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Unique Existence and Nonexistence of Limit Cycles for a Classical Liénard System

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Abstract

J. Graef in 1971 has studied the uniformly boundedness of the solution orbits under some condition and further proved the existence of limit cycles by the Poincaré–Bendixson's theorem. Recently, M. Cioni and G. Villari in 2015 gave the same result as that of J. Graef under weaker conditions. In this paper, the unique existence and the nonexistence of limit cycles for a classical Liénard system are investigated under the recent conditions. It shall be shown that our results are applied to two concrete examples.

AMS Subject Classifications: 34C07, 34C25, 34C26, 34D20.
Keywords: Liénard system, uniform boundedness, Limit cycles, global stability.

1 Introduction

In this paper, sufficient conditions in order that a classical Liénard system

$$\frac{dx}{dt} = y - F(x), \qquad \frac{dy}{dt} = -g(x) \tag{L}$$

has a unique limit cycle or no limit cycles are discussed, where $F(x)$ and $g(x)$ are continuous on an open interval I which contains the origin. The functions $F(x)$ and $g(x)$ satisfy smoothness conditions for the uniqueness of solutions of initial value problems.

The qualitative property of the limit cycles of system (L) has studied by many mathematicians, physicists, economists and engineers and so on. Thus, the results play an important role to resolve the scientific phenomenon. Our purpose is to discuss the following two cases under the recent condition in [1].

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- Nonexistence of limit cycles and Global unstability of solution orbits.
- Unique existence of limit cycles.

Throughout, we assume that the standard conditions for system (L)

(H1) $\dfrac{g(x)}{x} > 0,$ (H2) $g(x)F(x) < 0$ for $|x| < \epsilon$ and ϵ is small

are satisfied. We see from these conditions that the equilibrium point is the origin only and it is unstable.

J. Graef [4] in 1971 has studied the uniformly boundedness of the solution orbits under the condition

(C1) $\lim\limits_{x\to\pm\infty} F(x) \pm G(x) = \pm\infty$, where $G(x) = \displaystyle\int_0^x g(\xi)d\xi$.

Further, the existence of limit cycles was proved under the conditions (C1) and

(C2) $\exists k > 0$ such that $F(x) \geq A$ $(x \geq k)$ and $F(x) \leq -A$ $(x \leq -k)$

for some fixed constant A.

Recently, M. Cioni and G. Villari [1] in 2015 gave by the energy level comparison method the following.

Proposition 1.1. *System* (L) *has at least one limit cycles if the conditions*

(C3) $\limsup\limits_{x\to\pm\infty} G(x) \pm F(x) = +\infty,$

(C4) $\exists K_1 > \exists K_2$ *such that* $F(x) \geq K_1$ $(x > \beta > 0)$, $F(x) \leq K_2$ $(x < \alpha < 0)$

are satisfied.

This is the same result as in [4], but remark that (C3) and (C4) are an improvement of (C1) and (C2), respectively.

We are interesting in the case of which (C3) is satisfied, but (C4) is not satisfied. As the result, we give a simple criterion for the nonexistence of limit cycles of system (L) with these conditions. Further, we state the unique existence of the limit cycle under the conditions (C3) and (C4) by using the uniqueness theorem in the author's method [6, 8]. This is a partial improvement of [1]. These results are stated and proved in the next section. In Section 3, our results are applied to the interesting examples and a criterion for the nonexistence of limit cycles is given simply. We also note that the results for these systems are not given in the well-known textbooks such as [9].

2 Main Results and Proofs

From (H2), we divide the results to four cases by whether the curve $y = F(x)$ intersect the x-axis.

(i) $\nexists a_1, \nexists a_2$, (ii) $\exists a_2 > 0$, (iii) $\exists a_1 < 0$, (iv) $\exists a_1 < 0 < \exists a_2$,

where a_i $(i = 1, 2)$ are solutions of the equation $F(x) = 0$.

First, we give our results for the nonexistence of limit cycles. The following plays a fundamental role for our purpose.

Lemma 2.1 (See [3]). *If the plane curve* $(F(x), G(x))$ *has no intersecting points with itself for all* $x \in (a, b)$ *and* $a < 0 < b$*, system* (L) *has no limit cycles contained in the domain*

$D_1 = \{(x, y) \mid a < x < b, y \in \mathbb{R}\}$ *or* $D_2 = \{(x, y) \mid x < b, y \in \mathbb{R}\}$ *or*
$D_3 = \{(x, y) \mid a < x, y \in \mathbb{R}\}$.

The following is clear from the above lemma or [5, Lemma 3.1].

Theorem 2.2 (Case (i)). *If the curve* $y = F(x)$ *doesn't intersect the* x*-axis except the origin, system* (L) *has no limit cycles and the equilibrium point* $(0, 0)$ *is global unstability.*

The following results are the case of which the condition (C4) are not satisfied.

Theorem 2.3 (Case (ii)). *Let* α_1 *be a negative number such that* $G(a_2) = G(\alpha_1)$*. If there exist* K_1 *and* K_2 *such that* $F(x) \leq K_1$ $(x > a_2)$*,* $F(x) \geq K_2$ $(x < \alpha < 0)$*, further system* (L) *satisfies the conditions* (C3) *and*

(C5) $(0 <) K_1 < M = \min(F(\alpha_1), K_2)$,

then the system has no limit cycles.

Theorem 2.4 (Case (iii)). *Let* α_2 *be a positive number such that* $G(a_1) = G(\alpha_2)$*. If there exist* K_1 *and* K_2 *such that* $F(x) \leq K_1$ $(x > \beta > 0)$*,* $F(x) \geq K_2$ $(x < a_1)$*, further system* (L) *satisfies the conditions* (C3) *and*

(C6) $N = \max(F(\alpha_2), K_1) < K_2 \, (< 0)$,

then the system has no limit cycles.

Proof of Theorem 2.3. For all u_1 and u_2 such that $G(u_1) = G(u_2)$, $\alpha_1 \leq u_1 < 0$ and $0 < u_2 \leq a_2$, we have $F(u_1) > 0$ and $F(u_2) < 0$. Thus, we get $F(u_1) - F(u_2) > 0$. On the other hand, for all u_1 and u_2 such that $G(u_1) = G(u_2)$, $u_1 < \alpha_1$ and $a_2 < u_2$, we have from the condition (C5) that $F(u_1) - F(u_2) > M - K_1 > 0$. These facts means that the plane curve $(F(x), G(x))$ has no intersecting points with itself. Therefore, we conclude from Lemma 2.1 that system (L) has no limit cycles. □

The proof of Theorem 2.4 is given by the same discussion as Theorem 2.3.

Corollary 2.5. *Under the conditions in Theorem* 2.3 *or Theorem* 2.4 *the equilibrium point* $(0, 0)$ *of system* (L) *is global unstability.*

Proof. Under the conditions in Theorem 2.3 or Theorem 2.4 system (L) has no limit cycles. Further, from the condition (H2) system (L) has no homoclinic orbits (for instance see [7]) and the equilibrium point $(0, 0)$ is unstable. Thus, the origin is global unstable. □

Remark 2.6. In the case (iv), we see from [1] that system (L) has at least one limit cycles under the conditions (C3) and (C4).

Next, we discuss the unique existence of limit cycles.

Theorem 2.7 (Case(ii))**.** *Let α_1 be a negative number such that $G(a_2) = G(\alpha_1)$ and $p = \{x \mid \max_{x\in(-\infty,0)} F(x)\}$. If system* (L) *satisfies the conditions* (C3)*,* (C4)*,* (C7) $\alpha_1 \le p$ *and besides*

(C8) $F(x)$ *is monotone increasing for $x < \alpha_1$ and $x > \alpha_2$,*

then the system has a unique limit cycle. Also it is stable and hyperbolic.

Theorem 2.8 (Case (iii))**.** *Let α_2 be a positive number such that $G(a_1) = G(\alpha_2)$ and $q = \{x \mid \min_{x\in(0,+\infty)} F(x)\}$. If system* (L) *satisfies the conditions* (C3)*,* (C4)*,* (C9) $q \le \alpha_2$ *and besides*

(C10) $F(x)$ *is monotone increasing for $x < a_1$ and $x > \alpha_2$,*

then the system has a unique limit cycle. Also it is stable and hyperbolic.

Proof of Theorem 2.7. The existence of limit cycles of system (L) with the conditions (H1) and (H2) are guaranteed by the conditions (C3) and (C4) (see [1]). From Lemma 2.1 or [5, Lemma 3.1] the limit cycles must intersect the lines $x = \alpha_1$ and $x = a_2$. On the other hand, from [8] or [6, appendix], if system (L) satisfies the conditions (C7) and (C8), the limit cycle must be at most one. Thus, the proof is completed now by these facts. □

Similarily, Theorem 2.8 is proved.

The case (iv) has been studied in many papers. For instance see [2, 5, 6, 9].

3 Several Examples

We present two phase portraits of the concrete examples for system (L) as applications of our results.

Example 3.1. Consider system (L) in the form

$$F(x) = \begin{cases} (x^2 - x)e^{-x} & \text{for } x \ge -1 \\ 5(x^2 + x)e^{x+2} + 2e & \text{for } x \le -1, \end{cases} \qquad g(x) = x. \tag{E1}$$

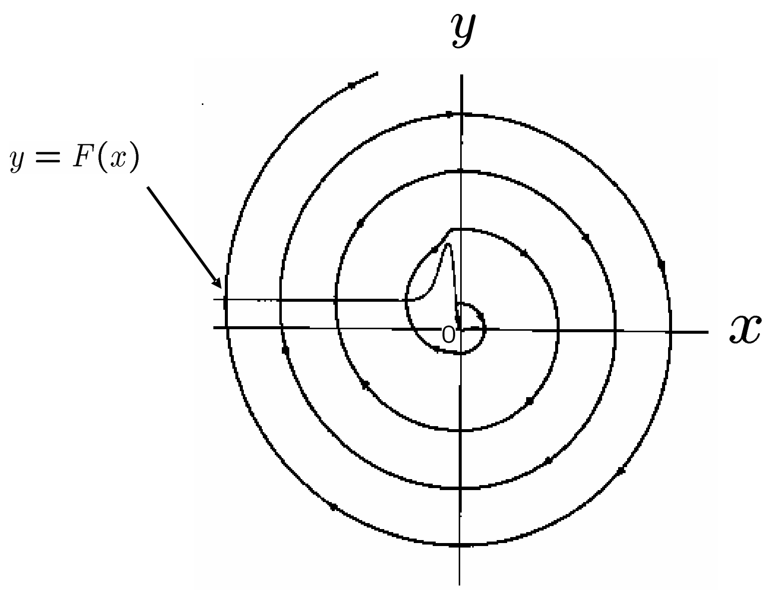

Figure 3.1: Nonexistence of limit cycles for system (E1)

We can take from $G(x) = x^2/2$ that $a_2 = 1$, $\alpha_1 = -1$, $K_1 = 4.3/e^{2.6} < K_2 = 2e$ and $F(-1) = 2e$. Since $K_1 < M = 2e$, the system satisfies all conditions of Theorem 2.3. Thus, we see that system (E1) has no limit cycles as is shown in Figure 3.1.

Example 3.2. Consider system (L) in the form

$$F(x) = \begin{cases} 3(x-3)e^{x-6} & \text{for } x \geq 3 \\ (x^2 - 3x)e^{-x} & \text{for } -1 \leq x \leq 3 \\ 9(x^2 + x)e^{x+2} + 4e & \text{for } x \leq -1, \qquad g(x) = x. \end{cases} \tag{E2}$$

We can take from $G(x) = x^2/2$ that $a_2 = 3$, $\alpha_1 = -3 < p = -3 - \sqrt{5}/2$, $K_2 = 38.7/e^{0.6} + 4e < K_1 = 38.7 + 4e$. Also, we have $F^{'}(x) = 9(x^2 + 3x + 1)e^{x+2} > 0$ for $x < -3$, $F^{'}(x) = 3(x-2)e^{x-6} > 0$ for $x > 3$. Thus, we see that the system satisfies all conditions of Theorem 2.7. Therefore, we also have that system (E2) has a unique limit cycle, and it is stable and hyperbolic. See Figure 3.2.

4 Appendix

In this section, we shall consider the case of which the condition

(H3) $g(x)F(x) > 0$ for $|x| < \epsilon$ and ϵ is small

is satisfied instead of (H2). Then we see that the only equilibrium point $(0, 0)$ is stable. We apply the tool in [7] to this case. We assume that

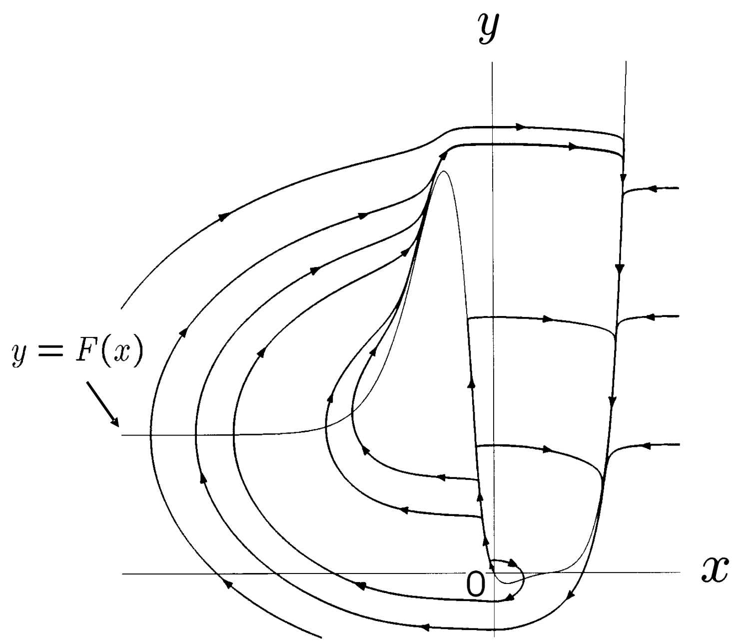

Figure 3.2: Unique existence of limit cycles for system (E2)

(C11) $\exists a_1 < 0$ such that such that $x(x - a_1)F(x) > 0$ for $x \neq 0$ and $x \neq a_1$.

and the conditions in the Proposition 1.1.

Consider a function $\varphi(x)$ with the condition

$$\varphi \in C^1,\ \varphi(\alpha) = 0 \text{ and } \varphi'(x) > 0 \text{ for } x > \alpha \in (0, \alpha_2],$$

where α_2 is a positive number satisfying the equation $G(\alpha_2) = G(a_1)$. We note from the condition (H1) that α_2 is unique.

Our result is the following.

Theorem 4.1. *Assume the conditions* (H1), (H3),(C3),(C4) *and* (C11). *System* (L) *has no limit cycles if there exists a function* $\varphi(x)$ *such that*

$$F(x) > \varphi(x) > 0 \ \text{ and } \ g(x) \leq \varphi'(x)[F(x) - \varphi(x)]$$

for all $x \geq \alpha \in (0, \alpha_2]$.

Corollary 4.2. *Under the conditions in Theorem* 4.1 *for system* (L) *the equilibrium point* $(0, 0)$ *of the system is globally asymptotic stable.*

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