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Exponential, Trigonometric and Hyperbolic Functions Associated with a General Quantum Difference Operator

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Abstract

In this paper, we present the β-exponential and β-trigonometric functions based on the general quantum difference operator D_β defined by

$$D_\beta f(t) = \frac{f(\beta(t)) - f(t)}{\beta(t) - t}, \;\; \beta(t) \neq t,$$

which are the solutions of the first and second order β-difference equations, respectively. Here, β is a strictly increasing continuous function defined on an interval $I \subseteq \mathbb{R}$. Furthermore, we establish many properties of these functions. Finally, the β-hyperbolic functions and their properties are introduced.

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1 Introduction

Quantum calculus is recently subject to an increase number of investigations, due to its applications. It substitutes the usual derivative by a difference operator, which allows one to deal with sets of non-differentiable functions [3, 10]. Quantum calculus has applications in many areas, for instance calculus of variations, economical systems, and several fields of physics such as black holes, quantum mechanics, nuclear and high energy physics [7–9, 16–19]. In [14] we have constructed a general quantum difference operator D_β, by considering a strictly increasing continuous function $\beta : I \to I$, where I is an interval of $\mathbb{R}$ containing a fixed point s_0 of β. The β-difference operator is defined by

$$D_\beta f(t) = \begin{cases} \dfrac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ \\ f'(s_0), & t = s_0. \end{cases}$$

where f is an arbitrary function defined on I, and is differentiable at $t = s_0$ in the usual sense. As particular cases, we obtain the Hahn difference operator $D_{q,\omega}$ when $\beta(t) = qt + \omega$, $q \in (0, 1)$, $\omega > 0$ are fixed numbers, the Jackson q-difference operator D_q when $\beta(t) = qt$, $q \in (0, 1)$, the n, q-power quantum difference operator when $\beta(t) = qt^n$, $q \in (0, 1)$, n is a fixed odd positive integer and the forward difference operator $\Delta_{a,b}$ when $\beta(t) = at + b$, $a \geq 1$, $b \geq 0$ and $a + b > 1$. For more details about these operators we refer the reader to [1, 2, 4–6, 11–13, 15].

In a first step towards the development of the general quantum difference calculus, in [14], we considered our function β when it has only one fixed point $s_0 \in I$ and satisfies the following condition

$$(t - s_0)(\beta(t) - t) \leq 0 \ \text{ for all } t \in I,$$

and gave a rigorous analysis of the calculus based on D_β and its associated integral operator. Some basic properties of such a calculus were stated and proved. For instance, the chain rule, Leibniz' formula, the mean value theorem and the fundamental theorem of β-calculus.

This paper is organized as follows. In Section 2, the β-exponential functions are defined and some of their properties are introduced. Also, we prove that they are the unique solutions of the first order β-difference equations. In Section 3, the β-trigonometric functions are presented and their properties are established. In Section 4, the β-hyperbolic functions are exhibited and their properties are shown. Throughout the paper $\mathbb{X}$ is a Banach space, I is an interval of $\mathbb{R}$ containing only one fixed point s_0 of β and

$$\beta^k(t) := \underbrace{\beta \circ \beta \circ \cdots \circ \beta}_{k-times}(t).$$

We need the following results from [14] to prove our main results.

Theorem 1.1. *If $f : I \to \mathbb{X}$ is continuous at s_0, then the series $\sum_{k=0}^{\infty} ||(\beta^k(t) - \beta^{k+1}(t))f(\beta^k(t))||$ is uniformly convergent on every compact interval $J \subseteq I$ containing s_0.*

Definition 1.2. For a function $f : I \to \mathbb{X}$, we define the β-difference operator of f as

$$D_\beta f(t) = \begin{cases} \dfrac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ \\ f'(s_0), & t = s_0 \end{cases}$$

provided that f' exists at s_0. In this case, we say that $D_\beta f(t)$ is the β-derivative of f at t. We say that f is β-differentiable on I if $f'(s_0)$ exists.

Theorem 1.3. *Assume that $f : I \to \mathbb{X}$ and $g : I \to \mathbb{R}$ are β-differentiable at $t \in I$. Then:*

(i) The product $fg : I \to \mathbb{X}$ is β-differentiable at t and

$$\begin{aligned} D_\beta(fg)(t) &= (D_\beta f(t))g(t) + f(\beta(t))D_\beta g(t) \\ &= (D_\beta f(t))g(\beta(t)) + f(t)D_\beta g(t). \end{aligned}$$

(ii) f/g is β-differentiable at t and

$$D_\beta\big(f/g\big)(t) = \frac{(D_\beta f(t))g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))},$$

provided that $g(t)g(\beta(t)) \neq 0$.

Lemma 1.4. *Let $f : I \to \mathbb{X}$ be β-differentiable and $D_\beta f(t) = 0$ for all $t \in I$, then $f(t) = f(s_0)$ for all $t \in I$.*

Theorem 1.5. *Assume $f : I \to \mathbb{X}$ is continuous at s_0. Then the function F defined by*

$$F(t) = \sum_{k=0}^{\infty} \big(\beta^k(t) - \beta^{k+1}(t)\big) f(\beta^k(t)), \ t \in I, \tag{1.1}$$

is a β-antiderivative of f with $F(s_0) = 0$. Conversely, a β-antiderivative F of f vanishing at s_0 is given by the formula (1.1).

Definition 1.6. Let $f : I \to \mathbb{X}$ and $a, b \in I$. We define the β-integral of f from a to b by

$$\int_a^b f(t)d_\beta t = \int_{s_0}^b f(t)d_\beta t - \int_{s_0}^a f(t)d_\beta t, \tag{1.2}$$

where

$$\int_{s_0}^{x} f(t)d_\beta t = \sum_{k=0}^{\infty} \big(\beta^k(x) - \beta^{k+1}(x)\big) f(\beta^k(x)), \quad x \in I. \tag{1.3}$$

provided that the series converges at $x = a$ and $x = b$. f is called β-integrable on I if the series converges at a, b for all $a, b \in I$. Clearly, if f is continuous at $s_0 \in I$, then f is β-integrable on I.

Theorem 1.7. *Let $f : I \to \mathbb{X}$ be continuous at s_0. Define the function*

$$F(x) = \int_{s_0}^{x} f(t)d_\beta t, \quad x \in I. \tag{1.4}$$

Then F is continuous at s_0, $D_\beta F(x)$ exists for all $x \in I$ and $D_\beta F(x) = f(x)$.

2 β-Exponential Functions

In this section, we define the β-exponential functions and we study some of their properties.

Definition 2.1 (β-Exponential Functions)**.** Assume that $p : I \to \mathbb{C}$ is a continuous function at s_0. We define the β-exponential functions $e_{p,\beta}(t)$ and $E_{p,\beta}(t)$ by

$$e_{p,\beta}(t) = \frac{1}{\prod_{k=0}^{\infty}\Big[1 - p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))\Big]} \tag{2.1}$$

and

$$E_{p,\beta}(t) = \prod_{k=0}^{\infty}\Big[1 + p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))\Big], \tag{2.2}$$

It is worth mentioning that both products in (2.1) and (2.2) are convergent since

$$\sum_{k=0}^{\infty} |p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))|$$

is uniformly convergent by Theorem 1.1. For the case when p is a constant function $p(t) = z$, $z \in \mathbb{C}$ and $\beta(t) = qt + \omega$, $\omega > 0$ and $q \in (0, 1)$, we obtain the Hahn exponential functions, see [4]. From (2.1), (2.2) we have

$$e_{p,\beta}(t) = \frac{1}{E_{-p,\beta}(t)}. \tag{2.3}$$

Theorem 2.2. *The β-exponential functions $e_{p,\beta}(t)$ and $E_{p,\beta}(t)$ are the unique solutions of the first order β-difference equations*

$$D_\beta y(t) = p(t)y(t), \qquad y(s_0) = 1, \tag{2.4}$$

and

$$D_\beta y(t) = p(t)y(\beta(t)), \qquad y(s_0) = 1, \tag{2.5}$$

respectively.

Proof. It is obvious that $e_{p,\beta}(s_0) = E_{p,\beta}(s_0) = 1$. We have

$$\begin{aligned} D_\beta e_{p,\beta}(t) &= \frac{e_{p,\beta}(\beta(t)) - e_{p,\beta}(t)}{\beta(t) - t} \\ &= \frac{1}{\beta(t) - t}\Big[\frac{1}{\prod_{k=0}^{\infty}(1 - p(\beta^{k+1}(t))(\beta^{k+1}(t) - \beta^{k+2}(t)))} \\ &\qquad - \frac{1}{\prod_{k=0}^{\infty}(1 - p(\beta^{k}(t))(\beta^{k}(t) - \beta^{k+1}(t)))}\Big] \\ &= \frac{p(t)}{\prod_{k=o}^{\infty}(1 - p(\beta^{k}(t))(\beta^{k}(t) - \beta^{k+1}(t)))} = p(t)e_{p,\beta}(t). \end{aligned}$$

Similarly, we see that $E_{p,\beta}(t)$ is a solution of (2.5). Finally, to prove the uniqueness of the solution $e_{p,\beta}(t)$, let $x(t)$ be another solution of (2.4). We have

$$D_\beta\left(\frac{x(t)}{e_{p,\beta}(t)}\right) = \frac{e_{p,\beta}(t)D_\beta x(t) - x(t)D_\beta e_{p,\beta}(t)}{e_{p,\beta}(t)e_{p,\beta}(\beta(t))} = 0, \;\; t \in I.$$

By Lemma 1.4, $\dfrac{x(t)}{e_{p,\beta}(t)}$ is a constant function and $\dfrac{x(t)}{e_{p,\beta}(t)} = \dfrac{x(s_0)}{e_{p,\beta}(s_0)} = 1$, i.e., $x(t) = e_{p,\beta}(t)$ for all $t \in I$. Similarly $E_{p,\beta}(t)$ is the unique solution of (2.5). $\square$

Consider the non-homogeneous first order linear β-difference equation

$$D_\beta y(t) = p(t)y(t) + f(t), \quad y(s_0) = y_0 \in \mathbb{X}. \tag{2.6}$$

Theorem 2.3. *Let $f : I \to \mathbb{X}$ be continuous function at s_0. then*

$$y(t) = e_{p,\beta}(t)\Big(y_0 + \int_{s_0}^{t} f(\tau)E_{-p,\beta}(\beta(\tau))d_\beta\tau\Big) \tag{2.7}$$

is a solution of equation (2.6).

Proof. We have

$$\begin{aligned}D_\beta y(t) &= D_\beta e_{p,\beta}(t)y_0 + D_\beta e_{p,\beta}(t)\int_{s_0}^{t} f(\tau)E_{-p,\beta}(\beta(\tau))d_\beta\tau \\ &\quad + e_{p,\beta}(\beta(t))f(t)E_{-p,\beta}(\beta(t)) \\ &= p(t)e_{p,\beta}(t)y_0 + p(t)e_{p,\beta}(t)\int_{s_0}^{t} f(\tau)E_{-p,\beta}(\beta(\tau))d_\beta\tau + f(t) \\ &= p(t)y(t) + f(t).\end{aligned}$$

Also, $y(s_0) = y_0$. □

In the following two theorems we introduce some important properties of the β-exponential function.

Theorem 2.4. *Let $p : I \to \mathbb{C}$ be a continuous function at s_0. Then the following properties hold:*

(i) $e_{p,\beta}(\beta(t)) = [1 + (\beta(t) - t)p(t)]e_{p,\beta}(t), \quad t \in I$,

(ii) $D_\beta(\frac{1}{e_{p,\beta}(t)}) = \frac{-p(t)}{e_{p,\beta}(\beta(t))}$,

(iii) $\frac{1}{e_{p,\beta}(t)}$ *is the unique solution of the first order β-difference equation*

$$D_\beta y(t) = \frac{-p(t)e_{p,\beta}(t)}{e_{p,\beta}(\beta(t))}y(t), \qquad y(s_0) = 1. \tag{2.8}$$

Proof. (i) From the definition of D_β, we have

$$\begin{aligned}e_{p,\beta}(\beta(t)) &= e_{p,\beta}(t) + (\beta(t) - t)D_\beta e_{p,\beta}(t) \\ &= e_{p,\beta}(t)[1 + (\beta(t) - t)p(t)].\end{aligned}$$

(ii) By Theorem 1.3 (ii), we get

$$D_\beta(\frac{1}{e_{p,\beta}(t)}) = \frac{-D_\beta e_{p,\beta}(t)}{e_{p,\beta}(t)e_{p,\beta}(\beta(t))} = \frac{-p(t)}{e_{p,\beta}(\beta(t))}.$$

(iii) We can see that

$$\frac{1}{e_{p,\beta}(s_0)} = 1.$$

By part (ii), we get $\frac{1}{e_{p,\beta}(t)}$ is a solution of (2.8). To show that the solution is unique, suppose that $x(t)$ is another solution of (2.8), then

$$\begin{aligned}D_\beta(x(t)e_{p,\beta}(t)) &= x(t)D_\beta e_{p,\beta}(t) + D_\beta x(t)e_{p,\beta}(\beta(t))\\ &= x(t)p(t)e_{p,\beta}(t) - \frac{p(t)e_{p,\beta}(t)}{e_{p,\beta}(\beta(t))}x(t)e_{p,\beta}(\beta(t)) = 0.\end{aligned}$$

Hence, $x(t)e_{p,\beta}(t) = x(s_0)e_{p,\beta}(s_0) = 1$. Therefore, $x(t) = \frac{1}{e_{p,\beta}(t)}$.

□

Theorem 2.5. *Suppose $p, q : I \to \mathbb{C}$ are continuous functions at s_0. Then the following properties hold:*

(i) $\frac{1}{e_{p,\beta}(t)} = e_{\frac{-p(t)}{1-p(t)(t-\beta(t))},\beta}(t)$,

(ii) $e_{p,\beta}(t)e_{q,\beta}(t) = e_{[p(t)+(\beta(t)-t)p(t)q(t)+q(t)],\beta}(t)$,

(iii) $\frac{e_{p,\beta}(t)}{e_{q,\beta}(t)} = e_{\frac{p(t)-q(t)}{1-q(t)(t-\beta(t))},\beta}(t)$.

Proof. (i) Clearly, $e_{\frac{-p(t)}{1-p(t)(t-\beta(t))},\beta}(t)$ is a solution of equation (2.8), then $\frac{1}{e_{p,\beta}(t)} = e_{\frac{-p(t)}{1-p(t)(t-\beta(t))},\beta}(t)$.

(ii) We have

$$\begin{aligned}D_\beta\big(e_{p,\beta}(t)e_{q,\beta}(t)\big) &= D_\beta e_{p,\beta}(t)e_{q,\beta}(t) + e_{p,\beta}(\beta(t))D_\beta e_{q,\beta}(t)\\ &= p(t)e_{p,\beta}(t)e_{q,\beta}(t) + q(t)e_{p,\beta}(\beta(t))e_{q,\beta}(t)\\ &= p(t)e_{p,\beta}(t)e_{q,\beta}(t) + q(t)e_{q,\beta}(t)[1 + (\beta(t)-t)p(t)]e_{p,\beta}(t)\\ &= [p(t) + (\beta(t)-t)p(t)q(t) + q(t)]e_{p,\beta}(t)e_{q,\beta}(t).\end{aligned}$$

(iii) This is a consequence of (i) and (ii).

The proof is complete. □

Example 2.6. Let $p(t) = \frac{2}{t}$ and $\beta(t) = \frac{1}{2}t + \frac{1}{2}$, for $t \in [1, 2]$. The unique fixed point of the function β is $s_0 = 1$. One can check that

$$e_{\frac{2}{t},\frac{1}{2}t+\frac{1}{2}}(t) = \frac{1}{\prod_{k=0}^{\infty}[1 - \frac{t-1}{t-1+2^k}]}.$$

Clearly, $e_{\frac{2}{t},\frac{1}{2}t+\frac{1}{2}}(1) = 1$. So, $e_{\frac{2}{t},\frac{1}{2}t+\frac{1}{2}}(t)$ is the unique solution of the equation

$$D_\beta y(t) = \frac{2}{t}y(t), \quad y(1) = 1.$$

Example 2.7. Let $p(t) = t(1+i)$ and $\beta(t) = \frac{1}{2}t$, for $t \in [0, 2]$. The unique fixed point of the function β is $s_0 = 0$. Clearly, $e_{t(1+i),\frac{1}{2}t}(0) = 1$. One can check that

$$e_{t(1+i),\frac{1}{2}t}(t) = \frac{1}{\prod_{k=0}^{\infty}\left[1 - \frac{t^2(1+i)}{2^{2k+1}}\right]},$$

is the unique solution of the equation $D_\beta y(t) = t(1+i)y(t), \quad y(0) = 1$.

3 β-Trigonometric Functions

In this Section we define the β-trigonometric functions and study some of their properties.

Definition 3.1 (β-Trigonometric Functions)**.** We define the β-trigonometric functions by

$$\cos_{p,\beta}(t) = \frac{e_{ip,\beta}(t) + e_{-ip,\beta}(t)}{2}, \tag{3.1}$$

$$\sin_{p,\beta}(t) = \frac{e_{ip,\beta}(t) - e_{-ip,\beta}(t)}{2i}, \tag{3.2}$$

$$\mathrm{Cos}_{p,\beta}(t) = \frac{E_{ip,\beta}(t) + E_{-ip,\beta}(t)}{2}, \tag{3.3}$$

and

$$\mathrm{Sin}_{p,\beta}(t) = \frac{E_{ip,\beta}(t) - E_{-ip,\beta}(t)}{2i}. \tag{3.4}$$

Simple calculations show that the β-trigonometric functions satisfy the relations in the following theorem.

Theorem 3.2. *For all $t \in I$. The following relations are true:*

(1) $D_\beta \sin_{p,\beta}(t) = p(t)\cos_{p,\beta}(t)$,

(2) $D_\beta \cos_{p,\beta}(t) = -p(t)\sin_{p,\beta}(t)$,

(3) $\cos_{p,\beta}(t) + i\sin_{p,\beta}(t) = e_{ip,\beta}(t)$,

(4) $\cos^2_{p,\beta}(t) + \sin^2_{p,\beta}(t) = e_{ip,\beta}(t)e_{-ip,\beta}(t)$ *(at $t = s_0$, $\cos^2_{p,\beta}(t) + \sin^2_{p,\beta}(t) = 1$),*

(5) $D_\beta \mathrm{Sin}_{p,\beta}(t) = p(t)\mathrm{Cos}_{p,\beta}(\beta(t))$,

(6) $D_\beta \mathrm{Cos}_{p,\beta}(t) = -p(t)\mathrm{Sin}_{p,\beta}(\beta(t))$,

(7) $\mathrm{Sin}^2_{p,\beta}(t) + \mathrm{Cos}^2_{p,\beta}(t) = E_{ip,\beta}(t)E_{-ip,\beta}(t)$,

(8) $\mathrm{Cos}_{p,\beta}(t) + i\mathrm{Sin}_{p,\beta}(t) = E_{ip,\beta}(t)$,

(9) $\sin_{p,\beta}(t)\mathrm{Sin}_{p,\beta}(t) + \cos_{p,\beta}(t)\mathrm{Cos}_{p,\beta}(t) = 1$,

(10) $\sin_{p,\beta}(t)\mathrm{Cos}_{p,\beta}(t) - \cos_{p,\beta}(t)\mathrm{Sin}_{p,\beta}(t) = 0$.

In the following theorem it can be easily seen that the β-trigonometric functions are solutions of the second order β-difference equations.

Theorem 3.3. *Let $p : I \to \mathbb{C}$ be a continuous function at s_0. Then $\cos_{p,\beta}(t)$, $\sin_{p,\beta}(t)$, $\mathrm{Cos}_{p,\beta}(t)$ and $\mathrm{Sin}_{p,\beta}(t)$ are solutions of the following second order β-difference equations, respectively.*

i) $D^2_\beta\, x(t) = -p^2(t)x(t) - D_\beta p(t)\, \sin_{p,\beta}(\beta(t))$,

$$x(s_0) = 1,\ D_\beta x(s_0) = 0.$$

ii) $D^2_\beta\, x(t) = -p^2(t)x(t) + D_\beta p(t)\, \cos_{p,\beta}(\beta(t))$,

$$x(s_0) = 0,\ D_\beta x(s_0) = p(s_0).$$

iii) $D^2_\beta\, x(t) = -p^2(\beta(t))\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t}x(\beta^2(t)) - D_\beta p(t)\, \mathrm{Sin}_{p,\beta}(\beta(t))$,

$$x(s_0) = 1,\ D_\beta x(s_0) = 0.$$

iv) $D^2_\beta\, x(t) = -p^2(\beta(t))\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t}x(\beta^2(t)) + D_\beta p(t)\, \mathrm{Cos}_{p,\beta}(\beta(t))$,

$$x(s_0) = 0,\ D_\beta x(s_0) = p(s_0).$$

Corollary 3.4. *Let $z \in \mathbb{C}$. Then $\cos_{z,\beta}(t)$, $\sin_{z,\beta}(t)$, $\mathrm{Cos}_{z,\beta}(t)$ and $\mathrm{Sin}_{z,\beta}(t)$ are solutions of the following second order β-difference equations, respectively.*

i) $D^2_\beta\, x(t) = -z^2\, x(t), \quad x(s_0) = 1,\ D_\beta x(s_0) = 0.$

ii) $D^2_\beta\, x(t) = -z^2\, x(t), \quad x(s_0) = 0,\ D_\beta x(s_0) = z.$

iii) $D^2_\beta\, x(t) = -z^2\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t}x(\beta^2(t)), \quad x(s_0) = 1,\ D_\beta x(s_0) = 0.$

iv) $D^2_\beta\, x(t) = -z^2\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t}x(\beta^2(t)), \quad x(s_0) = 0,\ D_\beta x(s_0) = z.$

Example 3.5. Let $\beta(t) = qt + \omega$, $q \in (0,1)$, $\omega > 0$, and let $p(t) = z$, $z \in \mathbb{C}$ be a constant. Then, $s_0 = \dfrac{\omega}{1-q}$ and $\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t} = q$. Consequently, $\cos_{z,\beta}(t)$, $\sin_{z,\beta}(t)$, $\text{Cos}_{z,\beta}(t)$ and $\text{Sin}_{z,\beta}(t)$ are solutions of the following second order q,ω-difference equations, respectively.

i) $D^2_{q,\omega}\, x(t) = -z^2\, x(t), \quad x(s_0) = 1,\ D_{q,\omega}x(s_0) = 0.$

ii) $D^2_{q,\omega}\, x(t) = -z^2\, x(t), \quad x(s_0) = 0,\ D_{q,\omega}x(s_0) = z.$

iii) $D^2_{q,\omega}\, x(t) = -z^2 q\, x(q^2 t + \omega(1+q)), \quad x(s_0) = 1,\ D_{q,\omega}x(s_0) = 0.$

iv) $D^2_{q,\omega}\, x(t) = -z^2 q\, x(q^2 t + \omega(1+q)), \quad x(s_0) = 0,\ D_{q,\omega}x(s_0) = z.$

Example 3.6. Let $\beta(t) = qt$, $q \in (0,1)$, and $p(t) = z$, $z \in \mathbb{C}$ be a constant. Then, $s_0 = 0$ and $\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t} = q$. Consequently, $\cos_{z,\beta}(t)$, $\sin_{z,\beta}(t)$, $\text{Cos}_{z,\beta}(t)$ and $\text{Sin}_{z,\beta}(t)$ are solutions of the following second order q,ω-difference equations, respectively.

i) $D^2_q\, x(t) = -z^2\, x(t), \quad x(s_0) = 1,\ D_q x(0) = 0.$

ii) $D^2_q\, x(t) = -z^2\, x(t), \quad x(s_0) = 0,\ D_q x(0) = z.$

iii) $D^2_q\, x(t) = -z^2 q\, x(q^2 t), \quad x(s_0) = 1,\ D_q x(0) = 0.$

iv) $D^2_q\, x(t) = -z^2 q\, x(q^2 t), \quad x(s_0) = 0,\ D_q x(0) = z.$

4 β-Hyperbolic Functions

In this Section we define the β-hyperbolic functions and study some of their properties.

Definition 4.1 (β-Hyperbolic Functions)**.** We define the β-hyperbolic functions by

$$\cosh_{p,\beta}(t) = \frac{e_{p,\beta}(t) + e_{-p,\beta}(t)}{2}, \tag{4.1}$$

$$\sinh_{p,\beta}(t) = \frac{e_{p,\beta}(t) - e_{-p,\beta}(t)}{2}, \tag{4.2}$$

$$\text{Cosh}_{p,\beta}(t) = \frac{E_{p,\beta}(t) + E_{-p,\beta}(t)}{2}, \tag{4.3}$$

and

$$\text{Sinh}_{p,\beta}(t) = \frac{E_{p,\beta}(t) - E_{-p,\beta}(t)}{2}. \tag{4.4}$$

The following theorem introduces some properties of the β-hyperbolic functions. Its proof is straightforward.

Theorem 4.2. *The β-hyperbolic functions satisfy the following properties:*

(1) $D_\beta \cosh_{p,\beta}(t) = p(t)\sinh_{p,\beta}(t)$,

(2) $D_\beta \sinh_{p,\beta}(t) = p(t)\cosh_{p,\beta}(t)$,

(3) $\cosh^2_{p,\beta}(t) - \sinh^2_{p,\beta}(t) = e_{p,\beta}(t)e_{-p,\beta}(t)$
(at $t = s_0$, $\cosh^2_{p,\beta}(t) - \sinh^2_{p,\beta}(t) = 1$*)*,

(4) $\cosh_{p,\beta}(t) + \sinh_{p,\beta}(t) = e_{p,\beta}(t)$,

(5) $\cosh_{p,\beta}(t) - \sinh_{p,\beta}(t) = e_{-p,\beta}(t)$,

(6) $D_\beta \operatorname{Cosh}_{p,\beta}(t) = p(t)\operatorname{Sinh}_{p,\beta}(\beta(t))$,

(7) $D_\beta \operatorname{Sinh}_{p,\beta}(t) = p(t)\operatorname{Cosh}_{p,\beta}(\beta(t))$,

(8) $\operatorname{Cosh}^2_{p,\beta}(t) - \operatorname{Sinh}^2_{p,\beta}(t) = E_{p,\beta}(t)E_{-p,\beta}(t)$,

(9) $\operatorname{Cosh}_{p,\beta}(t) + \operatorname{Sinh}_{p,\beta}(t) = E_{p,\beta}(t)$,

(10) $\operatorname{Cosh}_{p,\beta}(t) - \operatorname{Sinh}_{p,\beta}(t) = E_{-p,\beta}(t)$.

In the following Theorem, simple calculations show that the β-hyperbolic functions are solutions of second order β-difference equations.

Theorem 4.3. *Let $p : I \to \mathbb{C}$ be a continuous function at s_0. Then* $\cosh_{p,\beta}(t)$, $\sinh_{p,\beta}(t)$, $\operatorname{Cosh}_{p,\beta}(t)$ *and* $\operatorname{Sinh}_{p,\beta}(t)$ *are solutions of the following second order β-difference equations, respectively.*

i) $D^2_\beta\, x(t) = p^2(t)x(t) + D_\beta p(t) \sinh_{p,\beta}(\beta(t))$,

$$x(s_0) = 1, \ D_\beta\, x(s_0) = 0.$$

ii) $D^2_\beta\, x(t) = p^2(t)x(t) + D_\beta p(t) \cosh_{p,\beta}(\beta(t))$,

$$x(s_0) = 0, \ D_\beta x(s_0) = p(s_0).$$

iii) $D^2_\beta\, x(t) = p^2(\beta(t))\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t}x(\beta^2(t)) + D_\beta p(t) \operatorname{Sinh}_{p,\beta}(\beta(t))$,

$$x(s_0) = 1, \ D_\beta x(s_0) = 0.$$

iv) $D^2_\beta x(t) = p^2(\beta(t))\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t}x(\beta^2(t)) + D_\beta p(t) \operatorname{Cosh}_{p,\beta}(\beta(t))$,

$$x(s_0) = 0, \ D_\beta x(s_0) = p(s_0).$$

Corollary 4.4. *Let $z \in \mathbb{C}$. Then* $\cosh_{z,\beta}(t)$, $\sinh_{z,\beta}(t)$, $\text{Cosh}_{z,\beta}(t)$ *and* $\text{Sinh}_{z,\beta}(t)$ *are solutions of the following second order β-difference equations, respectively.*

i) $D_\beta^2\, x(t) = z^2\, x(t), \quad x(s_0) = 1,\ D_\beta\, x(s_0) = 0.$

ii) $D_\beta^2\, x(t) = z^2\, x(t), \quad x(s_0) = 0,\ D_\beta x(s_0) = z.$

iii) $D_\beta^2\, x(t) = z^2 \dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t} x(\beta^2(t)), \quad x(s_0) = 1,\ D_\beta x(s_0) = 0.$

iv) $D_\beta^2 x(t) = z^2 \dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t} x(\beta^2(t)), \quad x(s_0) = 0,\ D_\beta x(s_0) = z.$

Example 4.5. Let $\beta(t) = qt + \omega$, $q \in (0,1)$, $\omega > 0$, and let $p(t) = z$, $z \in \mathbb{C}$ be a constant. Then, $\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t} = q$. Consequently, $\cosh_{z,\beta}(t)$, $\sinh_{z,\beta}(t)$, $\text{Cosh}_{z,\beta}(t)$ and $\text{Sinh}_{z,\beta}(t)$ are solutions of the following second order q,ω-difference equations, respectively.

i) $D_{q,\omega}^2\, x(t) = z^2\, x(t), \quad x(s_0) = 1,\ D_{q,\omega} x(s_0) = 0.$

ii) $D_{q,\omega}^2\, x(t) = z^2\, x(t), \quad x(s_0) = 0,\ D_{q,\omega} x(s_0) = z.$

iii) $D_{q,\omega}^2\, x(t) = z^2 q\, x(q^2 t + \omega(1+q)), \quad x(s_0) = 1,\ D_{q,\omega} x(s_0) = 0.$

iv) $D_{q,\omega}^2\, x(t) = z^2 q\, x(q^2 t + \omega(1+q)), \quad x(s_0) = 0,\ D_{q,\omega} x(s_0) = z.$

Example 4.6. Let $\beta(t) = qt$, $q \in (0,1)$ and $p(t) = z$, $z \in \mathbb{C}$ be a constant. Then, $\dfrac{\beta^2(t) - \beta(t)}{\beta(t) - t} = q$. Consequently, $\cosh_{z,\beta}(t)$, $\sinh_{z,\beta}(t)$, $\text{Cosh}_{z,\beta}(t)$ and $\text{Sinh}_{z,\beta}(t)$ are solutions of the following second order q-difference equations, respectively.

i) $D_q^2\, x(t) = z^2\, x(t), \quad x(s_0) = 1,\ D_q x(0) = 0.$

ii) $D_q^2\, x(t) = z^2\, x(t), \quad x(s_0) = 0,\ D_q x(0) = z.$

iii) $D_q^2\, x(t) = z^2 q\, x(q^2 t), \quad x(s_0) = 1,\ D_q x(0) = 0.$

iv) $D_q^2\, x(t) = z^2 q\, x(q^2 t), \quad x(s_0) = 0,\ D_q x(0) = z.$

Conclusion

In this paper, we defined the β-exponential functions and proved that they are a unique solutions for the first order β-difference equations. Also, the β-trigonometric functions and their properties were introduced and that they satisfy the second order β-difference equations. Finally, the β-hyperbolic functions and their properties were introduced.

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