Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 11, Number 2, pp. 113–123 (2016)
http://campus.mst.edu/adsa

Positive Solutions of Nonlinear Hahn Difference Equations

Alaa E. Hamza and **S. D. Makharesh**
Cairo University
Department of Mathematics
Giza, Egypt
hamzaaeg2003@yahoo.com
sameeermakarish@yahoo.com

Abstract

In this paper, we study the existence of positive solutions of the Hahn difference equation

$$-\frac{1}{q}D^2_{q,\omega}u(t) = a(qt+\omega)f(u(qt+\omega)), \;\; t \in [\omega_0, d],$$

with linear boundary conditions. We apply a fixed point theorem in cones to show the existence of at least one positive solution, in either the superlinear or sublinear case. Here the Hahn difference operator $D_{q,\omega}$ is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, t \neq \omega_0,$$

where $\omega_0 = \dfrac{\omega}{1-q}$ with $0 < q < 1, \omega > 0$, and $d > \omega_0$.

AMS Subject Classifications: 39A13, 39A70.
Keywords: Boundary-value problem, Krasnoselskii's fixed point theorem, Green function, Hahn difference equation, positive solution.

1 Preliminaries

The existence of nonnegative solutions is important in studying of mathematical models in science such as chemical, physical models, population or concentration models in biology, and economical models. As we know, the cone sets, i.e., closed convex sets

Received June 21, 2016; Accepted August 9, 2016
Communicated by Martin Bohner

K of a Banach space X such that $\lambda K \subset K$ for all $\lambda \geqslant 0$ and $K \cap (-K) = \{0\}$ can describe nonnegativity.

Recently many authors studied positive solutions of different types of boundary value problems. Most of their results are based on Krasnoselskii's work [6]. He worked on nonlinear operator equations by using the theory of cones in Banach spaces.

In [2], the authors studied the existence of positive solutions of the second order boundary value problem

$$-u''(t) = a(t)f(u(t)), \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

$$\left.\begin{aligned} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{aligned}\right\} \tag{1.2}$$

with some conditions imposed on f, a and the constants of (1.2). It is shown that there is a positive solution in both of the superlinear and the sublinear cases. A function f is said to be superlinear (sublinear) if

$$f_0 = 0 \text{ and } f_\infty = \infty \ \ (f_0 = \infty \text{ and } f_\infty = 0),$$

where

$$f_0 := \lim_{u \to 0} \frac{f(u)}{u}, \quad f_\infty := \lim_{u \to \infty} \frac{f(u)}{u}.$$

The authors used a fixed point theorem of Krasnoselskii, see [6]. More precisely, they used a modified version of Krasnoselskii due to Guo [3], it reads as follows.

Theorem 1.1. *Let M_1 and M_2 be two bounded open sets in a Banach space E such that $0 \in M_1$, $\overline{M_1} \subset M_2$. Let*

$$A : K \cap \left(\overline{M_2} \setminus M_1\right) \longrightarrow K$$

be completely continuous and let one of the following conditions

$$(1) \|Ax\| \leqslant \|x\|, \forall x \in K \cap \partial M_1, \textit{and } \ \|Ax\| \geqslant \|x\|, \forall x \in K \cap \partial M_2.$$

$$(2) \|Ax\| \geqslant \|x\|, \forall x \in K \cap \partial M_1, \textit{and } \ \|Ax\| \leqslant \|x\|, \forall x \in K \cap \partial M_2.$$

be satisfied. Then A has at least one fixed point in $K \cap \left(\overline{M_2} \setminus M_1\right)$.

This paper is devoted to investigating positive solutions of (1.1)–(1.2) in the q,ω-difference operator setting. More specifically, we show the existence of positive solutions of the nonlinear Hahn difference equations of the form

$$-\frac{1}{q} D^2_{q,\omega} u(t) = a(qt + \omega) f(u(qt + \omega)), \quad t \in [\omega_0, d],$$

with boundary conditions

$$a_{11} u(\omega_0) - a_{12} D_{q,\omega} u(\omega_0) = 0,$$

$$a_{21} u(d) + a_{22} D_{q,\omega} u(d) = 0.$$

In the following section we state the main concepts of the q,ω-calculus which we use in the subsequent sections.

2 Introduction

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}_0$ and $0 < q < 1$, we define the q-numbers

$$[k]_q := \frac{1-q^k}{1-q}.$$

Let I be an interval of $\mathbb{R}$ containing ω_0, where $\omega_0 := \omega/(1-q)$, and h denote the transformation

$$h(t) := qt + \omega, \; t \in I.$$

One can see that

$$h(t) \begin{cases} > \; t, & \text{for } t < \omega_0, \\ = \; t, & \text{for } t = \omega_0, \\ < \; t, & \text{for } t > \omega_0. \end{cases}$$

The transformation h has the inverse $h^{-1}(t) = (t-\omega)/q, t \in I$. The kth order iteration of h is given by

$$h^k(t) \;:=\; \underbrace{h \circ h \circ \cdots \circ h}_{k-\text{times}}(t) = q^k t + \omega[k]_q, \quad t \in I, \tag{2.1}$$

$$(h^k(t))^{-1} \;:=\; h^{-k}(t) := \underbrace{h^{-1} \circ h^{-1} \circ \cdots \circ h^{-1}}_{k-\text{times}}(t) = \frac{t-\omega[k]_q}{q^k}, \quad t \in I. \tag{2.2}$$

Furthermore, $\{h^k(t)\}_{k=1}^{\infty}$ is a decreasing (an increasing) sequence in k when $t > \omega_0$ ($t < \omega_0$) with

$$\omega_0 = \begin{cases} \inf\limits_{k \in \mathbb{N}} h^k(t), & t > \omega_0, \\ \sup\limits_{k \in \mathbb{N}} h^k(t), & t < \omega_0. \end{cases} \tag{2.3}$$

The sequence $\{h^{-k}(t)\}_{k=1}^{\infty}$ is increasing (decreasing), $t > \omega_0$ ($t < \omega_0$) with

$$\infty = \begin{cases} \sup\limits_{k \in \mathbb{N}} h^{-k}(t), & t > \omega_0, \\ -\inf\limits_{k \in \mathbb{N}} h^{-k}(t)), & t < \omega_0. \end{cases} \tag{2.4}$$

Let f be a function defined on I. The *Hahn difference operator* is defined in [4] by

$$D_{q,\omega} f(t) := \frac{f(qt+\omega) - f(t)}{(qt+\omega) - t}, \quad \text{if } \; t \neq \omega_0, \tag{2.5}$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$, provided that f is differentiable at ω_0, where $q \in (0,1)$ and $\omega > 0$. In this case, we call $D_{q,\omega}f$, the q,ω-derivative of f. Finally, we say that f is q,ω-differentiable, i.e., throughout I, if $D_{q,\omega}f(\omega_0)$ exists.

The right inverse for $D_{q,\omega}$ is defined in [1] in terms of Jackson–Nörlund sums as follows. For $a, b \in I$, the q,ω-integral of f from a to b is defined to be

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t, \tag{2.6}$$

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(xq^k + \omega[k]_q), \qquad x \in I, \tag{2.7}$$

provided that the series converges at $x = a$ and $x = b$. It is known that if f is continuous at ω_0, then the series in (2.7) is uniformly convergent.

We summarize the results of the q,ω-calculus from [1] in the following theorem.

Theorem 2.1. *Let f,g be functions defined on I. The following statements are satisfied on every compact subinterval of I which contains ω_0.*
(i) *If f,g are q,ω-differentiable at $t \in I$, then*

$$D_{q,\omega}(fg)(t) = D_{q,\omega}(f(t))g(t) + f(qt+\omega)D_{q,\omega}g(t).$$

(ii) *If f is continuous at ω_0, then the function*

$$F(x) := \int_{\omega_0}^x f(t)d_{q,\omega}t, \quad x \in I,$$

is continuous at ω_0. Furthermore, $D_{q,\omega}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}f(t)d_{q,\omega}t = f(b) - f(a) \quad \textit{for all} \;\; a, b \in I.$$

Also, it is not difficult to see that the following statements hold.

(1) For $k_1, k_2 \in \mathbb{N}$, we have

$$\int_{\omega_0}^a f(t)d_{q,\omega}t \geqslant \int_{h^{k_1}(a)}^{h^{k_2}(a)} f(t)d_{q,\omega}t, \quad k_1 > k_2. \tag{2.8}$$

(2) If $t \neq \omega_0$, then

$$(D_{q,\omega}f)(h^{-1}(t)) = D_{\frac{1}{q},\frac{-\omega}{q}}f(t). \tag{2.9}$$

(3)

$$D_{\frac{1}{q},\frac{-\omega}{q}}\left(\int_{\omega_0}^{h(t)} f(x)d_{q,\omega}x\right) = qf(t). \tag{2.10}$$

$$D_{\frac{1}{q},\frac{-\omega}{q}}\left(\int_{\omega_0}^{t} f(x)d_{q,\omega}x\right) = f(h^{-1}(t)). \tag{2.11}$$

Our purpose here is to give an existence result for positive solutions to the nonlinear second order q,ω-difference equation of the form

$$\frac{-1}{q}D^2_{q,\omega}u(t) = a(h(t))f(u(h(t))), \quad t \in I = [\omega_0, h^{-1}(b)], \tag{2.12}$$

with certain linear boundary conditions. The following lemma indicates that equation (2.12) is equivalent to the equation

$$\frac{-1}{q}D_{\frac{1}{q},\frac{-\omega}{q}}D_{q,\omega}u(t) = a(t)f(u(t)), \quad t \in I = [\omega_0, b]. \tag{2.13}$$

Lemma 2.2. *u is a solution of equation* (2.13)*, if and only if u is a solution of equation* (2.12).

Proof. Let u be a solution of equation (2.13). Set $G = D_{q,\omega}u$. Simple calculations, using (2.9), show that

$$\begin{aligned}
-\frac{1}{q}D_{\frac{1}{q},\frac{-\omega}{q}}D_{q,\omega}u(t) &= -\frac{1}{q}D_{\frac{1}{q},\frac{-\omega}{q}}G(t)\\
&= -\frac{1}{q}\frac{G(h^{-1}(t)) - G(t)}{h^{-1}(t) - t}\\
&= -\frac{D_{q,\omega}u(h^{-1}(t)) - D_{q,\omega}u(t)}{t(1-q) - \omega}\\
&= -\frac{D_{\frac{1}{q},\frac{-\omega}{q}}u(t) - D_{\frac{1}{q},\frac{-\omega}{q}}u(h(t))}{t(1-q) - \omega}\\
&= a(t)f(u(t)), \quad t \in [\omega_0, b].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
-\frac{1}{q}D^2_{q,\omega}u(t) &= -\frac{1}{q}D_{q,\omega}G(t)\\
&= -\frac{1}{q}\frac{G(t) - G(h(t))}{t - h(t)}\\
&= -\frac{1}{q}\frac{D_{\frac{1}{q},\frac{-\omega}{q}}u(h(t)) - D_{\frac{1}{q},\frac{-\omega}{q}}u(h^2(t))}{t - h(t)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
-\frac{1}{q}D^2_{q,\omega}u(h^{-1}(t)) &= -\frac{1}{q}\frac{D_{\frac{1}{q},\frac{-\omega}{q}}u(t) - D_{\frac{1}{q},\frac{-\omega}{q}}u(h(t))}{h^{-1}(t)-t} \\
&= -\frac{D_{\frac{1}{q},\frac{-\omega}{q}}u(t) - D_{\frac{1}{q},\frac{-\omega}{q}}u(h(t))}{t(1-q)-\omega} \\
&= a(t)f(u(t)), \quad t \in [\omega_0, b],
\end{aligned}$$

from which we obtain

$$-\frac{1}{q}D^2_{q,\omega}u(t) = a(h(t))f(u(h(t))), \quad t \in [\omega_0, h^{-1}(b)].$$

It is not difficult to see the converse is true. □

So, we establish the existence of a positive solution of (2.13), under the boundary conditions

$$\left.\begin{aligned}
&\alpha u(\omega_0) - \beta D_{\frac{1}{q},\frac{-\omega}{q}}u(\omega_0) = 0, \\
&\gamma u(b) + \delta D_{\frac{1}{q},\frac{-\omega}{q}}u(b) = 0,
\end{aligned}\right\} \tag{2.14}$$

where $\rho := \gamma\beta + \alpha\delta + \alpha\gamma b - \gamma\alpha\omega_0 > 0,\ \ \alpha,\beta,\gamma,\delta \geqslant 0$. The functions a and f are assumed to be nonnegative continuous functions for $t \in [\omega_0, b]$, and $a(t) \neq 0$, on any subinterval of $[\omega_0, b]$. We also assume that f is either superlinear or sublinear.

3 Green's Function

In this section, we write the solution of (2.13)–(2.14) in a form of q,ω-integral involving the Green's function. This Green's function takes the same form of the classical case if $q \to 1,$ and $\omega \to 0$.

Lemma 3.1. *Any solution of* (2.13)–(2.14) *satisfies the integral equation*

$$u(t) = \int_{\omega_0}^{b} G(t,s)a(s)f(u(s))d_{q,\omega}s, \tag{3.1}$$

where $G(t,s)$ is giving by

$$G(t,s) = \frac{1}{\rho}\begin{cases} ((\gamma b+\delta)-\gamma t)(\beta - \alpha\omega_0 + \alpha s), & \omega_0 \leqslant s \leqslant t \leqslant b, \\ \\ ((\gamma b+\delta)-\gamma s)(\beta - \alpha\omega_0 + \alpha t), & \omega_0 \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.2}$$

$G(t,s)$ is called the Green's function.

Proof. The solution of (2.13) takes the form

$$u(t) = c_1 t + c_2 + \int_{\omega_0}^{t} [s - t] a(s) f(u(s)) d_{q,\omega} s, \tag{3.3}$$

where c_1 and c_2 are arbitrary constants. For a solution u that satisfies (2.14), we get

$$c_1 = -\frac{\alpha}{\rho} \int_{\omega_0}^{b} (\gamma s - (\gamma b + \delta)) a(s) f(u(s)) d_{q,\omega} s, \tag{3.4}$$

$$c_2 = \frac{-\beta + \alpha\omega_0}{\rho} \int_{\omega_0}^{b} (\gamma s - (\gamma b + \delta)) a(s) f(u(s)) d_{q,\omega} s. \tag{3.5}$$

Substituting in (3.3) we obtain the form (3.1). □

In the following we give some estimates for the Green's function which will be the main tools in getting the positive solution of (2.13)–(2.14). Clearly $G(t, s) \geqslant 0$.

Lemma 3.2. *Let $k_1 > k_2$, where $k_1, k_2 \in \mathbb{N}$, and let*

$$\sigma = \min\left\{ \frac{\gamma(b - h^{k_2}(b)) + \delta}{b\gamma + \delta}, \frac{\beta - \alpha\omega_0 + \alpha h^{k_1}(b)}{\beta - \alpha\omega_0 + \alpha b} \right\}.$$

Then

$$G(t, s) \geqslant \sigma G(s, s), \quad h^{k_1}(b) \leqslant t \leqslant h^{k_2}(b), \quad \omega_0 \leqslant s \leqslant b \tag{3.6}$$

and

$$G(t, s) \leqslant G(s, s), \quad \omega_0 \leqslant t, s \leqslant b. \tag{3.7}$$

Proof. If $h^{k_1}(b) \leqslant t \leqslant h^{k_2}(b)$, then

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{b\gamma + \delta - \gamma t}{b\gamma + \delta - \gamma s} \geqslant \dfrac{\gamma(b - h^{k_2}(b)) + \delta}{b\gamma + \delta}, & s \leqslant t, \\[2ex] \dfrac{\beta - \alpha\omega_0 + \alpha t}{\beta - \alpha\omega_0 + \alpha s} \geqslant \dfrac{\beta - \alpha\omega_0 + \alpha h^{k_1}(b)}{\beta - \alpha\omega_0 + \alpha b}, & t \leqslant s. \end{cases} \tag{3.8}$$

Also (3.7) follows from

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{b\gamma + \delta - \gamma t}{b\gamma + \delta - \gamma s} \leqslant 1, & s \leqslant t, \\[2ex] \dfrac{\beta - \alpha\omega_0 + \alpha t}{\beta - \alpha\omega_0 + \alpha s} \leqslant 1, & t \leqslant s. \end{cases} \tag{3.9}$$

The proof is complete. □

4 Existence of Positive Solutions

In this section, we establish the existence of a positive solution of (2.13)–(2.14) by applying Theorem 1.1.

Throughout the rest of the paper, we denote by $X = C[\omega_0, b]$ the space of all continuous complex valued functions with maximum norm $\|\cdot\|_\infty$. Let $A : X \longrightarrow X$, be defined by

$$Au(t) = \int_{\omega_0}^{b} G(t,s)a(s)f(u(s))d_{q,\omega}s.$$

Define the cone K in X by

$$K = \left\{ u \in X : u \geqslant 0, \quad \min_{h^{k_1}(b) \leqslant t \leqslant h^{k_2}(b)} u(t) \geqslant \sigma \|u\|_\infty \right\}, \tag{4.1}$$

where $k_1 > k_2$ such that $k_1, k_1 \in \mathbb{N}$, and σ be as in Lemma 3.2.

Lemma 4.1. *A is a positive operator in K, that is,*

$$A(K) \subset K. \tag{4.2}$$

Proof. Clearly $A(u) \geqslant 0$ for $u \in K$. Using Lemma 3.2, we get

$$\begin{aligned}
\min_{h^{k_1}(b) \leqslant t \leqslant h^{k_2}(b)} A(u(t)) &= \min_{h^{k_1}(b) \leqslant t \leqslant h^{k_2}(b)} \int_{\omega_0}^{b} G(t,s)a(s)f(u(s))d_{q,\omega}s \\
&\geqslant \sigma \int_{\omega_0}^{b} G(s,s)a(s)f(u(s))d_{q,\omega}s \\
&\geqslant \sigma \int_{\omega_0}^{b} G(t,s)a(s)f(u(s))d_{q,\omega}s \quad \forall t \in [\omega_0, b] \\
\min_{h^{k_1}(b) \leqslant t \leqslant h^{k_2}(b)} A(u(t)) &\geqslant \sigma \|Au\|_\infty.
\end{aligned}$$

The proof is complete. □

Theorem 4.2. *The problem* (2.13)–(2.14) *has at least one positive solution in the superlinear and sublinear cases.*

Proof. First we consider the superlinear case. Since $f_0 = 0$, for ϵ satisfying

$$0 < \epsilon \int_{\omega_0}^{b} G(s,s)a(s)d_{q,\omega}s \leqslant 1,$$

we can choose $\delta_1 > 0$ such that $f(u) \leqslant \epsilon u$, for $0 < u \leqslant \delta_1$. Define the set

$$M_1 := \{u \in X : \|u\|_\infty < \delta_1\}. \tag{4.3}$$

Assume $u \in K$ and $\|u\|_\infty = \delta_1$. Then from Lemma 3.2, we have

$$\begin{aligned} Au(t) &= \int_{\omega_0}^{b} G(t,s)a(s)f(u(s))d_{q,\omega}s \\ &\leqslant \int_{\omega_0}^{b} G(s,s)a(s)\epsilon u(s)d_{q,\omega}s \\ &\leqslant \|u\|_\infty. \end{aligned}$$

Hence

$$\|Au\|_\infty \leq \|u\|_\infty. \quad u \in K \cap \partial M_1. \tag{4.4}$$

In view of $f_\infty = \infty$, there exists δ^* such that $f(u) \geqslant \mu u,\ \ u \geqslant \delta^*$. where μ satisfies

$$\mu\sigma \int_{h^{k_1}(b)}^{h^{k_2}(b)} G(h^{k_0}(b), s)a(s)d_{q,\omega}s \geqslant 1. \tag{4.5}$$

Here $k_0 \in \mathbb{N}$, is such that $k_1 \leqslant k_0 \leqslant k_2$. Let $\delta_2 = \max\{2\delta_1, \delta^*/\sigma\}$, and let

$$M_2 := \{u \in X : \|u\|_\infty < \delta_2\}. \tag{4.6}$$

For $u \in K$ and $\|u\|_\infty = \delta_2$, we have

$$\min_{h^{k_1}(b)\leqslant t\leqslant h^{k_2}(b)} u(t) \geqslant \sigma\|u\|_\infty \geqslant \delta^*.$$

We deduce that

$$\begin{aligned} Au(h^{k_0}(b)) &= \int_{\omega_0}^{b} G(h^{k_0}(b), s)a(s)f(u(s))d_{q,\omega}s \\ &\geqslant \int_{h^{k_1}(b)}^{h^{k_2}(b)} G(h^{k_0}(b), s)a(s)f(u(s))d_{q,\omega}s \\ &\geqslant \mu \int_{h^{k_1}(b)}^{h^{k_2}(b)} G(h^{k_0}(b), s)a(s)u(s)d_{q,\omega}s \\ &\geqslant \mu\sigma\|u\|_\infty \int_{h^{k_1}(b)}^{h^{k_2}(b)} G(h^{k_0}(b), s)a(s)d_{q,\omega}s \\ &\geqslant \|u\|_\infty. \end{aligned}$$

This means $\|Au\|_\infty \geqslant \|u\|_\infty$ for $u \in K \cap \partial M_2$. Therefore, by the fixed point Theorem1.1, there exists a fixed point u of A,
where $u \in K \cap (\overline{M_2} \setminus M_1)$, i.e., $\delta_1 \leqslant \|u\| \leqslant \delta_2$.

For the sublinear case ($f_0 = \infty$, $f_\infty = 0$), we can choose $\delta' > 0$ such that $f(u) \geqslant \mu' u$ for $0 < u \leqslant \delta'$, where

$$\mu'\sigma \int_{h^{k_1}(b)}^{h^{k_2}(b)} G(h^{k_0}(b), s)a(s)d_{q,\omega}s \geqslant 1. \tag{4.7}$$

Set

$$M_1' := \{u \in X : \|u\|_\infty < \delta'\}.$$

In a similar way to the first case, if $u \in K$ and $\|u\|_\infty = \delta'$, we get

$$\begin{aligned} Au(h^{k_0}(b)) &= \int_{\omega_0}^{b} G(h^{k_0}(b), s)a(s)f(u(s))d_{q,\omega}s \\ &\geqslant \mu'\sigma\|u\|_\infty \int_{h^{k_1}(b)}^{h^{k_2}(b)} G(h^{k_0}(b), s)a(s)d_{q,\omega}s \\ &\geqslant \|u\|_\infty. \end{aligned}$$

Thus

$$\|Au\|_\infty \geqslant \|u\|_\infty, \quad u \in K \cap \partial M_1'. \tag{4.8}$$

Since $f_\infty = 0$, there is a $\tilde{\delta}$, such that $f(u) \leqslant \epsilon' u$, for $u \geqslant \tilde{\delta}$, where

$$0 < \epsilon' \int_{\omega_0}^{b} G(s, s)a(s)d_{q,\omega}s \leqslant 1. \tag{4.9}$$

Set

$$M_2' := \{u \in X : \|u\|_\infty < \xi\},$$

where ξ will be specified according to the following cases.

Case (1). f is bounded, say $f(u) \leqslant N$ for all $u \in (0, \infty)$. In this case, choose

$$\xi := \max\left\{2\delta', N \int_{\omega_0}^{b} G(s, s)a(s)d_{q,\omega}s\right\},$$

so that for $u \in K$ with $\|u\|_\infty = \xi$, we have

$$Au(t) = \int_{\omega_0}^{b} G(t, s)a(s)f(u(s))d_{q,\omega}s \leqslant N \int_{\omega_0}^{b} G(s, s)a(s)d_{q,\omega}s \leqslant \xi,$$

and consequently $\|Au\|_\infty \leqslant \|u\|_\infty$.

Case (2). f is unbounded. Then choose $\xi \geqslant \max\{2\delta', \tilde{\delta}\}$ and such that

$$f(u) \leqslant f(\xi) \text{ for } 0 \leqslant u \leqslant \xi.$$

For $u \in K$ and $\|u\|_\infty = \xi$, we have

$$\begin{aligned} Au(t) &= \int_{\omega_0}^{b} G(t, s)a(s)f(u(s))d_{q,\omega}s \\ &\leqslant \int_{\omega_0}^{b} G(s, s)a(s)f(\xi)d_{q,\omega}s \\ &\leqslant \epsilon'\xi \int_{\omega_0}^{b} G(s, s)a(s)d_{q,\omega}s \leqslant \xi. \end{aligned}$$

Hence $\|Au\|_\infty \leqslant \|u\|_\infty$ and for $u \in K \cap \partial M_2'$, we have $\|Au\|_\infty \leqslant \|u\|_\infty$. By the second part of the fixed point Theorem 1.1, it follows that problem (2.13)–(2.14) has a positive solution and this completes the proof. □

References

[1] M. H. Annaby, A. E. Hamza and K. A. Aldwoah, Hahn difference operator and associated Jackson–Norlund integrals, Journal of optimization theory and applications, 154 (2012), 133–153.

[2] L.H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc, 120 (1994), 743–748.

[3] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego,(1988).

[4] W. Hahn, Über Orthogonalpolynome, die q-Differenzenlgleichungen genügen, Math. Nachr, 2(1949).

[5] M. A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Groningen, (1964).

[6] M. K. Kwong, On Krasnoselskii's cone fixed point theorem, fixed point theory and applications, 2008 (2008), 1–18.