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New Conformable Fractional Derivative Definition for Positive and Increasing Functions and its Generalization

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Abstract

We present a new definition of fractional derivative. We check its basic properties, to show that it is the most natural and best suitable.

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1 Introduction

The theory of fractional derivation has known great importance in mathematical research these last decades. The definition of fractional derivative don't have a standard form. But the most commonly used definitions are those from the integration: Riemann–Liouville definition and Caputo definition [5].

The fractional derivative can also be seen as an approximation of the classical derivative. This is not the case in general. The definitions of Khalil *and Al* [4] and that of Katugampola [3] are more suited to the approximation of classical derivation, but less natural, since $\lim\limits_{\alpha \to 0} f^{(\alpha)}(t) = tf(t)$.

In [1], the authors offer a good definition, which played a big role in the development of this article. In this paper, we propose a new definition that is more natural, since $\lim\limits_{\alpha \to 0} f^{(\alpha)}(t) = f(t)$ and $\lim\limits_{\alpha \to 1} f^{(\alpha)}(t) = f'(t)$.

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2 The Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing and positive function, i.e., $f(t) > 0,\ \forall t \in \mathbb{R}$. For $0 < \alpha \leq 1$, we define the α-fractional derivative of f at $t \in \mathbb{R}$, denoted by $D^\alpha f\,(t)$, by

$$D^\alpha f\,(t) = \lim_{\varepsilon \to 0} \left(\frac{f(t + \varepsilon f(t)^{\frac{1-\alpha}{\alpha}}) - f(t)}{\varepsilon} \right)^\alpha .$$

f is said to be α-differentiable over $I \subset \mathbb{R}$ if $D^\alpha f\,(t)$ exists for all $t \in I$. If $f\,(t_0) = 0$, we define the α-fractional derivative of f at $t \in \mathbb{R}$ by $D^\alpha f\,(t_0) = \lim_{t \to t_0} D^\alpha f\,(t)$. Our goal is not to obtain the same properties as classical derivative, we want to get properties that are the most natural and converging towards the classic as α tends to 1. We will check the basic properties of a derivative.

Theorem 2.1. *If a positive and increasing function $f : \mathbb{R} \to \mathbb{R}$ is α-differentiable at $t_0 \in \mathbb{R}$, $0 < \alpha \leq 1$, then f is continuous at t_0.*

Proof. We have

$$\begin{aligned} \lim_{h \to 0} [f\,(t_0 + h) - f\,(t_0)] &= \lim_{\varepsilon \to 0} \left[f\left(t_0 + \varepsilon f(t_0)^{\frac{1-\alpha}{\alpha}}\right) - f\,(t_0) \right] \\ &= \lim_{\varepsilon \to 0} \left(\frac{f(t_0 + \varepsilon f(t_0)^{\frac{1-\alpha}{\alpha}}) - f(t_0)}{\varepsilon} \right) \varepsilon \\ &= D^\alpha f\,(t_0)^{\frac{1}{\alpha}}\ 0 \\ &= 0. \end{aligned}$$

Then, f is continuous at t_0. This concludes the proof. $\square$

Theorem 2.2. *Let $0 < \alpha \leq 1$ and f, g be positive, increasing and α-differentiable at $t \in \mathbb{R}$. Then*

1. $D^\alpha(af) = aD^\alpha(f)$*, for all* $a > 0$.
2. $D^\alpha(f+g) = \left(\left(\frac{f+g}{f}\right)^{\frac{1-\alpha}{\alpha}} (D^\alpha f)^{\frac{1}{\alpha}} + \left(\frac{f+g}{g}\right)^{\frac{1-\alpha}{\alpha}} (D^\alpha g)^{\frac{1}{\alpha}} \right)^\alpha .$
3. $D^\alpha(t^n) = n^\alpha t^{n-\alpha}$*, for all* $t \in \mathbb{R}$ *if* n *is even or for* $t \geq 0$.
4. $D^\alpha(a) = 0$*, for all* $a \geq 0$.
5. $D^\alpha(fg)(t) = \left(f(t)^{\frac{1}{\alpha}} (D^\alpha g(t))^{\frac{1}{\alpha}} + g(t)^{\frac{1}{\alpha}} (D^\alpha f(t))^{\frac{1}{\alpha}} \right)^\alpha .$

6. *If $\beta < \alpha$, then f is β-differentiable at t and*

$$D^{\beta}f(t) = f(t)^{1-\frac{\beta}{\alpha}} \left(D^{\alpha}f(t)\right)^{\frac{\beta}{\alpha}}.$$

In particular, if f is differentiable, then $D^{\alpha}f(t) = f(t)^{1-\alpha} f'(t)^{\alpha}$.

Proof. 1, 3 and 4 are clear. We have

$$\begin{aligned}
&(D^{\alpha}(f+g))^{\frac{1}{\alpha}} \\
=\ & \lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon(f(t)+g(t))^{\frac{1-\alpha}{\alpha}})+g(t+\varepsilon(f(t)+g(t))^{\frac{1-\alpha}{\alpha}})-f(t)-g(t)}{\varepsilon}\right) \\
=\ & \lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon(f(t)+g(t))^{\frac{1-\alpha}{\alpha}})-f(t)}{\varepsilon}+\frac{g(t+\varepsilon(f(t)+g(t))^{\frac{1-\alpha}{\alpha}})-g(t)}{\varepsilon}\right)
\end{aligned}$$

In the first part, we take $\varepsilon = h\dfrac{f(t)^{\frac{1-\alpha}{\alpha}}}{(f(t)+g(t))^{\frac{1-\alpha}{\alpha}}}$. In the second part, we take $\varepsilon = h\dfrac{g(t)^{\frac{1-\alpha}{\alpha}}}{(f(t)+g(t))^{\frac{1-\alpha}{\alpha}}}$. Then, we get 2.

Next, we have

$$\begin{aligned}
&(D^{\alpha}(fg)(t))^{\frac{1}{\alpha}} \\
=\ & \lim_{\varepsilon\to 0}\left(\frac{(f(t+\varepsilon(f(t)g(t))^{\frac{1-\alpha}{\alpha}})g(t+\varepsilon(f(t)g(t))^{\frac{1-\alpha}{\alpha}}-f(t)g(t)}{\varepsilon}\right) \\
=\ & \lim_{\varepsilon\to 0} f(t+\varepsilon f(t)^{\frac{1-\alpha}{\alpha}}g(t)^{\frac{1-\alpha}{\alpha}})\lim_{\varepsilon\to 0}\left(\frac{g(t+\varepsilon f(t)^{\frac{1-\alpha}{\alpha}}g(t)^{\frac{1-\alpha}{\alpha}})-g(t))}{\varepsilon f(t)^{\frac{1-\alpha}{\alpha}}}\right)f(t)^{\frac{1-\alpha}{\alpha}} \\
+\ & \lim_{\varepsilon\to 0} g(t+\varepsilon f(t)^{\frac{1-\alpha}{\alpha}}g(t)^{\frac{1-\alpha}{\alpha}})\lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon f(t)^{\frac{1-\alpha}{\alpha}}g(t)^{\frac{1-\alpha}{\alpha}})-f(t))}{\varepsilon g(t)^{\frac{1-\alpha}{\alpha}}}\right)g(t)^{\frac{1-\alpha}{\alpha}} \\
=\ & \left(f(t)^{\frac{1}{\alpha}}(D^{\alpha}g(t))^{\frac{1}{\alpha}}+g(t)^{\frac{1}{\alpha}}(D^{\alpha}f(t))^{\frac{1}{\alpha}}\right).
\end{aligned}$$

We have

$$\begin{aligned}
D^{\beta}f(t) &= \lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon f(t)^{\frac{1-\beta}{\beta}})-f(t)}{\varepsilon}\right)^{\beta} \\
&= \lim_{h\to 0}\left(\frac{f(t+hf(t)^{\frac{1-\alpha}{\alpha}})-f(t)}{h}f(t)^{\frac{1}{\alpha}-\frac{1}{\beta}}\right)^{\beta}.
\end{aligned}$$

If $h = \varepsilon f(t)^{\frac{1}{\beta}-\frac{1}{\alpha}}$, then

$$D^{\beta}f(t) = f(t)^{1-\frac{\beta}{\alpha}}\left(D^{\alpha}f(t)\right)^{\frac{\beta}{\alpha}}.$$

This concludes the proof. □

If we put $\alpha = 1$, we get the classical derivative formulas. The last equality shows that our definition is better than those proposed in [3,4] because if f is differentiable at $t > 0$, then,

$$\lim_{\alpha \to 0} D^{\alpha} f(t) = f(t), \ \lim_{\alpha \to 1} D^{\alpha} f(t) = f'(t).$$

Unlike those proposed in [3,4], which yield

$$\lim_{\alpha \to 0} D^{\alpha} f(t) = t f(t), \ \lim_{\alpha \to 1} D^{\alpha} f(t) = f'(t).$$

This means that our definition is more natural. Using the last equality of the previous theorem, we calculate the α-fractional derivative of certain functions as

1. $D^{\alpha}\left(e^{ct}\right) = c^{\alpha} e^{ct}$, $c \in \mathbb{R}$. In particular, $D^{\alpha}\left(e^{t}\right) = e^{t}$.
2. $D^{\alpha}(\sin(ct)) = c^{\alpha} \sin(ct)^{1-\alpha} \cos(ct)^{\alpha}$, $c \in \mathbb{R}$, $t \in \left[0, \frac{\pi}{2}\right]$.
3. $D^{\alpha}(t^{\alpha}) = \alpha^{\alpha}$, $t \geq 0$.

Like in [3,4], we can build a function which has no derivative in a given point, but which admits a α-fractional derivative in the same point. Take $f(t) = \sqrt{t}$, $t \geq 0$. Then, $D^{\frac{1}{2}}(f)(0) = \dfrac{1}{\sqrt{2}}$ and $f'(0) = \infty$.

In the same way as in [3,4], we can define the α-fractional derivative, $n < \alpha \leq n+1$, $n \in \mathbb{N}$, for a function $f : \mathbb{R} \to \mathbb{R}$, such that $f^{(n)}$ is positive and increasing, by

$$D^{\alpha} f(t) = \lim_{\varepsilon \to 0} \left(\frac{f^{(n)}(t + \varepsilon f^{(n)}(t)^{\frac{1-\alpha'}{\alpha'}}) - f^{(n)}(t)}{\varepsilon} \right)^{\alpha'},$$

where $\alpha' = \alpha - n$. This gives, if f is $n+1$ differentiable,

$$D^{\alpha} f(t) = f^{(n)}(t)^{1-\alpha'} f^{(n+1)}(t)^{\alpha'}.$$

In the same manner as [3,4], we can demonstrate the following results.

Theorem 2.3. *Let $f\,[a,b] \to \mathbb{R}$ be a positive, increasing function and α-differentiable for some $\alpha \in \,]0,1[$ such that $f(a) = f(b)$. Then, it exist $c \in \,]a,b[$, such that $D^{\alpha} f(c) = 0$.*

Using the previous theorem, we show the following result.

Theorem 2.4. *Let $0 \leq a < b$ and $f : [a,b] \to \mathbb{R}$ be a positive, increasing function and α-differentiable for some $\alpha \in \,]0,1[$. Then, it exists $c \in \,]a,b[$, such that,*

$$D^{\alpha} f(c) = \alpha^{\alpha} \left(\frac{f(b) - f(a)}{b^{\alpha} - a^{\alpha}} \right)^{\alpha} \left(\frac{f(c)}{c^{\alpha}} \right)^{1-\alpha}.$$

3 Fractional Integral

For a continuous, positive function $f : \mathbb{R} \to \mathbb{R}$, we define the α-fractional integral, $0 < \alpha \le 1$, $a \in \mathbb{R}$, by

$$G_\alpha^a(f)(t) = \left(\frac{1}{\alpha} \int_a^t f(s)^{\frac{1}{\alpha}} ds \right)^\alpha,$$

where the integral is the usual Riemann integral.

Theorem 3.1. $D^\alpha (G_\alpha^a(f))(t) = f(t),\ t \ge a.$

Proof. Since f is continuous, then $G_\alpha^a(f)(t)$ is positive, increasing and differentiable. Hence,

$$\begin{aligned} D^\alpha (G_\alpha^a(f))(t) &= (G_\alpha^a(f)(t))^{1-\alpha} \left(\frac{d}{dt} G_\alpha^a(f)(t) \right)^\alpha \\ &= (G_\alpha^a(f)(t))^{1-\alpha} f(t) (G_\alpha^a(f)(t))^{\alpha-1} \\ &= f(t). \end{aligned}$$

This concludes the proof. □

Unlike [3,4], the α-fractional integral defined here is not improper and more natural.

4 Generalization

In this section, we will define the fractional derivative for functions that are not necessarily positive and increasing. Let $\alpha = \dfrac{2p+1}{2q+1}$, $p, q \in \mathbb{N}$ and $q \ge p$. With this form of α the fractional derivative proposed in this paper can be used for more general function because

$$(-1)^\alpha = \left((-1)^{2q+1}\right)^\alpha = (-1)^{2p+1} = 1.$$

All the results obtained previously, remain accurate for functions that are not necessarily positive or increasing, and we get more.

Theorem 4.1. *If* $\alpha = \dfrac{2p+1}{2q+1}$, $p, q \in \mathbb{N}$, $q \ge p$ *and* $f, g : \mathbb{R} \to \mathbb{R}$, *then,*

1. $D^\alpha \left(\dfrac{1}{f} \right) = \dfrac{-D^\alpha f}{f^2}.$

2. $D^\alpha \left(\dfrac{f}{g} \right) = \dfrac{\left((D^\alpha f)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}} - (D^\alpha g)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}} \right)^\alpha}{g^2}.$

Proof. 1.

$$D^{\alpha}\left(\frac{1}{f}\right)(t) = \lim_{h\longrightarrow 0}\left(-\frac{f(t+hf(t)^{\frac{1-\alpha}{\alpha}})-f(t)}{h}\frac{f(t)^{\frac{2\alpha-2}{\alpha}}}{f(t+\varepsilon f(t)^{\frac{\alpha-1}{\alpha}})f(t)}\right)^{\alpha}$$
$$= \lim_{h\longrightarrow 0}\left(-\frac{f(t+hf(t)^{\frac{1-\alpha}{\alpha}})-f(t)}{h}\right)^{\alpha}\left(\frac{f(t)^{\frac{2\alpha-2}{\alpha}}}{f(t+\varepsilon f(t)^{\frac{\alpha-1}{\alpha}})f(t)}\right)^{\alpha}$$
$$= \frac{-D^{\alpha}(f(t))}{f(t)^2},$$

2. The previous result and 5 from Theorem 2.2 gives the result.

This concludes the proof. □

We have, if $\alpha = \dfrac{2p+1}{2q+1}$, $p, q \in \mathbb{N}$ and $q \geq p$,

1. $D^{\alpha}(\sin(ct)) = c^{\alpha}\sin(ct)^{1-\alpha}\cos(ct)^{\alpha}$, $c \in \mathbb{R}$, $t \in \mathbb{R}$.
2. $D^{\alpha}(\cos(ct)) = -c^{\alpha}\cos(ct)^{1-\alpha}\sin(ct)^{\alpha}$, $c \in \mathbb{R}$, $t \in \mathbb{R}$.

5 Relation with the Conformable Ratio Derivative

In [2], the author proposed a good fractional derivative definition, named the conformable ratio derivative, given for $f(t) \geq 0$ by

$$K_{\alpha}[f(t)] = \lim_{\varepsilon\to 0} f(t)^{1-\alpha}\left(\frac{f(t+\varepsilon)-f(t)}{\varepsilon}\right)^{\alpha},\ \alpha \in [0,1].$$

It is clear that our definition is more general that the conformable ratio derivative. In fact, if $f(t) \geq 0$ and $K_{\alpha}[f(t)]$ exists, then

$$D^{\alpha}f(t) = \lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon f(t)^{\frac{1-\alpha}{\alpha}})-f(t)}{\varepsilon}\right)^{\alpha}$$
$$= \lim_{h\to 0} f(t)^{1-\alpha}\left(\frac{f(t+h)-f(t)}{h}\right)^{\alpha} = K_{\alpha}[f(t)].$$

But, if we take

$$f_{\beta}(t) = t^{\beta},\ t \geq 0,\ \beta \in]0,1[,$$

then we see that $D^{\alpha}f_{\beta}(t) = \beta^{\alpha}t^{\beta-\alpha}$ which means that $D^{\alpha}f_{\beta}(0)$ exists only for $0 \leq \alpha < \beta$. But, $K_{\alpha}[f_{\beta}(0)]$ exists for $0 \leq \alpha < 1$ which is not really natural compared to the previous result.

6 Conclusion

We built the natural definition of fractional derivative. This definition will allow the new functional space construction. But the great difficulty with this definition is that it is not linear. What makes solving differential equations, harder but slightly more interesting.

Work is underway for the theoretical and numerical study of some fractional differential equations using this new definition. We arrived to transform the cauchy problem attached to this new fractional derivation, in a nonlinear Volterra equation. We get the existence of the solution and we are working on the unicity and numerical approximation.

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