Solvability of a Maximum Quadratic Integral Equation of Arbitrary Orders

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Abstract

We investigate a new quadratic integral equation of arbitrary orders with maximum and prove an existence result for it. We will use a fixed point theorem due to Darbo as well as the monotonicity measure of noncompactness due to Banaś and Olszowy to prove that our equation has at least one solution in $C[0,1]$ which is monotonic on $[0,1]$.

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1 Introduction

In several papers, among them [1,11], the authors studied differential and integral equations with maximum. In [6–9] Darwish et al. studied fractional integral equations with supremum. Also, in [4,5], Caballero et al. studied the Volterra quadratic integral equations with supremum. They showed that these equations have monotonic solutions in the space \( C[0,1] \). Darwish [7] generalized and extended the Caballero et al. [4] results to the case of quadratic fractional integral equations with supremum.

In this paper we will study the fractional quadratic integral equation with maximum

\[
y(t) = f(t) + \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{0 \leq \tau \leq s} |y(\tau)| (\varphi(t) - \varphi(s))^{1-\beta} ds, \quad t \in J = [0,1], \quad 0 < \beta < 1,
\]

where \( f, \varphi : J \rightarrow \mathbb{R}, T : C(J) \rightarrow C(J), \kappa : J \rightarrow J \) and \( \kappa : J \times J \rightarrow \mathbb{R}^+ \).

By using the monotonicity measure of noncompactness due to Banaś and Olszowy [3] as well as the Darbo fixed point theorem, we prove the existence of monotonic solutions to (1.1) in \( C[0,1] \).

Now, we assume that \((E, \| \cdot \|)\) is a real Banach space. We denote by \( B(x, r) \) the closed ball centred at \( x \) with radius \( r \) and \( B_{\theta} \equiv B(\theta, r), \) where \( \theta \) is a zero element of \( E \).

We let \( X \subset E \). The closure and convex closure of \( X \) are denoted by \( \overline{X} \) and \( \text{Conv} X \), respectively. The symbols \( X + Y \) and \( \lambda Y \) are using for the usual algebraic operators on sets and \( \mathcal{M}_E \) and \( \mathcal{N}_E \) stand for the families defined by \( \mathcal{M}_E = \{A \subset E : A \neq \emptyset, \ A \text{ is bounded}\} \) and \( \mathcal{N}_E = \{B \subset \mathcal{M}_E : B \text{ is relatively compact}\} \), respectively.

**Definition 1.1** (See [2]). A function \( \mu : \mathcal{M}_E \rightarrow [0, +\infty) \) is called a measure of noncompactness in \( E \) if the following conditions:

\begin{itemize}
  \item[1°] \( \emptyset \neq \{X \in \mathcal{M}_E : \mu(X) = 0\} = \ker \mu \subset \mathcal{N}_E, \)
  \item[2°] if \( X \subset Y \), then \( \mu(X) \leq \mu(Y), \)
  \item[3°] \( \mu(X) = \mu(\overline{X}) = \mu(\text{Conv} X), \)
  \item[4°] \( \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y), \ 0 \leq \lambda \leq 1 \)
  \item[5°] if \( (X_n) \) is a sequence of closed subsets of \( \mathcal{M}_E \) with \( X_n \supset X_{n+1} (n = 1, 2, 3, \ldots) \)
  and \( \lim_{n \to \infty} \mu(X_n) = 0 \) then \( X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset, \)
\end{itemize}

hold.

We will establish our result in the Banach space \( C(J) \) of all defined, real and continuous functions on \( J \equiv [0,1] \) with standard norm \( \|y\| = \max \{|y(\tau)| : \tau \in J\} \). Next, we define the measure of noncompactness related to monotonicity in \( C(J) \); see [2,3].

Let \( \emptyset \neq Y \subset C(J) \) be a bounded set. For \( y \in Y \) and \( \varepsilon \geq 0 \), the modulus of continuity of the function \( y \), denoted by \( \omega(y, \varepsilon) \), is defined by

\[
\omega(y, \varepsilon) = \sup \{|y(t) - y(s)| : t, s \in J, \ |t - s| \leq \varepsilon\}.
\]
Moreover, we let
\[ \omega(Y, \varepsilon) = \sup \{ \omega(y, \varepsilon) : y \in Y \} \]
and
\[ \omega_0(Y) = \lim_{\varepsilon \to 0} \omega(Y, \varepsilon). \]
Define
\[ d(y) = \sup_{t, s \in J, s \leq t} (|y(t) - y(s)| - [y(t) - y(s)]) \]
and
\[ d(Y) = \sup_{y \in Y} d(y). \]
Notice that all functions in \( Y \) are nondecreasing on \( J \) if and only if \( d(Y) = 0 \).

Now, we define the map \( \mu \) on \( M_{C(J)} \) as
\[ \mu(Y) = d(Y) + \omega_0(Y). \]
Clearly, \( \mu \) satisfies all conditions in Definition 3, and therefore, it is a measure of noncompactness in \( C(J) \) [3].

**Definition 1.2.** Let \( P : M \to E \) be a continuous mapping, where \( \emptyset \neq M \subset E \). Suppose that \( P \) maps bounded sets onto bounded sets. Let \( Y \) be any bounded subset of \( M \) with \( \mu(PY) \leq \alpha \mu(Y) \), \( \alpha \geq 0 \), then \( P \) is called verify the Darbo condition with respect to a measure of noncompactness \( \mu \).

In the case \( \alpha < 1 \), the operator \( P \) is said to be a contraction with respect to \( \mu \).

**Theorem 1.3** (See [10]). Let \( \emptyset \neq \Omega \subset E \) be a closed, bounded and convex set. If \( P : \Omega \to \Omega \) is a continuous contraction mapping with respect to \( \mu \), then \( P \) has a fixed point in \( \Omega \).

We will need the following two lemmas in order to prove our results [4].

**Lemma 1.4.** Let \( r : J \to J \) be a continuous function and \( y \in C(J) \). If, for \( t \in J \),
\[ (Fy)(t) = \max_{[0, \sigma(t)]} |y(\tau)|, \]
then \( Fy \in C(J) \).

**Lemma 1.5.** Let \( (y_n) \) be a sequence in \( C(J) \) and \( y \in C(J) \). If \((y_n)\) converges to \( y \in C(J) \), then \((Fy_n)\) converges uniformly to \( Fy \) uniformly on \( J \).
2 Main Theorem

Let us consider the following assumptions:

\((a_1)\) \(f \in C(J)\). Moreover, \(f\) is nondecreasing and nonnegative on \(J\).

\((a_2)\) The operator \(T : C(J) \to C(J)\) is continuous and satisfies the Darbo condition with a constant \(c\) for the measure of noncompactness \(\mu\). Moreover, \(Ty \geq 0\) if \(y \geq 0\).

\((a_3)\) There exist constants \(a, b \geq 0\) such that \(|(Ty)(t)| \leq a + b\|y\| \forall y \in C(J), t \in J\).

\((a_4)\) The function \(\varphi : J \to \mathbb{R}\) is \(C^1(J)\) and nondecreasing.

\((a_5)\) The function \(\kappa : J \times J \to \mathbb{R}_+\) is continuous on \(J \times J\) and nondecreasing \(\forall t\) and \(s\) separately. Moreover, \(\kappa^* = \sup_{(t,s) \in J \times J} \kappa(t, s)\).

\((a_6)\) The function \(\sigma : J \to J\) is nondecreasing and continuous on \(J\).

\((a_7)\) \(\exists r_0 > 0\) such that

\[\|f\| + \frac{\kappa^* r_0 (a + b r_0)}{\Gamma(\beta + 1)} (\varphi(1) - \varphi(0))^\beta \leq r_0 \tag{2.1}\]

and \(\frac{ck^* r_0}{\Gamma(\beta + 1)} < (\varphi(1) - \varphi(0))^{-\beta}\).

Now, we define two operators \(\mathcal{K}\) and \(\mathcal{F}\) on \(C(J)\) as follows

\[(\mathcal{K}y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s) \kappa(t, s) \max_{0 \leq \tau \leq \sigma(s)} |y(\tau)|}{(\varphi(t) - \varphi(s))^{1-\beta}} ds \tag{2.2}\]

and

\[(\mathcal{F}y)(t) = f(t) + (Ty)(t) \cdot (\mathcal{K}y)(t), \tag{2.3}\]

respectively. Solving (1.1) is equivalent to find a fixed point of the operator \(\mathcal{F}\).

Under the above assumptions, we will prove the following theorem.

**Theorem 2.1.** Assume the assumptions \((a_1) - (a_7)\) are satisfied. Then (1.1) has at least one solution \(y \in C(J)\) which is nondecreasing on \(J\).

**Proof.** First, we claim that the operator \(\mathcal{F}\) transforms \(C(J)\) into itself. For this, it is sufficient to show that if \(y \in C(J)\), then \(\mathcal{K}y \in C(J)\). Let \(y \in C(J)\) and \(t_1, t_2 \in J\) \((t_1 \leq t_2)\) such that \(|t_2 - t_1| \leq \varepsilon\) for fixed \(\varepsilon > 0\), then we have

\[|(\mathcal{K}y)(t_2) - (\mathcal{K}y)(t_1)|\]

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where we used

\[
\omega_\kappa(\varepsilon, \cdot) = \sup_{t, \tau \in I, |t-\tau| \leq \varepsilon} |\kappa(t, s) - \kappa(\tau, s)|
\]
and the fact that \( \varphi(t_1) - \varphi(0) \leq \varphi(t_2) - \varphi(0) \). Notice that, since the function \( \kappa \) is uniformly continuous on \( J \times J \) and the function \( \varphi \) is continuous on \( J \), then when \( \varepsilon \to 0 \), we have that \( \omega_\kappa(\varepsilon, \cdot) \to 0 \) and \( \omega(\varphi, \varepsilon) \to 0 \).

Therefore, \( \mathcal{K}_y \in C(J) \) and consequently, \( \mathcal{F}y \in C(J) \).

Now, for \( t \in J \), we have

\[
|\mathcal{(F}y(t))| \leq |f(t) + \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y(\tau)| ds| \\
\leq ||f|| + \frac{a + b||y||}{\Gamma(\beta)} \int_0^t \kappa(t, s) \max_{[0, \sigma(s)]} |y(\tau)| ds + \frac{(a + b||y||)\kappa^*||y||}{\Gamma(\beta + 1)} (\varphi(t) - \varphi(0))^\beta.
\]

Hence

\[
||\mathcal{F}y|| \leq ||f|| + \frac{(a + b||y||)\kappa^*||y||}{\Gamma(\beta + 1)} (\varphi(1) - \varphi(0))^\beta.
\]

By assumption \( (a_\tau) \), if \( ||y|| \leq r_0 \), we get

\[
||\mathcal{F}y|| \leq ||f|| + \frac{(a + b||y||)\kappa^*r_0}{\Gamma(\beta + 1)} (\varphi(1) - \varphi(0))^\beta \\
\leq r_0.
\]

Therefore, \( \mathcal{F} \) maps \( B_{r_0} \) into itself.

Next, we consider the operator \( \mathcal{F} \) on the set \( B_{r_0}^+ = \{ y \in B_{r_0} : y(t) \geq 0, \forall t \in J \} \). It is clear that \( B_{r_0}^+ \neq \emptyset \) is closed, convex and bounded. By these facts and our assumptions, we obtain \( \mathcal{F} \) maps \( B_{r_0}^+ \) into itself.

In what follows, we will show that \( \mathcal{F} \) is continuous on \( B_{r_0}^+ \). For this, let \( (y_n) \) be a sequence in \( B_{r_0}^+ \) such that \( y_n \to y \) and we will show that \( \mathcal{F}y_n \to \mathcal{F}y \). We have, for \( t \in J \),

\[
|\mathcal{(F}y_n(t)) - (\mathcal{F}y)(t)| = \left| \frac{(Ty_n)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y_n(\tau)| ds - \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y(\tau)| ds \right| \\
\leq \left| \frac{(Ty_n)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y_n(\tau)| ds \right| \\
- \left| \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y(\tau)| ds \right| \\
+ \left| \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y_n(\tau)| ds \right| \\
+ \left| \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \varphi'(s)\kappa(t, s) \max_{[0, \sigma(s)]} |y(\tau)| ds \right| \\
\]

\[
|\mathcal{(F}y_n(t)) - (\mathcal{F}y)(t)| \leq \frac{a + b||y_n||}{\Gamma(\beta)} \int_0^t \kappa(t, s) \max_{[0, \sigma(s)]} |y_n(\tau)| ds + \frac{a + b||y||}{\Gamma(\beta + 1)} (\varphi(t) - \varphi(0))^\beta \\
\leq ||f_n|| + \frac{a + b||y||}{\Gamma(\beta + 1)} (\varphi(t) - \varphi(0))^\beta.
\]

Hence,

\[
||\mathcal{F}y_n - \mathcal{F}y|| \leq ||f_n|| + \frac{a + b||y||}{\Gamma(\beta + 1)} (\varphi(t) - \varphi(0))^\beta.
\]

By assumption \( (a_\tau) \) and \( ||y_n|| \leq r_0 \), we get

\[
||\mathcal{F}y_n - \mathcal{F}y|| \leq ||f_n|| + \frac{a + b||y||}{\Gamma(\beta + 1)} (\varphi(t) - \varphi(0))^\beta \\
\leq r_0.
\]

Therefore, \( \mathcal{F} \) maps \( B_{r_0}^+ \) into itself.
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By the continuity of $f$, Next, let

$$
\|y_n - y\| \leq \frac{\varepsilon \Gamma(\beta + 1)}{2\kappa^* r_0 (\varphi(1) - \varphi(0))^\beta} , \forall n \geq n_1.
$$

Also, there exists $n_2 \in \mathbb{N}$ such that

$$
\|y_n - y\| \leq \frac{\varepsilon \Gamma(\beta + 1)}{2\kappa^* (a + br_0) (\varphi(1) - \varphi(0))^\beta} , \forall n \geq n_2.
$$

Now, take $n \geq \max\{n_1, n_2\}$, then (2.5) gives us that

$$
\|Fy_n - Fy\| \leq \varepsilon.
$$

This shows that $F$ is continuous in $B_{r_0}^+$.  

Next, let $Y \subseteq B_{r_0}^+$ be a nonempty set. Let us choose $y \in Y$ and $t_1, t_2 \in J$ with $|t_2 - t_1| \leq \varepsilon$ for fixed $\varepsilon > 0$. Since no generality will loss, we will assume that $t_2 \geq t_1$. Then, by using our assumptions and (2.4), we obtain

$$
\begin{align*}
|\langle Fy(t_2), y(t_1) \rangle | & \leq |\langle f(t_2) - f(t_1), y(t_1) \rangle | + |\langle (Ty)(t_2)(K_y)(t_2) - (Ty)(t_2)(K_y)(t_1) \rangle | \\
& + |\langle (Ty)(t_2)(K_y)(t_1) - (Ty)(t_1)(K_y)(t_1) \rangle | \\
& \leq \omega(f, \varepsilon) + |\langle Ty(t_2)(K_y)(t_2) - (K_y)(t_1) \rangle | + |\langle (Ty)(t_2) - (Ty)(t_1) \rangle | |\langle (K_y)(t_1) \rangle | \\
& \leq \omega(f, \varepsilon) + \frac{(a + b\|y\|\|y\|)}{\Gamma(\beta + 1)} \left[ \omega_\kappa(\varepsilon, \cdot)(\varphi(1) - \varphi(0))^\beta + 2\kappa^*(\omega(\varphi, \varepsilon))^\beta \right] \\
& + \frac{\omega(Ty, \varepsilon)\|y\|\kappa^*(\varphi(t_1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} \\
& \leq \omega(f, \varepsilon) + \frac{r_0(a + br_0)}{\Gamma(\beta + 1)} \left[ \omega_\kappa(\varepsilon, \cdot)(\varphi(1) - \varphi(0))^\beta + 2\kappa^*(\omega(\varphi, \varepsilon))^\beta \right]
\end{align*}
$$
Hence,

\[ \omega(Fy, \varepsilon) \leq \omega(f, \varepsilon) + \frac{r_0(a + br_0)}{\Gamma(\beta + 1)} \left[ \omega(\varepsilon, \cdot)(\varphi(1) - \varphi(0))^\beta + 2\kappa^* (\omega(\varphi, \varepsilon))^\beta \right] \]

\[ + \frac{\kappa^* r_0(\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} \omega(Ty, \varepsilon). \]

Consequently,

\[ \omega(FY, \varepsilon) \leq \omega(f, \varepsilon) + \frac{r_0(a + br_0)}{\Gamma(\beta + 1)} \left[ \omega(\varepsilon, \cdot)(\varphi(1) - \varphi(0))^\beta + 2\kappa^* (\omega(\varphi, \varepsilon))^\beta \right] \]

\[ + \frac{\kappa^* r_0(\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} \omega(TY, \varepsilon). \]

The uniform continuity of the function \( \kappa \) on \( J \times J \) and the continuity of the functions \( f \) and \( \varphi \) on \( J \), implies the last inequality becomes

\[ \omega_0(FY) \leq \frac{\kappa^* r_0(\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} \omega_0(TY). \quad (2.6) \]

In the next step, fix arbitrary \( y \in Y \) and \( t_1, t_2 \in J \) with \( t_2 > t_1 \). Then, by our assumptions, we have

\[ |(Fy)(t_2) - (Fy)(t_1)| - [(Fy)(t_2) - (Fy)(t_1)] \]

\[ = \left| f(t_2) + \frac{(Ty)(t_2)}{\Gamma(\beta)} \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds 
\]

\[ - f(t_1) - \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds \right| \]

\[ - \left[ f(t_2) + \frac{(Ty)(t_2)}{\Gamma(\beta)} \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds 
\]

\[ - f(t_1) - \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds \right| \]

\[ \leq \left\{ |f(t_2) - f(t_1)| - [f(t_2) - f(t_1)] \right\} \]

\[ + \left| \frac{(Ty)(t_2)}{\Gamma(\beta)} \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds 
\]

\[ - \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds \right| \]

\[ + \left| \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds 
\]

\[ - \frac{(Ty)(t_2)}{\Gamma(\beta)} \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds \right| \]
\[-\frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[-\left\{ \left( \frac{(Ty)(t_2)}{\Gamma(\beta)} \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \right) \right\} \]

\[-\frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[+ \left\{ \left( \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \right) \right\} \]

\[-\frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[-\frac{(Tx)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[\leq \frac{((Ty)(t_2) - (Ty)(t_1)) - ((Ty)(t_1) - (Ty)(t_1))}{\Gamma(\beta)} \]

\[\times \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[+ \frac{(Ty)(t_1)}{\Gamma(\beta)} \left\{ \left[ \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \right) \right\} \]

\[\leq \left\{ \left[ \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \right) \right\} . \]  

(2.7)

But

\[\int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{0, \sigma(s)} |y(\tau)| \, ds - \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[= \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{0, \sigma(s)} |y(\tau)| \, ds - \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[+ \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds - \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[+ \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds - \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[= \int_0^{t_2} \varphi'(s) \kappa(t_2, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[+ \int_0^{t_1} \varphi'(s) \kappa(t_1, s) \max_{0, \sigma(s)} |y(\tau)| \, ds \]

\[+ \int_0^{t_1} \kappa(t_1, s) ((\varphi(t_2) - \varphi(s))^\beta - (\varphi(t_1) - \varphi(s))^\beta - 1) \max_{0, \sigma(s)} |y(\tau)| \, ds . \]
Since \( \kappa(t_2, s) \geq \kappa(t_1, s) \) (\( \kappa(t, s) \) is nondecreasing with respect to \( t \)), we have

\[
\int_{0}^{t_2} \frac{\varphi'(s)(\kappa(t_2, s) - \kappa(t_1, s)) \max_{[0, \sigma(s)]} |y(\tau)|}{(\varphi(t_2) - \varphi(s))^{1-\beta}} \, ds \geq 0 \tag{2.8}
\]

and, since \( \frac{1}{(\varphi(t_2) - \varphi(s))^{1-\beta}} \geq \frac{1}{(\varphi(t_1) - \varphi(s))^{1-\beta}} \) for \( s \in [0, t_1] \) then

\[
\int_{0}^{t_1} \varphi'(s) \kappa(t_1, s) [\left(\frac{\varphi(t_2) - \varphi(s)}{\varphi(t_1) - \varphi(s)}\right)^{\beta - 1} - \left(\frac{\varphi(t_1) - \varphi(s)}{\varphi(t_1) - \varphi(s)}\right)^{\beta - 1}] \max_{[0, \sigma(t_1)]} |y(\tau)| \, ds
\]

\[
+ \int_{t_1}^{t_2} \varphi'(s) \kappa(t_1, s) \max_{[0, \sigma(s)]} |y(\tau)| \, ds
\]

\[
\geq \int_{0}^{t_1} \varphi'(s) \kappa(t_1, t_1) [\left(\frac{\varphi(t_2) - \varphi(s)}{\varphi(t_1) - \varphi(s)}\right)^{\beta - 1} - \left(\frac{\varphi(t_1) - \varphi(s)}{\varphi(t_1) - \varphi(s)}\right)^{\beta - 1}] \max_{[0, \sigma(t_1)]} |y(\tau)| \, ds
\]

\[
+ \int_{t_1}^{t_2} \varphi'(s) \kappa(t_1, t_1) \max_{[0, \sigma(s)]} |y(\tau)| \, ds
\]

\[
= \kappa(t_1, t_1) \max_{[0, \sigma(t_1)]} |y(\tau)| \left[ \int_{0}^{t_2} \frac{\varphi'(s) \, ds}{(\varphi(t_2) - \varphi(s))^{1-\beta}} - \int_{0}^{t_1} \frac{\varphi'(s) \, ds}{(\varphi(t_1) - \varphi(s))^{1-\beta}} \right]
\]

\[
= \kappa(t_1, t_1) \frac{(\varphi(t_2) - \varphi(0))^{\beta} - (\varphi(t_1) - \varphi(0))^{\beta}}{\beta} \max_{[0, \sigma(t_1)]} |y(\tau)|
\]

\[
\geq 0. \tag{2.9}
\]

Finally, (2.8) and (2.9) imply that

\[
\int_{0}^{t_2} \frac{\varphi'(s) \kappa(t_2, s) \max_{[0, \sigma(s)]} |y(\tau)|}{(\varphi(t_2) - \varphi(s))^{1-\beta}} \, ds - \int_{0}^{t_1} \frac{\varphi'(s) \kappa(t_1, s) \max_{[0, \sigma(s)]} |y(\tau)|}{(\varphi(t_1) - \varphi(s))^{1-\beta}} \, ds \geq 0.
\]

The above inequality and (2.7) leads us to

\[
\frac{|(\mathcal{F}y)(t_2) - (\mathcal{F}y)(t_1)|}{|(Ty)(t_2) - (Ty)(t_1)|} \leq \frac{\Gamma(\beta)}{\Gamma(\beta + 1)} \times \int_{0}^{t_2} \frac{\varphi'(s) \kappa(t_2, s) \max_{[0, \sigma(s)]} |y(\tau)|}{(\varphi(t_2) - \varphi(s))^{1-\beta}} \, ds
\]

\[
\leq \frac{\kappa^* r_0 (\varphi(1) - \varphi(0))^{\beta}}{\Gamma(\beta + 1)} \, d(Ty).
\]

Thus,

\[
d(\mathcal{F}y) \leq \frac{\kappa^* r_0 (\varphi(1) - \varphi(0))^{\beta}}{\Gamma(\beta + 1)} \, d(Ty)
\]

and therefore,

\[
d(\mathcal{F}Y) \leq \frac{\kappa^* r_0 (\varphi(1) - \varphi(0))^{\beta}}{\Gamma(\beta + 1)} \, d(TY). \tag{2.10}
\]
Finally, (2.6) and (2.10) give us that
\[
\omega_0(\mathcal{F}Y) + d(\mathcal{F}Y) \leq \frac{\kappa^* r_0 (\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} (\omega_0(\mathcal{F}Y) + d(TY))
\]
or
\[
\mu(\mathcal{F}Y) \leq \frac{r_0 \kappa^* (\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} \mu(TY)
\]
\[
\leq \frac{\kappa^* c r_0 (\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta + 1)} \mu(Y).
\]
Since \( \frac{\kappa^* r_0 c}{\Gamma(\beta + 1)} < (\varphi(1) - \varphi(0))^{-\beta} \), \( \mathcal{F} \) is a contraction operator with respect to \( \mu \).

Finally, by Theorem 1.3, \( \mathcal{F} \) has at least one fixed point, or equivalently, (1.1) has at least one nondecreasing solution in \( B_{r_0} \). This finishes our proof.

Next, we present the following numerical example in order to illustrate our results.

**Example 2.2.** Let us consider the following integral equation with maximum

\[
y(t) = \arctan t + \frac{y(t)}{5 \Gamma(1/2)} \int_0^t \frac{\sqrt{t^2 + s^2} \max_{0 \leq \ln(s+1)}}{2 \sqrt{s + 1} \sqrt{t + 1} - \sqrt{s + 1}} ds, \quad t \in J. \tag{2.11}
\]

Notice that (2.11) is a particular case of (1.1), where \( f(t) = \arctan t, (Ty)(t) = y(t)/5, \beta = 1/2, \varphi(s) = \sqrt{s + 1}, \kappa(t, s) = \sqrt{t^2 + s^2} \) and \( \sigma(t) = \ln(t + 1) \).

It is not difficult to see that assumptions (a1), (a2), (a3), (a4), (a5) and (a6) are verified with \( \|f\| = \pi/4, c = 1/5, a = 0, b = 1/5 \) and \( \kappa^* = \sqrt{2} \).

Now, the inequality (2.1) in assumption (a7) takes the expression

\[
\frac{\pi}{4} + \frac{\sqrt{2} \sqrt{2 - 1}}{5 \Gamma(3/2)} r_0^2 \leq r_0
\]

which is satisfied by \( r_0 = 1 \). Moreover,

\[
\frac{c \kappa^* r_0}{\Gamma(\beta + 1)} = \frac{\sqrt{2}}{5 \Gamma(3/2)} \leq 0.32 < (\varphi(1) - \varphi(0))^{-\beta} = \frac{1}{\sqrt{\sqrt{2} - 1}} \approx 1.56.
\]

Therefore, by Theorem 2.1, (2.11) has at least one continuous and nondecreasing solution which is located in the ball \( B_1 \).

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References


