Global Solutions of Higher-Order Functional Dynamic Equations

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Abstract
In this paper, we use the fixed point theorem of Schaefer type to establish the existence of global solutions for a class of higher-order functional dynamic equations on time scales. The uniqueness of solutions is also provided.

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1 Introduction and Preliminaries

In this paper, we are concerned with the global existence and uniqueness of solutions for the following functional dynamic equation

\[(a(t)(x^\Delta(t))^\Delta^{n-1} + f(t, x(q(t)))) = 0, \ t \in \mathbb{T},\]  

(1.1)

where \(n \geq 2\) is an integer, \(\mathbb{T}\) is a time scale with \(\inf \mathbb{T} = t_0\) and \(\sup \mathbb{T} = \infty\).

For the behavior of solutions of time scale dynamic equations, no matter whether it is of lower or higher order, there have drawn much attention from the researchers, see, e.g., the articles [1–3, 6, 7, 10–14] and their references. In particular, Graef et al. [10] considered the nonoscillation for the special case of (1.1). We observe, nevertheless, it is more rare for the existence results of global solutions of (1.1), and this brings us a chance to consider the present paper.

For the terminologies such as right-dense continuity, derivatives and integrations on time scales and so on, we refer the reader to the monographs [6, 7]. Here, we recall the generalized polynomials.

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For a given \( t_x \in \mathbb{T} \), let \([t_x, \infty)_{\mathbb{T}}\) denote the set \([t_x, \infty) \cap \mathbb{T}\), and \(C_{rd}[t_x, \infty)_{\mathbb{T}}\) the set of all functions \(\varphi : [t_x, \infty)_{\mathbb{T}} \to \mathbb{R}\) with \(\varphi\) is rd-continuous. Let \(h_0(t,s) \equiv 1\) on \(\mathbb{T} \times \mathbb{T}\) and
\[
h_m(t,s) = \int_t^s h_{m-1}(\tau,s) \Delta \tau, \quad m = 1, 2, 3, \ldots.
\]
Then \(h^\Delta_m(t,s) = h_{m-1}(t,s)\), where \(h^\Delta_m\) denotes the derivative with respect to the first variable. Furthermore, referring to [11, Corollary 1], for any \(a, t \in \mathbb{T}\) and \(U \in C_{rd}[t_0, \infty)_{\mathbb{T}}\), it follows that
\[
\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_n} U(\tau) \Delta \tau \Delta \tau_1 \cdots \Delta \tau_n = \int_a^t h_k(t,\sigma(\tau)) U(\tau) \Delta \tau,
\]
where \(\sigma : \mathbb{T} \to \mathbb{T}\) is defined by
\[
\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.
\]
Set \(t_1 = \inf\{t \in \mathbb{T} : g(t) \geq t_0\}\). By a solution of (1.1), we mean a real-valued function \(x(t) \in C_{rd}[t_0, \infty)\), with \((a(t)x^2(t))^{\Delta m} \in C_{rd}[t_1, \infty)_{\mathbb{T}}\) for \(m = 0, 1, 2, \ldots, n-2\), which satisfies (1.1) for all \(t \geq t_1\).

For given \(\phi \in C_{rd}[t_0, t_1]_{\mathbb{T}}\) and \(\eta_i \in \mathbb{R}\) for \(i = 1, 2, 3, \ldots, n-1\), by a solution \(x(t)\) of (1.1) with initial values \(\{\phi, \eta_1, \eta_2, \ldots, \eta_{n-1}\}\), we mean that \(x(t)\) is a solution of (1.1) satisfying \(x(t) = \phi(t)\) for \(t \in [t_0, t_1]_{\mathbb{T}}\), and \((a x^\Delta)^{\Delta i-1}(t_1) = \eta_i\) for \(i = 1, 2, 3, \ldots, n-1\).

Let \(k \in C_{rd}(\mathbb{T}, \mathbb{R}^+)\) be nondecreasing with \(k(t) \to \infty\) as \(t \to \infty\), and
\[
BC_{rd}[t_0, \infty)_{\mathbb{T}} := \left\{ x \in C_{rd}[t_0, \infty)_{\mathbb{T}} : \sup_{t \in \mathbb{T}} \frac{|x(t)|}{k(t)} < \infty \right\}.
\]
Then \(BC_{rd}[t_0, \infty)_{\mathbb{T}}\) is a Banach space under the norm defined by
\[
||x|| = \sup_{t \in \mathbb{T}} \frac{|x(t)|}{k(t)}.
\]
For any given \(\phi \in C_{rd}[t_0, t_1]\), by the subset \(X_{\phi}\) of \(BC_{rd}[t_0, \infty)_{\mathbb{T}}\), we define
\[
X_{\phi} := \{ x \in BC_{rd}[t_0, \infty)_{\mathbb{T}} : ||x|| \leq k_0 \text{ and } x \equiv \phi \text{ on } [t_0, t_1]_{\mathbb{T}} \},
\]
where \(k_0\) is some constant.

The subset \(X \subset BC_{rd}[t_0, \infty)_{\mathbb{T}}\) is said to be uniformly Cauchy if for any \(\varepsilon > 0\), there exists \(T \in \mathbb{T}\) such that
\[
\left| \frac{x(t')}{{k(t')}} - \frac{x(t'')}{k(t'')} \right| < \varepsilon
\]
for any \(x \in X\) and any \(t', t'' \in [T, \infty)_{\mathbb{T}}\). Besides, \(X\) is said to be equicontinuous on \([t_0, T]_{\mathbb{T}}\) if for any \(\varepsilon > 0\), there exists a \(\delta > 0\) such that for any \(x \in X\),
\[
\left| \frac{x(t')}{k(t')} - \frac{x(t'')}{k(t'')} \right| < \varepsilon \quad \text{for all } t', t'' \in [t_0, T]_{\mathbb{T}} \text{ with } |t' - t''| < \delta.
\]
In what follows, we will require a couple of important conclusions, in which we refer to [4, 8] for the first, and the second has a minor revision relative to Bihari type inequality (see [9, Lemma 1] and [10, Lemma 2.2]).

**Lemma 1.1** (Schaefer). Let $X$ be a Banach space and $\Psi: X \to X$ be a continuous compact map. If the set

$$\Omega = \{ x \in X : x = \lambda \Psi(x) \text{ for some } \lambda \in (0, 1) \}$$

is bounded, then $\Psi(x)$ has a fixed point.

**Lemma 1.2** (Bihari). Let $c > 0$ be a constant, $p \in C(\mathbb{R}, \mathbb{R}^+)$ with $p$ nondecreasing, and $h \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R}^+)$. Suppose that $P$ is defined by

$$P(u) = \int_{u_0}^{u} \frac{ds}{p(s)} \text{ for } u, u_0 > 0,$$

and satisfies

$$P(c) + \int_{t_0}^{t} \int_{t_0}^{\tau} h(s, \tau) \Delta s \Delta \tau \in \text{Dom}(P^{-1}).$$

Then

$$u(t) \leq c + \int_{t_0}^{t} \int_{t_0}^{\tau} h(s, \tau) p(u(s)) \Delta s \Delta \tau \text{ for } u \in C_{rd}(\mathbb{T}, \mathbb{R})$$

implies

$$u(t) \leq P^{-1} \left( P(c) + \int_{t_0}^{t} \int_{t_0}^{\tau} h(s, \tau) \Delta s \Delta \tau \right).$$

**Lemma 1.3** (See [14, Lemma 1.3]). Suppose that $X \subset BC_{rd}[t_0, \infty)_T$ is bounded and uniformly Cauchy. Suppose further that $X$ is equicontinuous on $[t_0, T]_T$ for any $T \in \mathbb{T}$. Then $X$ is relatively compact.

### 2 Main Results

In this section, we consider the existence and uniqueness on $\mathbb{T}$ for the solutions of (1.1). For this purpose, we assume that

(A1) $a \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $q \in C_{rd}(\mathbb{T}, \mathbb{T})$ with $q(t) \leq t$ and $\lim_{t \to \infty} q(t) = \infty$;

(A2) $f \in C(\mathbb{T}, \mathbb{R})$, and there exist $r_1, r_2 \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and $w \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $w$ being nondecreasing such that

$$|f(t, u)| \leq \begin{cases} 
  r_1(t) w(|u|) + r_2(t), & t \in \mathbb{T} \text{ and } u \in \mathbb{R}, \\
  r_1(t) w(||u||) + r_2(t), & t \in \mathbb{T} \text{ and } u \in BC_{rd}[t_0, \infty)_T.
\end{cases}$$

(2.1)
For the sake of convenience, we stipulate
\[
\int_{t_0}^{t} \frac{1}{a(\tau)} \int_{t_0}^{\tau} h_{-1}(\tau, \sigma(s)) \Delta s \Delta \tau := \int_{t_0}^{t} \frac{1}{a(s)} \Delta s
\]
and
\[
c_0 = c_0(\phi, \eta_1, \eta_2, \ldots, \eta_{n-1}) := |\phi(t_1)| + \sum_{i=1}^{n-1} \int_{t_1}^{\infty} \frac{|\eta_i|}{a(\tau)} \int_{t_1}^{\tau} h_{i-2}(\tau, \sigma(s)) \Delta s \Delta \tau
\]
whenever they are defined.

**Theorem 2.1.** Suppose that assumptions (A1)–(A2) hold and
\[
\int_{t_0}^{\infty} \frac{1}{a(\tau)} \int_{t_0}^{\tau} h_{n-2}(\tau, \sigma(s)) r_i(s) \Delta s \Delta \tau < \infty, \quad i = 1, 2, (2.2)
\]
as well as
\[
\int_{t_0}^{\infty} \frac{1}{a(\tau)} \int_{t_0}^{\tau} h_{i-2}(\tau, \sigma(s)) \Delta s \Delta \tau < \infty, \quad i = 1, 2, \ldots, n - 1. (2.3)
\]
Suppose further that for given \( \phi \in C_{rd}[t_0, t_1]_T \) and \( \eta_i \in \mathbb{R} \) for \( i = 1, 2, 3, \ldots, n - 1 \),
\[
\int_{t_1}^{\infty} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) r_1(s) \Delta s \Delta \tau \leq \int_{c_0}^{\infty} \frac{ds}{w(s)}. \quad (2.4)
\]
Then, Equation (1.1) admits a solution with the initial values \( \{\phi, \eta_1, \eta_2, \ldots, \eta_{n-1}\} \).

**Proof.** Let \( BC_{rd}[t_0, \infty)_T \) be defined as in (1.3). We now define an operator
\[
\Psi : BC_{rd}[t_0, \infty)_T \to BC_{rd}[t_0, \infty)_T
\]
as follows:
\[
(\Psi x)(t) = \begin{cases}
\phi(t_1) + H(t) - F(t), & t \in [t_1, \infty)_T, \\
\phi(t), & t \in [t_0, t_1)_T,
\end{cases}
\]
where \( H \) and \( F \) are defined, respectively, by
\[
H(t) := \sum_{i=1}^{n-1} \int_{t_1}^{t} \frac{\eta_i}{a(\tau)} \int_{t_1}^{\tau} h_{i-2}(\tau, \sigma(s)) \Delta s \Delta \tau
\]
and
\[
F(t) = \int_{t_1}^{t} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau.
\]
Then, it follows from the second part of (2.1) that

$$|(\Psi x)(t)| \leq |\phi(t_1)| + |\mathcal{H}(\infty)| + w(||x||) \int_{t_1}^{t} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) r_1(s) \Delta s \Delta \tau$$

$$+ \int_{t_1}^{t} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) r_2(s) \Delta s \Delta \tau, \quad t \in [t_1, \infty)_T, \quad (2.5)$$

which, together with conditions (2.2)–(2.3), induces $(\Psi x)(t) \in BC_{rd}[t_0, \infty)_T$, where $|\mathcal{H}(\infty)|$ represents

$$|\mathcal{H}(\infty)| := \sum_{i=1}^{n-1} \int_{t_1}^{\infty} \frac{|\eta_i|}{a(\tau)} \int_{t_1}^{\tau} h_{i-2}(\tau, \sigma(s)) \Delta s \Delta \tau.$$

Next we show $\Psi$ is continuous. To see this, we set $x_0, x_i \in BC_{rd}[t_0, \infty)_T$ with $||x_i - x_0|| \to 0$. Then we have for each $s \in [t_1, \infty)_T$,

$$|f(s, x_i(q(s))) - f(s, x_0(q(s)))| \to 0 \text{ as } i \to \infty. \quad (2.6)$$

Now by the definition of $\Psi$ it follows that

$$|(\Psi x_i)(t) - (\Psi x_0)(t)|$$

$$\leq \int_{t_1}^{\infty} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) |f(s, x_i(q(s))) - f(s, x_0(q(s)))| \Delta s \Delta \tau. \quad (2.7)$$

Note that the condition (2.1) implies that

$$|f(s, x_i(q(s))) - f(s, x_0(q(s)))| \leq r_1(s)[w(||x_i||) + w(||x_0||)] + 2r_2(s),$$

which, associated with (2.6)–(2.7) and the Lebesgue dominated convergence theorem [5], yields that

$$||(\Psi x_i) - (\Psi x_0)|| \to 0 \text{ as } i \to \infty.$$  

In other words, $\Psi$ is continuous.

Now we assert that $\Psi$ is compact. To this end, let $k_0 > 0$ be arbitrary and $X_\phi$ as in (1.4). Then, from (2.5), it is clear that $\Psi$ is bounded on $X_\phi$. Furthermore, since the function $k$ occurred in $BC_{rd}[t_0, \infty)_T$ satisfies $k(t) \to \infty$ as $t \to \infty$, for any given $\varepsilon > 0$, we choose $T \in [t_1, \infty)_T$ so that

$$\frac{1}{k(t)} < \varepsilon \text{ for all } t \in [T, \infty)_T.$$
Then, for any \( t', t'' \in [T, \infty) \) and \( x \in X_\phi \), we have

\[
\frac{1}{k(t')} \frac{\phi(t_1)}{k(t')} - \frac{1}{k(t'')} \frac{\phi(t_1)}{k(t'')} < \varepsilon,
\]

(2.9)

where we have imposed the condition (2.1) for the last inequality. Since \( \varepsilon > 0 \) is arbitrary, the inequality (2.8) implies that \( \Psi(X_\phi) \) is uniformly Cauchy.

To see the equicontinuity of \( \Psi(X_\phi) \) on the interval \([t_0, T] \), we choose \( \delta > 0 \) such that for \( t', t'' \in [t_0, T] \) with \(|t' - t''| < \delta\),

\[
\left| \frac{1}{k(t')} - \frac{1}{k(t'')} \right| < \varepsilon,
\]

(2.9)

as well as

\[
\sum_{i=1}^{n-1} \int_{t'}^{t''} \frac{|\eta_i|}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) \Delta s \Delta \tau < \varepsilon
\]

(2.10)

and

\[
\int_{t'}^{t''} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) r_i(s) \Delta s \Delta \tau < \varepsilon, \quad i = 1, 2.
\]

(2.11)
For simplicity, we consider $t' \leq t''$. Then, for any $t', t'' \in [t_1, T]$ and $x \in X_\phi$, we have

\[
\left| \frac{(\Psi x) (t')} {k(t')} - \frac{(\Psi x) (t'')} {k(t'')} \right| 
\leq \left| \frac{\phi(t_1)} {k(t')} - \frac{\phi(t_1)} {k(t'')} \right| + \sum_{i=1}^{n-1} \left| \frac{1} {k(t')} \int_{t_1}^{t'} \frac{\eta_i} {a(\tau)} \int_{t_1}^{\tau} h_{i-2}(\tau, \sigma(s)) \Delta s \Delta \tau \right| 
- \left| \frac{1} {k(t'')} \int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau \right| 
+ \left| \frac{1} {k(t'')} \int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau \right| 
- \left| \frac{1} {k(t'')} \int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau \right|
\]

\[
= \left| \frac{\phi(t_1)} {k(t')} - \frac{\phi(t_1)} {k(t'')} \right| + \sum_{i=1}^{n-1} \left| \frac{1} {k(t')} \int_{t_1}^{t'} \frac{\eta_i} {a(\tau)} \int_{t_1}^{\tau} h_{i-2}(\tau, \sigma(s)) \Delta s \Delta \tau \right| 
+ \left| \frac{1} {k(t'')} \int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau \right| 
+ \left| \frac{1} {k(t'')} \int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau \right|
\]

\[
(2.12)
\]

By condition (2.2), we denote

\[
M_i := \int_{t_0}^{\infty} \frac{1} {a(\tau)} \int_{t_0}^{\tau} h_{n-2}(\tau, \sigma(s)) r_i(s) \Delta s \Delta \tau, \quad i = 1, 2.
\]

Note that for $x \in X_\phi$,

\[
\int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) f(s, x(q(s))) \Delta s \Delta \tau 
\leq \int_{t_1}^{t'} \frac{1} {a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s)) [w(k_0) r_1(s) + r_2(s)] \Delta s \Delta \tau.
\]

Hence, with the aid of (2.9)–(2.12) it follows that when $t', t'' \in [t_1, T]$ and $|t' - t''| < \delta$,

\[
\left| \frac{(\Psi x) (t')} {k(t')} - \frac{(\Psi x) (t'')} {k(t'')} \right| \leq \phi(t_1) \varepsilon + \frac{\varepsilon} {k(t_0)} + |H(\infty)| \varepsilon
+ \frac{(w(k_0) + 1) \varepsilon} {k(t_0)} + (w(k_0) M_1 + M_2) \varepsilon \quad \text{for all } x \in X_\phi,
\]
which shows that $\Psi(X_\phi)$ is equicontinuous on $[t_0, T]_T$. Hence, Lemma 1.3 implies that $\Psi$ is a compact operator.

To impose Lemma 1.1, we need to show that $\Omega$ is bounded. Indeed, for any $x \in \Omega$, by the definition of $\Psi$, we have

$$|x(t)| \leq \lambda(|\Psi x|(t))$$

$$\leq |\phi(t_1)| + |\mathcal{H}(t)| + \int_{t_1}^t \frac{1}{a(\tau)} \int_{t_1}^\tau w(|x(s)|)h_{n-2}(\tau, \sigma(s))r_1(\tau)\Delta s \Delta \tau$$

$$+ \int_{t_1}^t \frac{1}{a(\tau)} \int_{t_1}^\tau h_{n-2}(\tau, \sigma(s))r_2(s)\Delta s \Delta \tau$$

$$\leq c_0 + \int_{t_1}^t \frac{1}{a(\tau)} \int_{t_1}^\tau w(|x(s)|)h_{n-2}(\tau, \sigma(s))r_1(\tau)\Delta s \Delta \tau, \quad t \in [t_1, \infty)_T,$$

where we have imposed the first part of (2.1) for the second step. Let $W(t) = \int_{c_0}^t \frac{ds}{w(s)}$.

Then, invoking (2.4), we have

$$\int_{t_1}^t \frac{1}{a(\tau)} \int_{t_1}^\tau h_{n-2}(\tau, \sigma(s))r_1(\tau)\Delta s \Delta \tau \in \text{Dom}(W^{-1}).$$

Therefore, Lemma 1.2 implies that

$$||x|| \leq W^{-1}\left(\int_{t_1}^\infty \frac{1}{a(\tau)} \int_{t_1}^\tau h_{n-2}(\tau, \sigma(s))r_1(\tau)\Delta s \Delta \tau\right).$$

Now by Schaefer’s fixed point theorem there exists $\tilde{x} \in BC_{rd}[t_0, \infty)_T$ such that

$$\tilde{x}(t) = \Psi(\tilde{x})(t), \quad t \in [t_0, \infty)_T,$$

which, in combination with the formula (1.2), indicates $\tilde{x}$ is a solution of (1.1) with the initial values $\{\phi, \eta_1, \eta_2, \ldots, \eta_{n-1}\}$, and this completes our proof. $\square$

**Theorem 2.2.** Suppose that there exists a constant $\alpha > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq \alpha|x_1 - x_2| \quad \text{for all } t \in T \text{ and } x_1, x_2 \in \mathbb{R}. \quad (2.14)$$

Suppose further that $x(t)$ is a solution of (1.1) with the initial condition $\{\phi, \eta_1, \eta_2, \ldots, \eta_{n-1}\}$ for $\phi \in C_{rd}[t_0, t_1]_T$ and $\eta_i \in \mathbb{R}, i = 1, 2, 3, \ldots, n - 1$. Then $x(t)$ is unique.
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Proof. Suppose that \( y(t) \) is another solution (1.1) with the initial values \( \{ \phi, \eta_1, \eta_2, \ldots, \eta_{n-1} \} \), and that
\[
\tilde{t} = \sup \{ s \in \mathbb{T} : x(t) - y(t) = 0 \text{ for } t \leq s \} < \infty.
\]
Then, in the case \( \sigma(\tilde{t}) = \tilde{t} \), we can choose \( \delta > 0 \) so that \( \tilde{t} + \delta \in \mathbb{T} \). Since \( x(t) \) and \( y(t) \) are two solutions of (1.1) with the same initial values, we have
\[
|x(t) - y(t)| \leq \int_{t_1}^{t} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s))|f(s, x(q(s))) - f(s, y(q(s)))| \Delta s \Delta \tau
\]
\[
\leq \alpha \int_{\tilde{t}}^{\tilde{t}+\delta} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s))|x(\tau) - y(\tau)| \Delta s \Delta \tau
\]
\[
\leq \int_{\tilde{t}}^{\tilde{t}+\delta} \int_{\tilde{t}}^{\tilde{t}+\delta} M \max_{t \leq \tilde{t} + \delta} |x(s) - y(s)| \Delta s \Delta \tau
\]
\[
= M \delta^2 \max_{t \leq \tilde{t} + \delta} |x(s) - y(s)|, \quad t \in \tilde{t}, \tilde{t} + \delta, \quad (2.15)
\]
where \( M \) denotes
\[
M := \max \left\{ \frac{\alpha h_{n-2}(\tilde{t}, \sigma(s))}{a(\tau)} : (s, \tau) \in \tilde{t}, \tilde{t} + \delta \right\}.
\]
Now we can choose \( \delta \) satisfying \( M \delta^2 < 1 \) due to \( \delta > 0 \) being arbitrary. Then we are led to a contradiction by (2.15). In other words, \( \tilde{t} < \infty \) is invalid when \( \sigma(\tilde{t}) = \tilde{t} \).

In the case \( \sigma(\tilde{t}) > \tilde{t} \), we have
\[
|x(\sigma(\tilde{t})) - y(\sigma(\tilde{t}))|\]
\[
\leq \int_{t_1}^{\sigma(\tilde{t})} \frac{1}{a(\tau)} \int_{t_1}^{\tau} h_{n-2}(\tau, \sigma(s))|f(s, x(q(s))) - f(s, y(q(s)))| \Delta s \Delta \tau
\]
\[
\leq \int_{\tilde{t}}^{\sigma(\tilde{t})} \frac{1}{a(\tau)} \int_{\tilde{t}}^{\tau} h_{n-2}(\tau, \sigma(s))|x(\tau) - y(\tau)| \Delta s \Delta \tau
\]
\[
= \int_{\tilde{t}}^{\sigma(\tilde{t})} \frac{1}{a(\tau)} h_{n-2}(\tau, \sigma(\tilde{t}))|x(\tilde{t}) - y(\tilde{t})| \Delta \tau
\]
\[
= 0,
\]
which contradicts our assumption for \( \tilde{t} \), where for the third step, we have used the conclusion [6, Theorem 1.75]
\[
\int_{t}^{\sigma(t)} g(s) \Delta s = g(t)(\sigma(t) - t).
\]
In summary, we have shown that
\[
x(t) = y(t) \text{ for all } t \in [t_0, \infty)_T,
\]
which completes our proof. \( \square \)
A simple example can illustrate our results, see the following.

**Example 2.3.** Let $\mathbb{T} = [0, \infty)$ and consider the equation

$$x'' + x' + e^{-2t}x = e^{-t}, \quad t \geq 0,$$

which is equivalent to

$$(e^t x') + e^{-t}x = 1, \quad t \geq 0. \quad (2.16)$$

Then the functions $a, f$ in (1.1) are given by

$$a(t) = e^t, \quad f(t, x) = e^{-t}x - 1.$$

Now if we choose $k(t) = e^t$ in $BC_{rd}[t_0, \infty)_T$, then

$$|f(t, u)| \leq \begin{cases} |u| + 1, & t \in \mathbb{T} \text{ and } u \in \mathbb{R}, \\ ||u|| + 1, & t \in \mathbb{T} \text{ and } u \in BC_{rd}[t_0, \infty)_T \end{cases}$$

and hence, all conditions in Theorems 2.1–2.2 are verified. Therefore, for any given $\phi, \eta \in \mathbb{R}$, equation (2.16) has a unique solution $x(t)$ with $x(0) = \phi$ and $x'(0) = \eta$.

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**References**


