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Sensitivity for Set-Valued Discrete Dynamical Systems

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Abstract

Any discrete topological dynamical system can be extended to some hyperspace dynamical system. So the natural question to know if a property of the base system transfers on the extended one, and viceversa, arises. This note investigates these connections relatively to various forms of *sensitivity*, a dynamical property which is essential in the definition of some kind of *chaos*. Some results about these relationships are given.

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1 Introduction

Given a dynamical system, a natural question arises: when its extension to some hyperspace satisfies a dynamical property of the base space, and viceversa. This question has a scientific starting point. In many fields, such as biology, demography, it is not enough to know the behavior of an individual point in the space but it is necessary to see the collective life, that is to say the movement of subsets in the space. This corresponds to study the set valued dynamical system associated to the base system. In recent years many authors investigated these kind of relationships by giving particular attention to topological properties such as transitivity, weakly mixing, mixing. In particular Roman Flores investigated the transitivity, and John Banks used the weakly mixing property to obtain the transitivity for the induced map (see [2, 8, 13]).

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Among these properties *sensitive dependence on initial condition*, briefly *sensitivity*, represents the central concept for the definition of *chaos* (see [1, 6, 9]). However, it is surprising that, when the space is infinite, the sensitivity is a redundant property in the definition of Devaney chaos (see [6] for the definition and [3] for the result). In [11], Hou, Liao and Liu studied the sensitivity of systems induced by M-systems. Stronger forms of sensitivity, formulated in terms of large subset of $\mathbb{N}$ have been introduced by Moothathu in [14]. Roughly speaking, a discrete dynamical system is sensitive when given a region in the space there are two points in the region such that at a time n the n-th iterates of the two points are separated. The largeness of the set of integers for which this happens gives an idea of the measure of the sensitivity. Therefore, other stronger forms of sensitivity are obtained by enlarging this measure.

In this note we restrict our attention on the property of sensitivity, even in its stronger forms. Namely, the aim of this paper is to study of the condition for which the sensitivity, or stronger sensitivity, transfers to the extended system, and viceversa. In other words, we investigate the relationships between the sensitivity of a dynamical system and the sensitivity of the associated set valued discrete system. We will ask, in particular, when a discrete dynamical sensitive (resp. strongly sensitive) space satisfies that its extension is also sensitive (resp. strongly sensitive). Some results about these relationships are given. We refer to [6] for definitions not explicitly given.

2 Preliminaries

A pair (X, f) where X is a topological space and $f : X \to X$ is a continuous map, is said to be topological dynamical system.

We need to recall some dynamical properties, well known in literature.

Definition 2.1. A dynamical system (X, f) is *(topologically) transitive* if for any pair of nonempty open sets U and V there is a $k \geq 1$ such that $f^k(U) \cap V \neq \emptyset$.

Stronger properties are the following.

Definition 2.2. A dynamical system (X, f) is *weakly (topologically) mixing* if for all nonempty open sets U_1, U_2, V_1 and V_2 there exists $k \geq 1$ such that $f^k(U_1) \cap V_1 \neq \emptyset$ and $f^k(U_2) \cap V_2 \neq \emptyset$.

Definition 2.3. A dynamical system (X, f) is *(topologically) mixing* if for any pair of non-empty open sets U and V there is a $N \geq 1$ such that, for all $k \geq N$ $f^k(U) \cap V \neq \emptyset$

Moreover, we define the following.

Definition 2.4. A point x is *periodic* if $f^k(x) = x$ for some $k \geq 1$. The least k such that this happens is called *period* of x.

If X is a metrizable space, and it is metrized by d, we say that f has *sensitive dependence on initial conditions*, in brief *sensitive*, if there is a constant $\delta > 0$ such that for every point x and every open set U containing x there is a $y \in U$ and a $k \geq 1$ such that $d(f^k(x), f^k(y)) \geq \delta$.

A map which is transitive, sensitive and has a dense set of periodic points is called *Devaney chaotic*, briefly *D-chaotic* (see [6]). However it is known that, when the space is infinite, transitivity and density of periodic points, both topological properties in nature, imply the sensitivity which is expressed in metric terms (see [3]). But sensitivity is a fundamental concept for the definition of other kind of chaos, such as for example, Kato's chaos, introduced in 1996 (see [12]).

Now, since we consider extensions of the dynamical system (X, f) on set-valued discrete topological dynamical systems, we need to recall how to topologize and to metrize the hyperspace. The set of all nonempty closed subset of a topological space X, denoted by 2^X, can be topologized by the Vietoris topology, denoted by τ_V, where V stands for Vietoris. A basis for τ_V is given by the collection of sets of the form

$$\langle U_1, U_2, ..., U_n \rangle = \{A \in 2^X : A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for all } i \leq n\},$$

where $U_1, U_2, ..., U_n$ are nonempty open subsets of X.

The Vietoris topology can be split in two topologies, that is to say it is the supremum of two topologies,

$$\tau_V = \tau_V^- \vee \tau_V^+$$

where τ_V^+ has as base the collection of sets of the form

$$U^+ = \{A \in 2^X : A \subseteq V\} \tag{2.1}$$

and τ_V^- has as subbase the collection of sets of the form

$$W^- = \{A \in 2^X : A \cap W \neq \emptyset\} \tag{2.2}$$

where U and W are nonempty open subsets of X.

Let (X, d) be a metric space. The set 2^X can be also metrized by the Hausdorff metric which we denote by H_d. Let $A, B \in 2^X$, we say *excess* the real number

$$e_d(A, B) = \sup\{d(a, B) : a \in A\}. \tag{2.3}$$

The *Hausdorff metric* on 2^X is defined by

$$H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}, \tag{2.4}$$

for every $A, B \in 2^X$.

If X is a compact space, the topology induced on 2^X by the Hausdorff metric (2.4) coincides with the Vietoris topology. Moreover these topologies agrees on $\mathcal{K}(X)$, the set of all nonempty compact subset of a topological space X.

The most studied extended topological dynamical system is $(\mathcal{K}(X), \overline{f})$, where the map $\overline{f}$ is defined by $\overline{f}(A) = f(A)$ for each $A \in \mathcal{K}(X)$.

Definition 2.5. A point x is said to be *almost periodic* if for any arbitrary neighborhood $U \ni x$ there is a positive integer k such that $\{f^l(x), f^{l+1}(x), f^{l+2}(x), ..., f^{l+k}(x)\} \cap U \neq \emptyset$ for every $l \in \mathbb{N}$.

Definition 2.6. A topological dynamical system (X, f) is called M-system if f is transitive and the set of almost periodic points is dense in X.

Theorem 2.7 (Hou, Liao, Liu theorem)**.** *If (X, f) is a nonminimal M-system, then $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is sensitive.*

We have the following result.

Corollary 2.8. *If (X, f) is Devaney chaotic, then $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is sensitive.*

This note investigates the connections between the strong sensitivity of a given topological discrete dynamical system (X, f) and the strong sensitivity of $(\mathcal{H}(X), \overline{f})$, where $\mathcal{H}(X)$ runs in the set of the extensions of X, in analogy with the some results already known about the sensitivity.

3 Extensions and Compatibility

Evidently, since 2^X contains the singletons of the space X, it can be seen as an extension of X. More generally, we say that a $\mathcal{H} \subset 2^X$ is an extension of X if $\mathcal{H}$ contains the singletons. For example, the set of all compact subsets of a topological space X, $\mathcal{K}(X)$, is an extension for X. Now, given a dynamical system (X, f) and an extension $\mathcal{H} \subset 2^X$, we want to know when a map $\overline{f} : \mathcal{H} \to \mathcal{H}$ extends the map f in such a way that $(\mathcal{H}, \overline{f})$ is a dynamical system.

Definition 3.1. Given a dynamical system (X, f), a dynamical system $(\mathcal{H}(X), F)$, where $\mathcal{H}(X) \subset 2^X$, is said to be an *extension* of (X, f) if it contains a dynamical system which is topologically conjugate to (X, f) that is to say:

1. $\mathcal{H}$ contains a subspace Y homeomorphic to X: $\exists Y \subset \mathcal{H}$ and a homeomorphism $h : X \longrightarrow Y$ such that
2. $F_{|Y} \circ h = h \circ f$.

If $(\mathcal{H}(X), F)$ is an extension of (X, f) and (X, f) embeds as a dense dynamical system in $(\mathcal{H}(X), F)$, that is to say the closure $\overline{Y} = \mathcal{H}$, then $\mathcal{H}$ is said to be *dense extension* of (X, f).

As an example, given a dynamical system (X, f), then $(\mathcal{K}(X), F)$, where $\mathcal{K}(X) = \{A \subseteq X : A \text{ is compact}\}$ equipped by the Vietoris topology and $F : \mathcal{K}(X) \to \mathcal{K}(X)$ is defined by $F(A) = f(A)$, is a dense extension of (X, f).

Definition 3.2. We say that $f : X \to X$ is compatible with $\mathcal{H}$ provided that $f(A) \in \mathcal{H}$ for every $A \in \mathcal{H}$.

Obviously, every map is compatible with $\mathcal{F}(X) = \{A \in 2^X : A \text{ is finite}\}$. Moreover, every continuous map is *compatible* with $\mathcal{K}(X) = \{A \in 2^X : A \text{ is compact}\}$, and with $\mathcal{C}(X) = \{A \in 2^X : A \text{ is connected}\}$.

Proposition 3.3. *If $f : X \to X$ is compatible with $\mathcal{H}$ then there exists an extended map*

$$\overline{f} : \mathcal{H} \longrightarrow \mathcal{H}$$

defined by $\overline{f}(A) = f(A)$ for every $A \in \mathcal{H}$.

The converse is not true. Indeed, if (X, f) is a dynamical system and $(\mathcal{H}, F)$ is an extension for (X, f), then the map f need not be compatible with $\mathcal{H}$.

It is easy to check that the map

$$e : X \hookrightarrow (2^X, \tau_V)$$

defined by $e(x) = \{x\}$ for every $x \in X$, is an homeomorphism when it is restricted to $e(X)$, and $e(X)$ is open. Moreover, if X is a metrizable space metrized by d, then the map

$$e : X \hookrightarrow (2^X, H_d)$$

defined by $e(x) = \{x\}$ for every $x \in X$, is an isometry when it is restricted to $e(X)$.

Now, it follows that, in both cases $(2^X, \tau_V)$ and $(2^X, H_d)$, the dynamical system $(2^X, F)$, where $F : 2^X \longrightarrow 2^X$ is defined by $F(A) = \overline{f(A)}$, for every $A \in 2^X$, is an extension for (X, f) but f is not compatible with 2^X.

4 Some Results

Let X be a metric space and let (X, f) be a discrete dynamical system. Moreover, let $(\mathcal{H}(X), \overline{f})$, where $(\mathcal{H}(X), H_d)$ is the Hausdorff metric space, be an extension for (X, f). We investigate when the dynamical system (X, f) inherits the sensitivity, or other stronger form of sensitivity, from $(\mathcal{H}(X), \overline{f})$, and, conversely, when these properties transfer from (X, f) to $(\mathcal{H}(X), \overline{f})$.

First of all we need to recall the following definition.

Definition 4.1. A map f of a metric space (X, d) to itself is sensitive iff there is a constant $\delta > 0$ such that for every point x and every open subset U of X containing x there exist an element $y \in U$ and an integer $n \geq 1$ such that $d(f^n(x), f^n(y)) \geq \delta$, for some $\delta > 0$.

It is known that if X is a compact metric space, then the sensitivity of the extended map on $\mathcal{K}(X)$ implies the sensitivity for the base map(see [10]). More generally, we have the following result.

Theorem 4.2. *If $\overline{f} : (\mathcal{H}(X), H_d) \to (\mathcal{H}(X), H_d)$ is sensitive, then so is $f : (X, d) \to (X, d)$.*

Proof. The results immediately comes from the fact that the map

$$e : (X, d) \to (\mathcal{H}(X), H_d)$$

defined by $e(x) = \{x\}$ for every $x \in X$, is an isometry.

Suppose that $\overline{f}$ is sensitive. This means that for every $A \in \mathcal{H}(X)$ and every subset U of $\mathcal{H}(X)$, open in the topology induced by H_d, there exist an element $B \in U$ and an integer $n \geq 1$ such that $H_d(\overline{f}^n(A), \overline{f}^n(B)) \geq \delta$, for some $\delta > 0$. Let $\epsilon > 0$ and $x \in X$. Then the open ball $B_{H_d}(\{x\}, \epsilon)$ is an open set in the topology induced by H_d. By the hypothesis there is $E \in B_{H_d}(\{x\}, \epsilon)$ and an integer $n \in \mathbb{N}$ such that $H_d(\overline{f}^n(\{x\}), \overline{f}^n(E)) \geq \delta$, for some $\delta > 0$. Then there is an $y \in E$ such that $H_d(\overline{f}^n(\{x\}), \overline{f}^n(\{y\})) \geq \delta$. Now, since e is an isometry, we have

$$H_d(\overline{f}^n(\{x\}), \overline{f}^n(\{y\}) = d(f^n(x), f^n(y)),$$

completing the proof. □

Stronger forms of sensitivity have been introduced by Subrahmonian Moothathu in [14]. He considers two notions of largeness for subsets of $\mathbb{N}$, the *syndeticity* and the *cofiniteness* and uses them to define *cofinite sensitivity* and *syndetical sensitivity*.

A subset $A \subset \mathbb{N}$ is called *cofinite* if $A \setminus \mathbb{N}$ is finite, and A is called *syndetic* if A is infinite and if $A = \{a_1 < a_2 < a_3 ...\}$ then there exists an $M \in \mathbb{N}$ such that $a_n - a_{n-1} < M$ for every $n \in \mathbb{N}$, where $a_0 = 0$.

For a dynamical system (X, f), take $U, V \subset X$ and let

$$N_f(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}. \tag{4.1}$$

This set enables Mothathu (see [14]) to reformulate the known properties of transitivity and mixing, and to introduce the new concept of *syndetical transivity*. Indeed, from this new point of view, we have:

M1) f is *transitive* if for every pair of nonempty open sets $U, V \subset X$ we have that $N_f(U, V) \neq \emptyset$.

M2) f is *syndetically transitive* if if for every pair of nonempty open sets $U, V \subset X$ we have that $N_f(U, V)$ is syndetic.

M3) f is *mixing* if for every pair of nonempty open sets $U, V \subset X$ we have that $N_f(U, V)$ is cofinite.

Moreover, Moothathu defines (see [14]) stronger forms of sensitivity in a similar way. If X is metrized by a metric d, for $U \subset X$ and $\delta > 0$, let

$$N_f(U, \delta) = \{n \in \mathbb{N} | \exists y, z \in U : d(f^n(y), f^n(z) > \delta\}. \tag{4.2}$$

Thus we have:

M4) f is *sensitive* if there is $\delta > 0$ with the property that for every nonempty open set $U \subset X$ we have that $N_f(U, \delta) \neq \emptyset$.

M5) f is *syndetically sensitive* if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$ we have that $N_f(U, \delta)$ is syndetic.

M6) f is *cofinitely sensitive* if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$ we have that $N_f(U, \delta)$ is cofinite.

We investigate the connections between syndetically sensitivity (resp. cofinitely sensitivity) of $(\mathcal{H}(X), \overline{f})$, where $\mathcal{H}(X)$ runs in the set of the extensions of X.

Relatively to these stronger forms of sensitivity, we have the following result.

Theorem 4.3. *If* $\overline{f} : (\mathcal{H}(X), H_d) \to (\mathcal{H}(X), H_d)$ *is cofinitely (resp. syndetically) sensitive, then so is* $f : (X, d) \to (X, d)$.

Proof. Let $\mathcal{H}(X)$ be an extension of the space X, and, let f be a compatible map. Suppose that $\overline{f} : (\mathcal{H}(X), H_d) \to (\mathcal{H}(X), H_d)$ is cofinitely (resp. syndetically) sensitive, with constant $\delta > 0$. Now consider $x \in X$, $\epsilon > 0$ and the open ball $U = B_d(x, \epsilon)$, then the ball $\overline{U} = B_{H_d}(\{x\}, \epsilon)$ is open in the topology induced by the Hausdorff metric. So, by the hypothesis, $N_{\overline{f}}(\overline{U}, \delta)$ is syndetic (resp. cofinite). This implies that $N_f(U, \delta)$ is syndetic (resp. cofinite). It is enough to observe that

1. $N_{\overline{f}}(\overline{U}, \delta) = N_f(U, \delta)$.

2. Syndeticity is an hereditary property, while for cofiniteness we have that a set containing a cofinite set is cofinite.

Indeed, if $n \in N_f(U, \delta)$, then there exist $y, z \in U$ such that

$$d(f^n(y), f^n(z)) \geq \delta.$$

Then, since $d(f^n(y), f^n(z)) = H_d(\overline{f}^n(\{x\}), \overline{f}^n(\{y\}))$, we have $n \in N_{\overline{f}}(\overline{U}, \delta)$. Conversely, if $n \in N_{\overline{f}}(\overline{U}, \delta)$, then there exist $A, B \in \overline{U}$ such that $H_d(\overline{f}^n(A), \overline{f}^n(B)) \geq \delta$. It follows that there exist $a \in A$ and $b \in B$ such that $d(f^n(a), f^n(b)) \geq \delta$. Thus, since $A, B \subset U$, we have $n \in N_f(U, \delta)$. In conclusion, the result immediately follows. □

If f is a cofinitely (resp. syndetically) sensitive map, then $\overline{f}$ need not be a cofinitely (resp. syndetically) sensitive map.

The converse of the previous result is not true, as the following example shows.

Example 4.4. *A cofinitely sensitive map with an extension which not a sensitive map.*

Let X be the cylinder $S^1 \times I$ with the usual metric

$$\rho((e^{2\pi i\alpha}, x), (e^{2\pi i\beta}, y) = \max\{| e^{2\pi i\alpha} - e^{2\pi i\beta} |, | x - y |\}.$$

Let $h : X \to X$ be the product map $f \times g$, where f is the rotation defined by $f(e^{2\pi i\alpha}) = e^{2\pi i(\alpha+1)}$, for every $0 \le \alpha < 1$ and g the *tent map* defined by $g(x) = 1- \mid 2x - 1 \mid$, for every $x \in I$. Now, since g is mixing, we have g is cofinitely sensitive. Therefore h is cofinitely sensitive. However $\overline{h} : \mathcal{K}(X) \to \mathcal{K}(X)$ is not sensitive, as Gu shows in [10, Example 3.7].

As seen, the converse of the previous theorem is not true. Anyway, if, in the definition, the open set $U \subset \mathcal{H}$ runs in the topology τ_V^+, we have the following result.

Theorem 4.5. *Let (X, f) be a discrete topological dynamical system, where X is a metrizable space, and $(\mathcal{H}, \overline{f})$ an extension of (X, f), where $\mathcal{H}$ is equipped with the topology τ_V^+. Then the following conditions are equivalent:*

1. *f is cofinitely (resp. syndetically) sensitive;*
2. *$\overline{f}$ is cofinitely (resp. syndetically) sensitive.*

Proof. $1. \Rightarrow 2.$ Let V be an open set in X. Then V^+ is an open set in the hyperspace $(\mathcal{H}, \tau_V^+)$ and we have

$$N_{\overline{f}}(V^+, \delta) = N_f(V, \delta).$$

It follows that, since by the hypothesis $\overline{f}$ is cofinitely (resp. syndetically) sensitive, then $N_f(V, \delta)$ is cofinite (resp. syndetical). $2. \Rightarrow 1.$ This follows from the previous theorem. □

Glasner and Weiss in [9] proved the following result.

Theorem 4.6. *A transitive system with a dense set of almost periodic points, that is to say an M-system, is either minimal or sensitive.*

Therefore, we have the following result.

Theorem 4.7. *If (X, f) is a nonminimal M-system, then it is sensitive*

This theorem is the starting point of our next results. Indeed, by using Moothathu definitions $M2, M3, M5, M6$, we ask the following questions:

1. *Is a mixing system with a dense set of almost periodic points either minimal or cofinitely sensitive?*

2. *Is a syndetically transitive system with a dense set of almost periodic points either minimal or syndetically sensitive?*

In anology with the definition of M-system ([11]) we introduce the definitions of cofinitely-M-system and syndetically-M-system.

Definition 4.8. We call *cofinitely-M-system* any mixing dynamical system (X, f) such that the set of almost periodic points is dense.

Definition 4.9. We call *syndetically-M-system* any syndetically transitive dynamical system (X, f) such that the set of almost periodic points is dense.

Theorem 4.10. *If (X, f) is a mixing (resp. syndetically transitive) dynamical system with at least two point, then f is cofinitely (resp. syndetically) sensitive.*

Proof. Let (X, f) a dynamical system where X is a metric space equipped with a metric d. Fix $u, v \in X$ such that $u \neq v$ and take $\delta = \dfrac{d(u,v)}{2}$. Since (X, f) is mixing (resp. syndetically transitive) then, by definition $M3$) (resp. $M2$), for each open set $U \subset X$ we deduce that $N_f(U, B(u, \delta))$ and $N_f(U, B(v, \delta))$ are cofinite (resp. syndetic). Then, since

$$N_f(U, \delta) = N_f(U, B(u, \delta)$$

or

$$N_f(U, \delta) = N_f(U, B(v, \delta),$$

it follows $N_f(U, \delta)$ is cofinite (resp. syndetical) and this means that f is cofinitely (resp. syndetically) sensitive, by definition $M5$) (resp. $M4$) . □

Immediately it follows that questions 1. and 2. have positive answers.

Corollary 4.11. *If $X, f)$ is a nonminimal cofinitely-M-system (resp. syndetically-M-system), then f is cofinitely (resp. syndetically) sensitive.*

From Theorems 4.6 and 4.11, we have the following result.

Theorem 4.12. *If (X, f) is a mixing dynamical system (resp. syndetically transitive dynamical system) with at least two points, then $\overline{f} : \mathcal{H}(X) \to \mathcal{H}(X)$, where $\mathcal{H}(X)$ is equipped by the Upper Vietoris topology, τ_{V^+}, is cofinitely (resp. syndetically) sensitive.*

Proposition 4.13. *If (X, f) is a mixing (resp. syndetically transitive) dynamical sy-stem, then $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is cofinitely (resp. syndetically) sensitive.*

Proof. If (X, f) is a mixing (resp. syndetically transitive) dynamical system, then $\overline{f}$: $\mathcal{K}(X) \to \mathcal{K}(X)$, $(\mathcal{K}(X), \tau_V\ )$ is mixing (resp. syndetically transitive). Now, by Theorem 4.10 $\overline{f}$ is cofinitely (resp. syndetically) sensitive. □

Our questions have positive answers, as the following corollary shows.

Corollary 4.14. *If (X, f) is a nonminimal, cofinitely-M-system (resp. syndetically-M-system), then $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is cofinitely (resp. syndetically) sensitive.*

Equivalently, this corollary says: If (X, f) is a nonminimal, mixing (resp. syndetically transitive) dynamical system such that the set of almost periodic points is dense, then $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is cofinitely (resp. syndetically) sensitive.

In an analogous way, it is possible to introduce new definitions of chaos by using the mixing (resp. syndetically-transitivity) property and to generalize Theorem 2.8.

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