Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 12, Number 1, pp. 51–62 (2017)
http://campus.mst.edu/adsa

# Periodic Solutions to a Class of Mixed Max-Type Nonlinear Difference Equations

**Jahmario L. Williams**
Texas Southern University
Department of Mathematics
williamsjl@tsu.edu

**Abstract**

We investigate the closed form solutions of a certain system of nonlinear mixed max-type difference equations. Under certain conditions, we show that the solutions to the system are periodic. Furthermore, we give graphical evidence that verifies the periodicity of the system being analyzed.

**AMS Subject Classifications:** 39A05, 39A23.
**Keywords:** Periodic solutions, mixed-max type difference equations.

## 1 Introduction

Difference equations are pervasive in mathematics and understanding the behavior of such equations gives insight to many interesting problems, see [4, 7, 11, 16]. Studying the periodic nature of certain difference equations has attracted many authors, see [1–3, 5, 6, 8–10, 12–14].

In 2015, Nouressadat Touafek and Nabila Haddad studied the closed form periodic solutions in [15] to the following mixed max-type rational system of difference equations

$$x_{n+1} = \frac{x_n y_n}{y_{n-1}}, y_{n+1} = \max\left\{\frac{A_n}{x_n}, y_{n-1}\right\}.$$

We study the periodic solutions of the system of difference equations

$$x_{n+1} = \frac{f(y_n)}{x_{n-1}}, y_{n+1} = \max\left\{x_{n-1}^2, \frac{A}{x_{n-1}}\right\} \quad \text{for } n \in N_0, \tag{1.1}$$

where $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, and $y_0 = \mu$ are positive numbers.

Received August 22, 2016; Accepted December 13, 2016
Communicated by Martin Bohner

# 2 Assumptions

The function $f$ will have one of the following forms:

$$f(z) = 1, \tag{2.1}$$

$$f(z) = \begin{cases} B, & \text{if } z > 0 \\ C, & \text{if } z < 0, \end{cases} \tag{2.2}$$

$$f(z) = \begin{cases} B, & \text{if } z > 0 \\ Cz, & \text{if } z < 0, \end{cases} \tag{2.3}$$

where $B, C \in \mathbb{R}$ such that $B^2 + C^2 \neq 0$.

# 3 Main Results

**Theorem 3.1.** *Assume that* (2.1) *holds with* $0 < x_{-1}, y_{-1}, x_0, y_0 < A < 1$. *Also, let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$. *Then all solutions of* (1.1) *are of the following:*

$$x_{4n-3} = \frac{1}{\alpha}, \quad y_{4n-3} = \frac{A}{\alpha}$$

$$x_{4n-2} = \frac{1}{\lambda}, \quad y_{4n-2} = \frac{A}{\lambda}$$

$$x_{4n-1} = \alpha, \quad y_{4n-1} = \frac{1}{\alpha^2}$$

$$x_{4n} = \lambda, \quad y_{4n} = \frac{1}{\lambda^2}.$$

*Proof.* For $n = 1$, we have

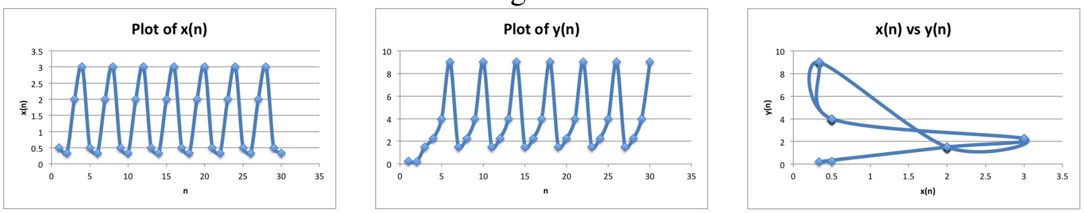
$$x_1 = \frac{f(y_0)}{x_{-1}} = \frac{1}{\alpha}, \quad y_1 = \max\left\{x_{-1}^2, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \frac{A}{\alpha}$$

$$x_2 = \frac{f(y_0)}{x_0} = \frac{1}{\lambda}, \quad y_2 = \max\left\{x_0^2, \frac{A}{x_0}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda}\right\} = \frac{A}{\lambda}$$

$$x_3 = \frac{f(y_2)}{x_1} = \frac{1}{1/\alpha} = \alpha, \quad y_3 = \max\left\{x_1^2, \frac{A}{x_1}\right\} = \max\left\{\frac{1}{\alpha^2}, A\alpha\right\} = \frac{1}{\alpha^2}$$

$$x_4 = \frac{f(y_3)}{x_2} = \frac{1}{1/\lambda} = \lambda, \quad y_4 = \max\left\{x_2^2, \frac{A}{x_2}\right\} = \max\left\{\frac{1}{\lambda^2}, A\lambda\right\} = \frac{1}{\lambda^2}.$$

Figure 3.1:




So the result holds for $n = 1$. Now suppose the result is true for some $k \in N$, that is,

$$x_{4k-3} = \frac{1}{\alpha}, \quad y_{4k-3} = \frac{A}{\alpha}$$

$$x_{4k-2} = \frac{1}{\lambda}, \quad y_{4k-2} = \frac{A}{\lambda}$$

$$x_{4k-1} = \alpha, \quad y_{4k-1} = \frac{1}{\alpha^2}$$

$$x_{4k} = \lambda, \quad y_{4k} = \frac{1}{\lambda^2}.$$

Then, for $k+1$ we have the following:

$$x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{1}{\alpha}, \quad y_{4k+1} = \max\left\{x_{4k-1}^2, \frac{A}{x_{4k-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \frac{A}{\alpha}$$

$$x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{1}{\lambda}, \quad y_{4k+2} = \max\left\{x_{4k}^2, \frac{A}{x_{4k}}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda}\right\} = \frac{A}{\lambda}$$

$$x_{4k+3} = \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{1}{1/\alpha} = \alpha$$

$$y_{4k+3} = \max\left\{x_{4k+1}^2, \frac{A}{x_{4k+1}}\right\} = \max\left\{\frac{1}{\alpha^2}, A\alpha\right\} = \frac{1}{\alpha^2}$$

$$x_{4k+4} = \frac{f(y_{4k+3})}{x_{4k+2}} = \frac{1}{1/\lambda} = \lambda$$

$$y_{4k+4} = \max\left\{x_{4k+2}^2, \frac{A}{x_{4k+2}}\right\} = \max\left\{\frac{1}{\lambda^2}, A\lambda\right\} = \frac{1}{\lambda^2}.$$

Therefore the result is true for every $k \in N$. This concludes the proof. $\square$

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.1 with $x_1 = 1/2$, $x_2 = 1/3$, $y_1 = 1/4$, $y_2 = 1/5$, and $A = 3/4$.

**Theorem 3.2.** *Assume that* (2.2) *holds with* $B, C < 0$ *and* $0 < x_{-1}, y_{-1}, x_0, y_0 < A < 1$. *Also, let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$. *Then all solutions of* (1.1) *are of the following:*

$$x_{4n-3} = \frac{B}{\alpha}, \quad y_{4n-3} = \frac{A}{\alpha}$$

$$x_{4n-2} = \frac{B}{\lambda}, \quad y_{4n-2} = \frac{A}{\lambda}$$

$$x_{4n-1} = \alpha, \quad y_{4n-1} = \left(\frac{B}{\alpha}\right)^2$$

$$x_{4n} = \lambda, \quad y_{4n} = \left(\frac{B}{\lambda}\right)^2.$$

*Proof.* For $n = 1$, we have

$$x_1 = \frac{f(y_0)}{x_{-1}} = \frac{B}{\alpha}, \quad y_1 = \max\left\{x_{-1}^2, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \frac{A}{\alpha}$$

$$x_2 = \frac{f(y_0)}{x_{-1}} = \frac{B}{\lambda}, \quad y_2 = \max\left\{x_0^2, \frac{A}{x_0}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda}\right\} = \frac{A}{\lambda}$$

$$x_3 = \frac{f(y_2)}{x_1} = \frac{B}{B/\alpha} = \alpha, \quad y_3 = \max\left\{x_1^2, \frac{A}{x_1}\right\} = \max\left\{\frac{B^2}{\alpha^2}, \frac{A}{B}\alpha\right\} = \left(\frac{B}{\alpha}\right)^2$$

$$x_4 = \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda, \quad y_4 = \max\left\{x_2^2, \frac{A}{x_2}\right\} = \max\left\{\frac{B^2}{\lambda^2}, \frac{A}{B}\lambda\right\} = \left(\frac{B}{\lambda}\right)^2.$$

So the result holds for $n = 1$. Now suppose the result is true for some $k \in N$, that is,

$$x_{4k-3} = \frac{B}{\alpha}, \quad y_{4k-3} = \frac{A}{\alpha}$$

$$x_{4k-2} = \frac{B}{\lambda}, \quad y_{4k-2} = \frac{A}{\lambda}$$

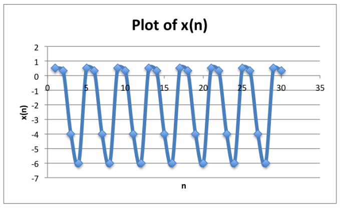
$$x_{4k-1} = \alpha, \quad y_{4k-1} = \left(\frac{B}{\alpha}\right)^2$$

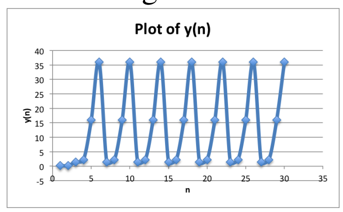
$$x_{4k} = \lambda, \quad y_{4k} = \left(\frac{B}{\lambda}\right)^2.$$

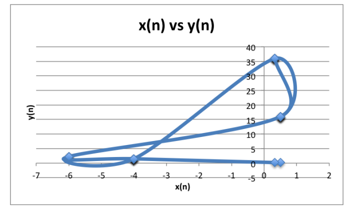
Then, for $k + 1$ we have the following:

$$x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{B}{\alpha}, \quad y_{4k+1} = \max\left\{x_{4k-1}^2, \frac{A}{x_{4k-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \frac{A}{\alpha}$$

Figure 3.2:








$$x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{B}{\lambda}, \quad y_{4k+2} = \max\left\{x_{4k}^2, \frac{A}{x_{4k}}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda}\right\} = \frac{A}{\lambda}$$

$$x_{4k+3} = \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{B}{B/\alpha} = \alpha$$

$$y_{4k+3} = \max\left\{x_{4k+1}^2, \frac{A}{x_{4k+1}}\right\} = \max\left\{\left(\frac{B}{\alpha}\right)^2, \frac{A}{B}\alpha\right\} = \left(\frac{B}{\alpha}\right)^2$$

$$x_{4k+4} = \frac{f(y_{4k+3})}{x_{4k+2}} = \frac{B}{B/\lambda} = \lambda$$

$$y_{4k+4} = \max\left\{x_{4k+2}^2, \frac{A}{x_{4k+2}}\right\} = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A}{B}\lambda\right\} = \left(\frac{B}{\lambda}\right)^2.$$

Therefore the result is true for every $k \in N$. This concludes the proof. □

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.2 with $x_1 = 1/2$, $x_2 = 1/3$, $y_1 = 1/4$, $y_2 = 1/5$, $A = 3/4$, $B = -2$, and $C = -1$.

**Theorem 3.3.** *Assume that* (2.2) *holds with* $A, B, C > 0$ *and* $x_{-1}, y_{-1}, x_0, y_0 \leq 0$. *Also, let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$. *Then all solutions of* (1.1) *are of the following:*

$$y_1 = \alpha^2.$$

*For* $n \in N$,

$$x_{4n-3} = \frac{C}{\alpha}, \quad y_{4n-2} = \lambda^2$$

$$x_{4n-2} = \frac{B}{\lambda}, \quad y_{4n-1} = \left(\frac{C}{\alpha}\right)^2$$

$$x_{4n-1} = \frac{B}{C}\alpha, \quad y_{4n} = \left(\frac{B}{\lambda}\right)^2$$

$$x_{4n} = \lambda, \quad y_{4n+1} = \left(\frac{B\lambda}{C}\right)^2.$$

*Proof.* First,

$$y_1 = \max\left\{x_{-1}^2, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \alpha^2.$$

Next, we shall proceed by induction on $n$. For $n = 1$, we have

$$x_1 = \frac{f(y_0)}{x_{-1}} = \frac{C}{\alpha}, \quad y_2 = \max\left\{\lambda^2, \frac{A}{\lambda^2}\right\} = \lambda^2$$

$$x_2 = \frac{f(y_1)}{x_0} = \frac{B}{\lambda}, \quad y_3 = \max\left\{\left(\frac{C}{\alpha}\right)^2, \frac{A}{C}\alpha\right\} = \left(\frac{C}{\alpha}\right)^2$$

$$x_3 = \frac{f(y_2)}{x_1} = \frac{B}{C/\alpha} = \frac{B}{C}\alpha, \quad y_4 = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A}{B}\lambda\right\} = \left(\frac{B}{\lambda}\right)^2$$

$$x_4 = \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda, \quad y_5 = \max\left\{\left(\frac{B\alpha}{C}\right)^2, \frac{AC}{B\alpha}\right\} = \left(\frac{B\alpha}{C}\right)^2.$$

So the result holds for $n = 1$. Now suppose the result is true for some $k > 0$, that is,

$$x_{4k-3} = \frac{C}{\alpha}, \quad y_{4k-2} = \lambda^2$$

$$x_{4k-2} = \frac{B}{\lambda}, \quad y_{4k-1} = \left(\frac{C}{\alpha}\right)^2$$

$$x_{4k-1} = \frac{B}{C}\alpha, \quad y_{4k} = \left(\frac{B}{\lambda}\right)^2$$

$$x_{4k} = \lambda, \quad y_{4k+1} = \left(\frac{B\lambda}{C}\right)^2.$$

Then, for $k + 1$ we have the following:

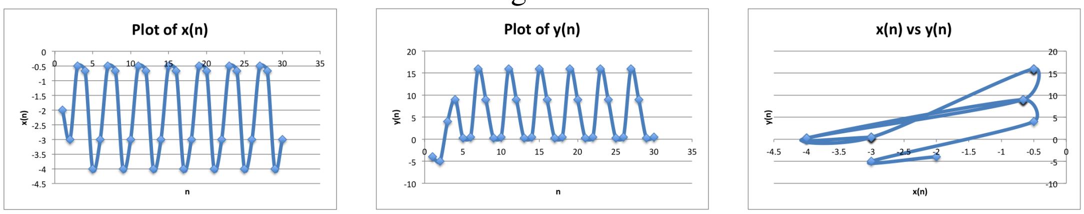
$$x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{f\left(\left(\frac{B}{\lambda}\right)^2\right)}{B\alpha/C} = \frac{B}{B\alpha/C} = \frac{C}{\alpha}$$

$$y_{4k+2} = \max\left\{x_{4k}^2, \frac{A}{x_{4k}}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda^2}\right\} = \lambda^2$$

$$x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{f\left(\left(\frac{B\lambda}{C}\right)^2\right)}{\lambda} = \frac{B}{\lambda}$$

$$y_{4k+3} = \max\left\{x_{4k+1}^2, \frac{A}{x_{4k+1}}\right\} = \max\left\{\left(\frac{C}{\alpha}\right)^2, \frac{A}{C}\alpha\right\} = \left(\frac{C}{\alpha}\right)^2$$

Figure 3.3:




$$x_{4k+3} = \frac{f\left(y_{4k+2}\right)}{x_{4k+1}} = \frac{f\left(\lambda^2\right)}{C/\alpha} = \frac{B}{C/\alpha} = \frac{B}{C}\alpha$$

$$y_{4k+4} = \max\left\{x_{4k+2}^2, \frac{A}{x_{4k+2}}\right\} = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A}{B}\lambda\right\} = \left(\frac{B}{\lambda}\right)^2$$

$$x_{4k+4} = \frac{f\left(y_{4k+3}\right)}{x_{4k+2}} = \frac{f\left(\left(\frac{C}{\alpha}\right)^2\right)}{B/\lambda} = \frac{B}{B/\lambda} = \lambda$$

$$y_{4k+5} = \max\left\{x_{4k+3}^2, \frac{A}{x_{4k+3}}\right\} = \max\left\{\left(\frac{B\alpha}{C}\right)^2, \frac{AC}{B\alpha}\right\} = \left(\frac{B\alpha}{C}\right)^2.$$

Therefore the result is true for every $k \in N$. This concludes the proof. □

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.3 with $x_1 = -2$, $x_2 = -3$, $y_1 = -4$, $y_2 = -5$, $A = 3/4$, $B = 2$, and $C = 1$.

**Theorem 3.4.** *Assume that* (2.3) *holds with* $A, B > 0$ *and* $x_{-1}, y_{-1}, x_0, y_0, C < 0$. *Also, let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$. *Then all solutions of* (1.1) *are of the following:*

$$y_1 = \alpha^2.$$

*For* $n \in N$,

$$x_{4n-3} = \frac{C\mu}{\alpha}, \quad y_{4n-2} = \lambda^2$$

$$x_{4n-2} = \frac{B}{\lambda}, \quad y_{4n-1} = \left(\frac{C\mu}{\alpha}\right)^2$$

$$x_{4n-1} = \frac{B\alpha}{C\mu}, \quad y_{4n} = \left(\frac{B}{\lambda}\right)^2$$

$$x_{4n} = \lambda, \quad y_{4n+1} = \left(\frac{B\alpha}{C\mu}\right)^2.$$

*Proof.* First,

$$y_1 = \max\left\{x_{-1}^2, \frac{A}{x_{-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \alpha^2.$$

Next, we shall proceed by induction on $n$. For $n = 1$, we have

$$x_1 = \frac{f(y_0)}{x_{-1}} = \frac{C\mu}{\alpha}, \quad y_2 = \max\left\{x_0^2, \frac{A}{x_0}\right\} = \lambda^2$$

$$x_2 = \frac{f(y_1)}{x_0} = \frac{B}{\lambda}$$

$$y_3 = \max\left\{x_1^2, \frac{A}{x_1}\right\} = \max\left\{\left(\frac{C\mu}{\alpha}\right)^2, \frac{A\alpha}{C\mu}\right\} = \left(\frac{C\mu}{\alpha}\right)^2$$

$$x_3 = \frac{f(y_2)}{x_1} = \frac{B}{C\mu/\alpha} = \frac{B\alpha}{C\mu}$$

$$y_4 = \max\left\{x_2^2, \frac{A}{x_2}\right\} = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A\lambda}{B}\right\} = \left(\frac{B}{\lambda}\right)^2$$

$$x_4 = \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda$$

$$y_5 = \max\left\{x_3^2, \frac{A}{x_3}\right\} = \max\left\{\left(\frac{B\alpha}{C\mu}\right)^2, \frac{AC\mu}{B\alpha}\right\} = \left(\frac{B\alpha}{C\mu}\right)^2.$$

So the result holds for $n = 1$. Now suppose the result is true for some $k \in N$, that is,

$$x_{4k-3} = \frac{C\mu}{\alpha}, \quad y_{4k-2} = \lambda^2$$

$$x_{4k-2} = \frac{B}{\lambda}, \quad y_{4k-1} = \left(\frac{C\mu}{\alpha}\right)^2$$

$$x_{4k-1} = \frac{B\alpha}{C\mu}, \quad y_{4k} = \left(\frac{B}{\lambda}\right)^2$$

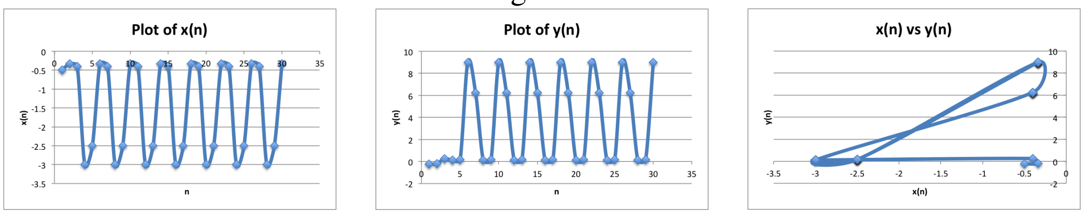
$$x_{4k} = \lambda, \quad y_{4k+1} = \left(\frac{B\alpha}{C\mu}\right)^2.$$

Then, for $k+1$ we have the following:

$$x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{f\left(\left(\frac{B}{\lambda}\right)^2\right)}{B\alpha/C\mu} = \frac{B}{B\alpha/C\mu} = \frac{C\mu}{\alpha}$$

$$y_{4k+2} = \max\left\{x_{4k}^2, \frac{A}{x_{4k}}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda}\right\} = \lambda^2$$

Figure 3.4:




$$x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{f\left(\left(\frac{B\alpha}{C\mu}\right)^2\right)}{\lambda} = \frac{B}{\lambda}$$

$$y_{4k+3} = \max\left\{x_{4k+1}^2, \frac{A}{x_{4k+1}}\right\} = \max\left\{\left(\frac{C\mu}{\alpha}\right)^2, \frac{A\alpha}{C\mu}\right\} = \left(\frac{C\mu}{\alpha}\right)^2$$

$$x_{4k+3} = \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{f(\lambda^2)}{C\mu/\alpha} = \frac{B}{C\mu/\alpha} = \frac{B\alpha}{C\mu}$$

$$y_{4k+4} = \max\left\{x_{4k+2}^2, \frac{A}{x_{4k+2}}\right\} = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A}{B}\lambda\right\} = \left(\frac{B}{\lambda}\right)^2$$

$$x_{4k+4} = \frac{f(y_{4k+3})}{x_{4k+2}} = \frac{f\left(\left(\frac{C\mu}{\alpha}\right)^2\right)}{B/\lambda} = \frac{B}{B/\lambda} = \lambda$$

$$y_{4k+5} = \max\left\{x_{4k+3}^2, \frac{A}{x_{4k+3}}\right\} = \max\left\{\left(\frac{B\alpha}{C\mu}\right)^2, \frac{AC\mu}{B\alpha}\right\} = \left(\frac{B\alpha}{C\mu}\right)^2.$$

Therefore the result is true for every $k \in N$. This concludes the proof. $\square$

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.4 with $x_1 = -1/2$, $x_2 = -1/3$, $y_1 = -1/4$, $y_2 = -1/5$, $A = 3/4$, $B = 1$, and $C = -1$.

**Theorem 3.5.** *Assume that* (2.2) *holds with* $B, C > 0$ *and let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$ *all positive. Furthermore, assume*

$$A < \alpha^3, \quad A < \lambda^3$$

*and*

$$B^3 > A\alpha^3, \quad B^3 > A\lambda^3.$$

*Then all solutions of* (1.1) *are of the following:*

$$x_{4n-3} = \frac{B}{\alpha}, \quad y_{4n-3} = \alpha^2$$

$$x_{4n-2} = \frac{B}{\lambda}, \quad y_{4n-2} = \lambda^2$$

$$x_{4n-1} = \alpha, \quad y_{4n-1} = \left(\frac{B}{\alpha}\right)^2$$

$$x_{4n} = \lambda, \quad y_{4n} = \left(\frac{B\alpha}{\lambda}\right)^2.$$

*Proof.* For $n = 1$, we have

$$x_1 = \frac{f(y_0)}{x_{-1}} = \frac{B}{\alpha}, \quad y_1 = \max\left\{x_{-1}^2, \frac{A}{x_{-1}}\right\} = \alpha^2$$

$$x_2 = \frac{f(y_1)}{x_0} = \frac{B}{\lambda}, \quad y_2 = \max\left\{x_0^2, \frac{A}{x_0}\right\} = \lambda^2$$

$$x_3 = \frac{f(y_2)}{x_1} = \frac{B}{B/\alpha} = \alpha, \quad y_3 = \max\left\{x_1^2, \frac{A}{x_1}\right\} = \max\left\{\left(\frac{B}{\alpha}\right)^2, \frac{A\alpha}{B}\right\} = \left(\frac{B}{\alpha}\right)^2$$

$$x_4 = \frac{f(y_3)}{x_2} = \frac{B}{B/\lambda} = \lambda, \quad y_4 = \max\left\{x_2^2, \frac{A}{x_2}\right\} = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A\lambda}{B}\right\} = \left(\frac{B}{\lambda}\right)^2.$$

So the result holds for $n = 1$. Now suppose the result is true for some $k \in N$, that is,

$$x_{4k-3} = \frac{B}{\alpha}, \quad y_{4k-3} = \alpha^2$$

$$x_{4k-2} = \frac{B}{\lambda}, \quad y_{4k-2} = \lambda^2$$

$$x_{4k-1} = \alpha, \quad y_{4k-1} = \left(\frac{B}{\alpha}\right)^2$$

$$x_{4k} = \lambda, \quad y_{4k} = \left(\frac{B\alpha}{\lambda}\right)^2.$$

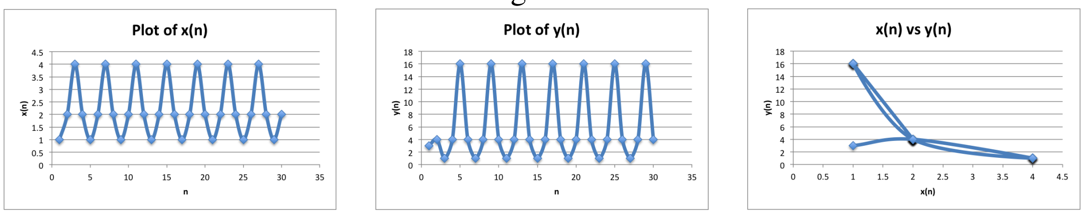
Then, for $k + 1$ we have the following:

$$x_{4k+1} = \frac{f(y_{4k})}{x_{4k-1}} = \frac{B}{\alpha}, \quad y_{4k+1} = \max\left\{x_{4k-1}^2, \frac{A}{x_{4k-1}}\right\} = \max\left\{\alpha^2, \frac{A}{\alpha}\right\} = \alpha^2$$

$$x_{4k+2} = \frac{f(y_{4k+1})}{x_{4k}} = \frac{B}{\lambda}, \quad y_{4k+2} = \max\left\{x_{4k}^2, \frac{A}{x_{4k}}\right\} = \max\left\{\lambda^2, \frac{A}{\lambda}\right\} = \lambda^2$$

$$x_{4k+3} = \frac{f(y_{4k+2})}{x_{4k+1}} = \frac{f(\lambda^2)}{B/\alpha} = \frac{B}{B/\alpha} = \alpha$$

Figure 3.5:




$$y_{4k+3} = \max\left\{x_{4k+1}^2, \frac{A}{x_{4k+1}}\right\} = \max\left\{\left(\frac{B}{\alpha}\right)^2, \frac{A\alpha}{B}\right\} = \left(\frac{B}{\alpha}\right)^2$$

$$x_{4k+4} = \frac{f\left(y_{4k+3}\right)}{x_{4k+2}} = \frac{f\left(\left(\frac{B}{\alpha}\right)^2\right)}{B/\lambda} = \frac{B}{B/\lambda} = \lambda$$

$$y_{4k+4} = \max\left\{x_{4k+2}^2, \frac{A}{x_{4k+2}}\right\} = \max\left\{\left(\frac{B}{\lambda}\right)^2, \frac{A\lambda}{B}\right\} = \left(\frac{B}{\lambda}\right)^2.$$

Therefore the result is true for every $k \in N$. This concludes the proof. $\square$

To see the periodic behavior of $\{x_n, y_n\}$, observe the three diagrams in Figure 3.5 with $x_1 = 1$, $x_2 = 2$, $y_1 = 3$, $y_2 = 4$, $A = 1/2$, and $B = 4$.

## Acknowledgements

This work is supported by Texas Southern University Research Seed Grant and the author would like to thank the anonymous referees for carefully reading the manuscript and providing valuable suggestions.

## References

[1] A. M. Ahmed and E. M. Elsayed. The expressions of solutions and the periodicity of some rational difference equations systems. *J. Appl. Math. Inform.*, 34(1-2):35–48, 2016.

[2] E. Camouzis, C. H. Gibbons, and G. Ladas. On period-two convergence in rational equations. *J. Difference Equ. Appl.*, 9(5):535–540, 2003. Dedicated to Professor George R. Sell on the occasion of his 65th birthday.

[3] K. A. Cunningham, G. Ladas, S. Valicenti, and J. Feuer. On the difference equation $x_{n+1} = \dfrac{\max\{x_n, A_n\}}{x_n^2 x_{n-1}}$. In *New trends in difference equations (Temuco, 2000)*, pages 79–98. Taylor & Francis, London, 2002.

[4] Q. Din, M. N. Qureshi, and A. Qadeer Khan. Dynamics of a fourth-order system of rational difference equations. *Adv. Difference Eq.*, 1(2012):1–15, 2012.

[5] H. El-Metwally, E. A. Grove, and G. Ladas. A global convergence result with applications to periodic solutions. *J. Math. Anal. Appl.*, 245(1):161–170, 2000.

[6] E. M. Elabbasy and S. M. Eleissawy. Periodicity and stability of solutions of rational difference systems. *J. Appl. Comput. Math.*, 1:114, 2012.

[7] E. M. Elsayed, M. M. El-Dessoky, and M. A. Alghamdi. On the solutions of some systems of difference equations. *Arch. Sciences*, 5:369–378, 2013.

[8] E. A. Grove and G. Ladas. Periodicity in nonlinear difference equations. *Cubo Mat. Educ.*, 4(1):195–230, 2002.

[9] E. A. Grove, G. Ladas, and M. Predescu. On the periodic character of the difference equation $x_n = \dfrac{p + x_{n-(2k+1)}}{1 + x_{n-2l}}$. *Math. Sci. Res. J.*, 6(5):221–233, 2002.

[10] E. A. Grove, G. Ladas, N. R. Prokup, and R. Levins. On the global behavior of solutions of a biological model. *Comm. Appl. Nonlinear Anal.*, 7(2):33–46, 2000.

[11] Tarek F. Ibrahim and Nouressadat Touafek. Max-type system of difference equations with positive two-periodic sequences. *Math. Methods Appl. Sci.*, 37(16):2541–2553, 2014.

[12] Banyat Sroysang. On the periodic solutions of some rational difference systems ii. *Mathematica Aeterna*, 3(8):701–708, 2013.

[13] Stevo Stević. Periodicity of a class of nonautonomous max-type difference equations. *Appl. Math. Comput.*, 217(23):9562–9566, 2011.

[14] Stevo Stević, Mohammed A. Alghamdi, Dalal A. Maturi, and Naseer Shahzad. On the periodicity of some classes of systems of nonlinear difference equations. In *Abst. Appl. Anal.* Hindawi Publishing Corporation, 2014.

[15] Nouressadat Touafek and Nabila Haddad. On a mixed max-type rational system of difference equations. *Electron. J. Math. Anal. Appl.*, 3(1):164–169, 2015.

[16] Y. Yazlik, D. T. Tollu, and N. Taskara. On the solutions of a max-type difference equation system. *Math. Methods Appl. Sciences*, 38(17):4388–4410, 2015.