_n, y_n\}$ be a solution of the system of equations (1.1) with $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, and $y_0 = \mu$. Then all solutions of (1.1) are of the following:

\[ y_1 = \alpha^2. \]

For $n \in \mathbb{N}$,

\[ x_{4n-3} = \frac{C}{\alpha}, \quad y_{4n-2} = \lambda^2 \]

\[ x_{4n-2} = \frac{B}{\lambda}, \quad y_{4n-1} = \left( \frac{C}{\alpha} \right)^2 \]

\[ x_{4n-1} = \frac{B}{C} \alpha, \quad y_{4n} = \left( \frac{B}{\lambda} \right)^2 \]

\[ x_{4n} = \lambda, \quad y_{4n+1} = \left( \frac{B \lambda}{C} \right)^2. \]
Proof. First,

\[ y_1 = \max \left\{ x_{-1}^2, \frac{A}{x_{-1}} \right\} = \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \alpha^2. \]

Next, we shall proceed by induction on \( n \). For \( n = 1 \), we have

\[ x_1 = \frac{f(y_0)}{x_{-1}} = \frac{C}{\alpha}, \quad y_2 = \max \left\{ \lambda^2, \frac{A}{\lambda^2} \right\} = \lambda^2. \]

\[ x_2 = \frac{f(y_1)}{x_0} = \frac{B}{\lambda}, \quad y_3 = \max \left\{ \left( \frac{C}{\alpha} \right)^2, \frac{A}{C} \right\} = \left( \frac{C}{\alpha} \right)^2. \]

\[ x_3 = \frac{f(y_2)}{x_1} = \frac{B}{C/\alpha} = \frac{B}{C^2}, \quad y_4 = \max \left\{ \left( \frac{B}{\lambda} \right)^2, \frac{A}{B} \right\} = \left( \frac{B}{\lambda} \right)^2. \]

\[ x_4 = \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda, \quad y_5 = \max \left\{ \left( \frac{B\alpha}{C} \right)^2, \frac{AC}{B\alpha} \right\} = \left( \frac{B\alpha}{C} \right)^2. \]

So the result holds for \( n = 1 \). Now suppose the result is true for some \( k > 0 \), that is,

\[ x_{4k-3} = \frac{C}{\alpha}, \quad y_{4k-2} = \lambda^2. \]

\[ x_{4k-2} = \frac{B}{\lambda}, \quad y_{4k-1} = \left( \frac{C}{\alpha} \right)^2. \]

\[ x_{4k-1} = \frac{B}{C^2}, \quad y_{4k} = \left( \frac{B}{\lambda} \right)^2. \]

\[ x_{4k} = \lambda, \quad y_{4k+1} = \left( \frac{B\alpha}{C} \right)^2. \]

Then, for \( k + 1 \) we have the following:

\[ x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{f \left( \left( \frac{B}{\lambda} \right)^2 \right)}{B\alpha/C} = \frac{B}{B\alpha/C} = \frac{C}{\alpha}. \]

\[ y_{4k+2} = \max \left\{ x_{4k}^2, \frac{A}{x_{4k}} \right\} = \max \left\{ \lambda^2, \frac{A}{\lambda^2} \right\} = \lambda^2. \]

\[ x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{f \left( \left( \frac{B\alpha}{C} \right)^2 \right)}{\lambda} = \frac{B}{\lambda}. \]

\[ y_{4k+3} = \max \left\{ x_{4k+1}^2, \frac{A}{x_{4k+1}} \right\} = \max \left\{ \left( \frac{C}{\alpha} \right)^2, \frac{A}{C\alpha} \right\} = \left( \frac{C}{\alpha} \right)^2. \]
Figure 3.3:

\[ x_{4k+3} = \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{f(\lambda^2)}{C/\alpha} = \frac{B}{C/\alpha} = \frac{B}{C} \alpha \]

\[ y_{4k+4} = \max \left\{ x_{4k+2}^2, \frac{A}{x_{4k+2}} \right\} = \max \left\{ \left( \frac{B}{\lambda} \right)^2, \frac{A}{B} \lambda \right\} = \left( \frac{B}{\lambda} \right)^2 \]

\[ x_{4k+4} = \frac{f(y_{4k+3})}{x_{4k+2}} = \frac{f\left( \left( \frac{C}{\alpha} \right)^2 \right)}{B/\lambda} = \frac{B}{B/\lambda} = \lambda \]

\[ y_{4k+5} = \max \left\{ x_{4k+3}^2, \frac{A}{x_{4k+3}} \right\} = \max \left\{ \left( \frac{Bo}{C} \right)^2, \frac{AC}{B}\alpha \right\} = \left( \frac{Bo}{C} \right)^2. \]

Therefore the result is true for every \( k \in \mathbb{N} \). This concludes the proof. \( \square \)

To see the periodic behavior of \( \{x_n, y_n\} \), observe the three diagrams in Figure 3.3 with \( x_1 = -2, x_2 = -3, y_1 = -4, y_2 = -5, A = 3/4, B = 2, \) and \( C = 1 \).

**Theorem 3.4.** Assume that (2.3) holds with \( A, B > 0 \) and \( x_{-1}, y_{-1}, x_0, y_0, C < 0 \). Also, let \( \{x_n, y_n\} \) be a solution of the system of equations (1.1) with \( x_{-1} = \alpha, y_{-1} = \beta, x_0 = \lambda, \) and \( y_0 = \mu \). Then all solutions of (1.1) are of the following:

\[ y_1 = \alpha^2. \]

For \( n \in \mathbb{N} \),

\[ x_{4n-3} = \frac{C\mu}{\alpha}, \quad y_{4n-2} = \lambda^2 \]

\[ x_{4n-2} = \frac{B}{\lambda}, \quad y_{4n-1} = \left( \frac{C\mu}{\alpha} \right)^2 \]

\[ x_{4n-1} = \frac{Bo}{C\mu}, \quad y_{4n} = \left( \frac{B}{\lambda} \right)^2 \]

\[ x_{4n} = \lambda, \quad y_{4n+1} = \left( \frac{Bo}{C\mu} \right)^2. \]
Proof. First, 

\[ y_1 = \max \left\{ x_{-1}^2, \frac{A}{x_{-1}} \right\} = \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \alpha^2. \]

Next, we shall proceed by induction on \( n \). For \( n = 1 \), we have

\[ x_1 = \frac{f(y_0)}{x_{-1}} = \frac{C\mu}{\alpha}, \quad y_2 = \max \left\{ x_0^2, \frac{A}{x_0} \right\} = \lambda^2 \]

\[ x_2 = \frac{f(y_1)}{x_0} = \frac{B}{\lambda} \]

\[ y_3 = \max \left\{ x_1^2, \frac{A}{x_1} \right\} = \max \left\{ \left( \frac{C\mu}{\alpha} \right)^2, \frac{A\alpha}{C\mu} \right\} = \left( \frac{C\mu}{\alpha} \right)^2 \]

\[ x_3 = \frac{f(y_2)}{x_1} = \frac{B}{C\mu/\alpha} = \frac{B\alpha}{C\mu} \]

\[ y_4 = \max \left\{ x_2^2, \frac{A}{x_2} \right\} = \max \left\{ \left( \frac{B}{\lambda} \right)^2, \frac{A\lambda}{B} \right\} = \left( \frac{B}{\lambda} \right)^2 \]

\[ x_4 = \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda \]

\[ y_5 = \max \left\{ x_3^2, \frac{A}{x_3} \right\} = \max \left\{ \left( \frac{B\alpha}{C\mu} \right)^2, \frac{AC\mu}{B\alpha} \right\} = \left( \frac{B\alpha}{C\mu} \right)^2. \]

So the result holds for \( n = 1 \). Now suppose the result is true for some \( k \in N \), that is,

\[ x_{4k-3} = \frac{C\mu}{\alpha}, \quad y_{4k-2} = \lambda^2 \]

\[ x_{4k-2} = \frac{B}{\lambda}, \quad y_{4k-1} = \left( \frac{C\mu}{\alpha} \right)^2 \]

\[ x_{4k-1} = \frac{B\alpha}{C\mu}, \quad y_{4k} = \left( \frac{B}{\lambda} \right)^2 \]

\[ x_{4k} = \lambda, \quad y_{4k+1} = \left( \frac{B\alpha}{C\mu} \right)^2 \]

Then, for \( k + 1 \) we have the following:

\[ x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{f \left( \left( \frac{B}{\lambda} \right)^2 \right)}{B\alpha/C\mu} = \frac{B}{B\alpha/C\mu} = \frac{C\mu}{\alpha} \]

\[ y_{4k+2} = \max \left\{ x_{4k}^2, \frac{A}{x_{4k}} \right\} = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \lambda^2 \]
Figure 3.4:

\[ x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{f\left(\left(\frac{B\alpha}{C\mu}\right)^2\right)}{\lambda} = \frac{B}{\lambda} \]

\[ y_{4k+3} = \max\left\{ x_{4k+1}^2, \frac{A}{x_{4k+1}} \right\} = \max\left\{ \left(\frac{C\mu}{\alpha}\right)^2, \frac{A\alpha}{C\mu} \right\} = \left(\frac{C\mu}{\alpha}\right)^2 \]

\[ x_{4k+3} = \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{f\left(\frac{\lambda^2}{C\mu/\alpha}\right)}{B/\lambda} = \frac{B\alpha}{C\mu} = \frac{B}{C\mu/\alpha} = \frac{B\alpha}{C\mu} \]

\[ y_{4k+4} = \max\left\{ x_{4k+2}^2, \frac{A}{x_{4k+2}} \right\} = \max\left\{ \left(\frac{B\lambda}{C\mu}\right)^2, \frac{A\lambda}{B\lambda} \right\} = \left(\frac{B\lambda}{C\mu}\right)^2 \]

\[ x_{4k+4} = \frac{f(y_{4k+3})}{x_{4k+2}} = \frac{f\left(\frac{C\mu}{\alpha}\right)^2}{B/\lambda} = \frac{B}{\lambda} = \lambda \]

\[ y_{4k+5} = \max\left\{ x_{4k+3}^2, \frac{A}{x_{4k+3}} \right\} = \max\left\{ \left(\frac{B\alpha}{C\mu}\right)^2, \frac{AC\mu}{B\alpha} \right\} = \left(\frac{B\alpha}{C\mu}\right)^2 . \]

Therefore the result is true for every \( k \in \mathbb{N} \). This concludes the proof.

To see the periodic behavior of \( \{x_n, y_n\} \), observe the three diagrams in Figure 3.4 with \( x_1 = -1/2, x_2 = -1/3, y_1 = -1/4, y_2 = -1/5, A = 3/4, B = 1, \) and \( C = -1 \).

**Theorem 3.5.** Assume that (2.2) holds with \( B, C > 0 \) and let \( \{x_n, y_n\} \) be a solution of the system of equations (1.1) with \( x_1 = \alpha, y_1 = \beta, x_0 = \lambda, \) and \( y_0 = \mu \) all positive. Furthermore, assume

\[ A < \alpha^3, \quad A < \lambda^3 \]

and

\[ B^3 > A\alpha^3, \quad B^3 > A\lambda^3. \]

Then all solutions of (1.1) are of the following:

\[ x_{4n-3} = B/\alpha, \quad y_{4n-3} = \alpha^2. \]
Proof. For $n = 1$, we have

\begin{align*}
x_4 &= \frac{f(y_0)}{x_1} = \frac{B}{\alpha}, \quad y_1 = \max \left\{ \frac{x_0^2}{x_1}, \frac{A}{x_1} \right\} = \alpha^2 \\
x_2 &= \frac{f(y_1)}{x_0} = \frac{B}{\lambda}, \quad y_2 = \max \left\{ \frac{x_0^2}{x_0}, \frac{A}{x_0} \right\} = \lambda^2 \\
x_3 &= \frac{f(y_2)}{x_1} = \frac{B}{B/\alpha} = \alpha, \quad y_3 = \max \left\{ \frac{x_1^2}{x_1}, \frac{A}{x_1} \right\} = \max \left\{ \frac{(B/\alpha)^2}{A/\alpha}, \frac{A\alpha}{B} \right\} = \left( \frac{B}{\alpha} \right)^2 \\
x_4 &= \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda, \quad y_4 = \max \left\{ \frac{x_2^2}{x_2}, \frac{A}{x_2} \right\} = \max \left\{ \frac{(B/\lambda)^2}{A/\lambda}, \frac{A\lambda}{B} \right\} = \left( \frac{B}{\lambda} \right)^2.
\end{align*}

So the result holds for $n = 1$. Now suppose the result is true for some $k \in \mathbb{N}$, that is,

\begin{align*}
x_{4k-3} &= \frac{B}{\alpha}, \quad y_{4k-3} = \alpha^2 \\
x_{4k-2} &= \frac{B}{\lambda}, \quad y_{4k-2} = \lambda^2 \\
x_{4k-1} &= \alpha, \quad y_{4k-1} = \left( \frac{B}{\alpha} \right)^2 \\
x_{4k} &= \lambda, \quad y_{4k} = \left( \frac{B\alpha}{\lambda} \right)^2.
\end{align*}

Then, for $k + 1$ we have the following:

\begin{align*}
x_{4k+1} &= \frac{f(y_{4k})}{x_{4k-1}} = \frac{B}{\alpha}, \quad y_{4k+1} = \max \left\{ \frac{x_{4k-1}^2}{x_{4k-1}}, \frac{A}{x_{4k-1}} \right\} = \max \left\{ \alpha^2, \frac{A}{\alpha} \right\} = \alpha^2 \\
x_{4k+2} &= \frac{f(y_{4k+1})}{x_{4k}} = \frac{B}{\lambda}, \quad y_{4k+2} = \max \left\{ \frac{x_{4k}^2}{x_{4k}}, \frac{A}{x_{4k}} \right\} = \max \left\{ \lambda^2, \frac{A}{\lambda} \right\} = \lambda^2 \\
x_{4k+3} &= \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{f(\lambda^2)}{B/\alpha} = \frac{B}{B/\alpha} = \alpha.
\end{align*}
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Figure 3.5:

\[ y_{4k+3} = \max \left\{ x_{4k+1}, \frac{A}{x_{4k+1}} \right\} = \max \left\{ \left( \frac{B}{\alpha} \right)^2, \frac{A\alpha}{B} \right\} = \left( \frac{B}{\alpha} \right)^2 \]

\[ x_{4k+4} = \frac{f(y_{4k+3})}{x_{4k+2}} = \frac{f\left( \left( \frac{B}{\alpha} \right)^2 \right)}{B/\lambda} = \frac{B}{B/\lambda} = \lambda \]

\[ y_{4k+4} = \max \left\{ x_{4k+2}, \frac{A}{x_{4k+2}} \right\} = \max \left\{ \left( \frac{B}{\lambda} \right)^2, \frac{A\lambda}{B} \right\} = \left( \frac{B}{\lambda} \right)^2. \]

Therefore the result is true for every \( k \in \mathbb{N} \). This concludes the proof. \( \square \)

To see the periodic behavior of \( \{x_n, y_n\} \), observe the three diagrams in Figure 3.5 with \( x_1 = 1, x_2 = 2, y_1 = 3, y_2 = 4, A = 1/2, \) and \( B = 4 \).

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